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On The Arithmetic-Geometric Means Of Positive Integers And The Number e^*

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Abstract

Assume that A_n and G_n denote the arithmetic and geometric means of the integers $1, 2, \ldots, n$, respectively. It this paper, we obtain some sharp inequalities and the asymptotic expansion of the ratio A_n/G_n.

1 Introduction

Assume that $(a_n)_{n\in\mathbb{N}}$ is a positive real sequence. Through the paper, we denote the arithmetic and geometric means of the numbers $a_1, a_2, \ldots, a_n$, respectively, by $A(a_1, \ldots, a_n)$ and $G(a_1, \ldots, a_n)$. A nice relation which connects the number e to the mean values

$$A_n := A_n(1, 2, \ldots, n) \text{ and } G_n := G_n(1, 2, \ldots, n)$$

asserts (see [2]) that

$$\lim_{n\to\infty} \frac{A_n}{G_n} = \frac{e}{2},$$

which is a consequence of the Stirling's approximation

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{1}$$

Motivated by this fact, recently we obtained similar asymptotic result concerning the sequence of prime numbers, by proving validity of

$$\frac{A(p_1, \ldots, p_n)}{G(p_1, \ldots, p_n)} = \frac{e}{2} + O\left(\frac{1}{\log n}\right),$$

where as usual p_n denotes the nth prime number. More precisely, we computed the value of constant of O-term for the case of prime numbers (see [1]).

In this paper, we obtain various properties of the ratio A_n/G_n, including sharp and explicit lower and upper bounds, precise asymptotic expansion, and monotonicity. More precisely, we show the following results.

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THEOREM 1. For any integers $m \geqslant 1$ and $n \geqslant 1$, let

$$J := J_m(n) = \sum_{r=1}^{m} \frac{B_{2r}}{(2r)(2r-1)n^{2r-1}} \quad \text{and} \quad u_m(n) = \frac{|B_{2m}|}{2m(2m-1)n^{2m-1}}, \tag{2}$$

where B_n denote the Bernoulli numbers. Then, for any integers $m \geqslant 1$ and $n \geqslant 1$,

$$\frac{e}{2}\left(1+\frac{1}{n}\right) e^{-\frac{1}{n}(\log\sqrt{2\pi n}+J+u_m(n))} \leqslant \frac{A_n}{G_n} \leqslant \frac{e}{2}\left(1+\frac{1}{n}\right) e^{-\frac{1}{n}(\log\sqrt{2\pi n}+J-u_m(n))}. \tag{3}$$

COROLLARY 2. For any integer $n \geqslant 1$, we have

$$\frac{A_n}{G_n} = \frac{e}{2}\left(1 - \frac{1}{n}\log\Big(\frac{\sqrt{2\pi n}}{e}\Big) + O\Big(\frac{\log^2 n}{n^2}\Big)\right) \tag{4}$$

and

$$\left(\frac{A_n}{G_n}\right)^n = \frac{e^{n+1}}{\sqrt{\pi n}\, 2^{n+\frac{1}{2}}}\left(1 + O\Big(\frac{1}{n}\Big)\right). \tag{5}$$

COROLLARY 3. For any integer $n \geqslant 1$, we have

$$\frac{A_n}{G_n} < \frac{e}{2}. \tag{6}$$

The proof of the above results is hidden in heart of the following precise form of Stirling's approximation for $n!$.

LEMMA 4. For any integers $m \geqslant 1$ and $n \geqslant 1$, we have

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{J-u_m(n)} \leqslant n! \leqslant \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{J+u_m(n)}. \tag{7}$$

Our last result concerning the ratio A_n/G_n asserts that the sequence with general term A_n/G_n is indeed strictly increasing.

THEOREM 5. For any integer $n \geqslant 1$, we have

$$\frac{A_{n+1}}{G_{n+1}} > \frac{A_n}{G_n}. \tag{8}$$

Finally, we note that in our proofs we will use the notion of Bernoulli functions $B_n(\{x\})$, where $\{x\}$ denotes the fractional part of the real x. Among the proofs we obtain an improper integral concerning the Bernoulli functions as follows.

COROLLARY 6. For any integer $m \geqslant 1$, we have

$$\frac{1}{m}\int_1^{\infty} \frac{B_{2m}(\{x\})}{x^{2m}} dx = \log\left(\frac{2\pi}{e^2}\right) + \sum_{r=1}^{m} \frac{B_{2r}}{r(2r-1)}.$$

2 Proofs

PROOF OF LEMMA 4. We apply Euler–Maclaurin summation formula (see [3]) by letting $g(k) = \log k$, from which we obtain

$$\log n! = n \log n - n + \frac{1}{2} \log n + 1 - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)(2r-1)} + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)(2r-1)n^{2r-1}} + R_m,$$

where $m \geqslant 1$ is any fixed integer and

$$R_m = \int_1^{\infty} \frac{B_{2m}(\{x\})}{2mx^{2m}} dx - \int_n^{\infty} \frac{B_{2m}(\{x\})}{2mx^{2m}} dx.$$

Thus, we obtain

$$\log n! = n \log n - n + \frac{1}{2} \log n + C_m + J - I \tag{9}$$

with

$$C_m = 1 + \int_1^{\infty} \frac{B_{2m}(\{x\})}{2mx^{2m}} dx - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)(2r-1)}, \tag{10}$$

a constant depending, at most, only on m. Also, the remainders J, defined as in (2), and

$$I = \int_n^{\infty} \frac{B_{2m}(\{x\})}{2mx^{2m}} dx \tag{11}$$

satisfy $J \ll \frac{1}{n}$ and $I \ll \frac{1}{n}$ as $n \to \infty$. So, if we let

$$D_n = \frac{n!}{\left(\frac{n}{e}\right)^n n^{\frac{1}{2}}} \text{ and } D = \lim_{n \to \infty} D_n,$$

then we have

$$C_m = \lim_{n \to \infty} \left[\log n! - \left(n \log n - n + \frac{1}{2} \log n \right) \right] = \lim_{n \to \infty} \log D_n = \log D.$$

A simple computation shows that

$$(D_n)^2 = \frac{n!^2 e^{2n}}{n^{2n+1}} \quad \text{and} \quad D_{2n} = \frac{(2n)! e^{2n}}{(2n)^{2n+\frac{1}{2}}}.$$

Hence, we obtain

$$\frac{(D_n)^2}{D_{2n}} = \frac{n!^2 2^{2n}}{(2n)!} \sqrt{\frac{2}{n}}.$$

We recall the Wallis product formula for π (see [5] for an elementary proof), which asserts that

$$\lim_{n \to \infty} \prod_{k=1}^{n} \left(\frac{2k}{2k-1} \times \frac{2k}{2k+1} \right) = \frac{\pi}{2}.$$

We note that

$$\prod_{k=1}^{n}\left(\frac{2k}{2k-1}\times\frac{2k}{2k+1}\right)=\left(\frac{n!^2 2^{2n}}{(2n)!}\right)^2\frac{1}{2n+1}$$
$$=\left(\frac{(D_n)^2}{D_{2n}}\sqrt{\frac{n}{2}}\right)^2\frac{1}{2n+1}=\left(\frac{(D_n)^2}{D_{2n}}\right)^2\frac{n}{2(2n+1)}.$$

Hence, we get

$$\frac{D^2}{4}=\lim_{n\to\infty}\left(\frac{(D_n)^2}{D_{2n}}\right)^2\frac{n}{2(2n+1)}=\frac{\pi}{2}.$$

Thus, we obtain $D=\sqrt{2\pi}$, and consequently

$$C_m=\log D=\log\sqrt{2\pi}\text{ for any integer } m\geqslant 1. \tag{12}$$

Therefore, by using (9), we imply that

$$n!=\left(\frac{n}{e}\right)^n\sqrt{n}\,e^{C_m}e^{J-I}=\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\,e^{J-I}. \tag{13}$$

In particular, we obtain Stirling's approximation for $n!$ as in (1). More precisely, we have

$$|I|\leqslant\int_n^\infty\frac{|B_{2m}(\{x\})|}{2mx^{2m}}dx\leqslant\frac{|B_{2m}|}{2m}\int_n^\infty\frac{dx}{x^{2m}}=u_m(n).$$

This completes the proof of Lemma 4.

We apply the relations (10) and (12) to obtain Corollary 6.

PROOF OF COROLLARY 2. By using (13), we obtain

$$\frac{A_n}{G_n}=\frac{e}{2}\left(1+\frac{1}{n}\right)(2\pi n)^{-\frac{1}{2n}}e^{-\frac{J-I}{n}}=\frac{e}{2}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}(\log\sqrt{2\pi n}+J-I)}. \tag{14}$$

Thus, we have

$$\left(\frac{A_n}{G_n}\right)^n=\left(\frac{e}{2}\right)^n\left(1+\frac{1}{n}\right)^n e^{-(\log\sqrt{2\pi n}+J-I)}.$$

We use the expansion $\left(1+\frac{1}{n}\right)^n=e\left(1+O(\frac{1}{n})\right)$ to conclude the proof of (5). To prove (4), we use (14) with the approximation

$$e^{-\frac{1}{n}(\log\sqrt{2\pi n}+J-I)}=1-\frac{1}{n}\log\sqrt{2\pi n}+O\Big(\frac{\log^2 n}{n^2}\Big).$$

This completes the proof of Corollary 2.

PROOF OF THEOREM 1. We start from the fact that

$$\frac{A_n}{G_n}=\frac{n+1}{2\,n!^{\frac{1}{n}}},$$

and then, we use the sharp inequalities in (7) to complete the proof.

PROOF OF COROLLARY 3. The assertion is valid for $n = 1$. We consider the right hand side of the inequalities in (3) with $m = 5$. In order to prove (6), we require to have

$$\left(1+\frac{1}{n}\right) e^{-\frac{1}{n}(\log\sqrt{2\pi n}+J_5(n)-u_5(n))} < 1. \tag{15}$$

Considering the inequality $\left(1+\frac{1}{n}\right)^n < e$, which is valid for any integer $n \geqslant 1$, we observe that the inequality (15) holds true, provided

$$f(n) := J_5(n) - u_5(n) - 1 + \log\sqrt{2\pi n} > 0.$$

The function $f(x)$, defined over $x \in [1, \infty)$, is strictly increasing and $f(1)f(2) < 0$. Thus, $f(n) > 0$ for $n \geqslant 2$, from which we imply validity of (6) for $n \geqslant 2$. This completes the proof of Corollary 3.

PROOF OF THEOREM 5. The inequality (8) is equivalent to

$$n! > (n+1)^n \left(\frac{n+1}{n+2}\right)^{n(n+1)}.$$

We prove the last inequality by induction on n. Clearly, it is ture for $n = 1$. To deduce the $(n+1)^{\text{th}}$ step from the n^{th} step, we require to have

$$(n+1)^{n+1}\left(\frac{n+1}{n+2}\right)^{n(n+1)} > (n+2)^{n+1}\left(\frac{n+2}{n+3}\right)^{(n+1)(n+2)},$$

or equivalently, we should have

$$(n+1)^{n+1}(n+3)^{n+2} > (n+2)^{2n+3}, \tag{16}$$

for any integer $n \geqslant 1$. Now, we note that (16) is equivalent by the assertion $e_{n+1} < e_{n+2}$ for any integer $n \geqslant 1$, where

$$e_n = \left(1+\frac{1}{n}\right)^n. \tag{17}$$

The sequence with general term e_n is strictly increasing, because if we apply the Arithmetic–Geometric mean inequality (see [4] for a very fast and elementary proof) on the numbers

$$1, \overbrace{\frac{1}{n+1}, \ldots, \frac{1}{n+1}}^{n\text{ times}},$$

we imply that

$$\frac{1+n\left(1+\frac{1}{n}\right)}{n+1} > \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n},$$

or equivalently

$$1 + \frac{1}{n+1} > \left(1 + \frac{1}{n}\right)^{\frac{n}{n+1}},$$

and the later inequality is $e_{n+1} > e_n$. The proof is complete.

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References

[1] M. Hassani, On the ratio of the arithmetic and geometric means of the prime numbers and the number e, Int. J. Number Theory, 9(2013), 1593–1603.

[2] B. J. McCartin, e: The master of all, Math. Intelligencer, 28(2006), 10–21.

[3] A. M. Odlyzko, Asymptotic Enumeration Methods, Handbook of combinatorics, Vol. 1, 2, 1063–1229, Elsevier, Amsterdam, 1995.

[4] O. A. S. Karamzadeh, One-line proof of the AM-GM inequality, Math. Intelligencer, 33(2011), Page 3.

[5] J. Wästlund, An elementary proof of the Wallis product formula for pi, Amer. Math. Monthly, 114(2007), 914–917.