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Scalar Wave Scattering By Two-Layer Radial Inhomogeneities*

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Abstract

It is shown that the iteration technique gives a better approximation for the problem with long wavelengths.

1 Introduction

This paper is based on the study of the scattering of a scalar plane wave by an inhomogeneous medium which can be applied in many interests such as scattering by inhomogeneous spheres and scattering of acoustic waves in the ocean. Since the analogous classical problems with scattering by spherically symmetric inhomogeneities have not been thoroughly studied, the purpose of this paper is to show that the classical problems and some simple solvable problems can be simply treated by a quantum-mechanical method. Moreover, in practice (in optical and industrial applications at least) the inhomogeneous scattering media will be piecewise constant continuous, so a useful approach to the problem may be to mimic the continuous cases for piecewise increasing or decreasing refractive index profiles. Therefore, the application of the Jost function formulation of potential scattering theory [1] to the scattering of a scalar plane wave by a medium with a piecewise constant two-layer spherical inhomogeneity is of interest.

2 Scattering From a Piecewise Constant by Multi-Layer Spherically Symmetric Inhomogeneities

We are now in a position to apply the method outlined in [1] to the problem of scattering from a piecewise constant in a multi-layer spherical inhomogeneities. For a three-layer inhomogeneity we define the following potential

$$\begin{aligned} &Region\ 1: V(r) = -V_1, k(r) = k_1, r < R_1; \\ &Region\ 2: V(r) = -V_2, k(r) = k_2, R_1 < r < R_2; \\ &Region\ 3: V(r) = 0, k(r) = k, r > R_2. \end{aligned} \tag{1}$$

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The solutions in the three regions are:

$$\begin{aligned}
&Region\ 1: u^{(1)}_{\lambda-\frac{1}{2}}(k_1,r) = r[Aj_{\lambda-\frac{1}{2}}(k_1r) + By_{\lambda-\frac{1}{2}}(k_1r)];\\
&Region\ 2: u^{(2)}_{\lambda-\frac{1}{2}}(k_2,r) = r[Cj_{\lambda-\frac{1}{2}}(k_2r) + Dy_{\lambda-\frac{1}{2}}(k_2r)];\\
&Region\ 3: u^{(3)}_{\lambda-\frac{1}{2}}(k,r) = r[Eh^{(1)}_{\lambda-\frac{1}{2}}(kr) + Fh^{(2)}_{\lambda-\frac{1}{2}}(kr)].
\end{aligned}$$

where again $j_{\lambda-\frac{1}{2}}(k_1r)$, $y_{\lambda-\frac{1}{2}}(k_2r)$, $h^{(1)}_{\lambda-\frac{1}{2}}(kr)$, and $h^{(2)}_{\lambda-\frac{1}{2}}(kr)$ are spherical Bessel, Neumann, and Hankel functions of the first kind and second kind, respectively.

Choosing $u^{(1)}_{\lambda-\frac{1}{2}}(k_1r)$ to be $\phi_1(\lambda,k_1,r)$ and imposing the boundary conditions at $r=0$ (see [1, (13)]), we find that $B=0$ and

$$\phi_1(\lambda,k_1,r) = 2^{\lambda+\frac{1}{2}}\pi^{-\frac{1}{2}}k_1^{-\lambda+\frac{1}{2}}\Gamma(\lambda+1)rj_{\lambda-\frac{1}{2}}(k_1r)$$

and

$$\phi_1^{'}(\lambda,k_1,r) = 2^{\lambda+\frac{1}{2}}\pi^{-\frac{1}{2}}k_1^{-\lambda+\frac{1}{2}}\Gamma(\lambda+1)\times[j_{\lambda-\frac{1}{2}}(k_1r) + k_1rj^{'}_{\lambda-\frac{1}{2}}(k_1r)],$$

where the prime denotes differentiation with respect to the argument of the function, Γ is the gamma function, and we have used the following series representation for $j_{\lambda-\frac{1}{2}}(k_1r)$ [Handbook of Mathematical Functions (McGraw Hill Book, p. 263)]:

$$j_{\lambda-\frac{1}{2}}(k_1r) = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{\frac{1}{2}}(k_1r/2)^{\lambda+2n-\frac{1}{2}}}{2n!\Gamma(\lambda+n+1)},\ \lambda-\frac{1}{2}\neq -1,-2,-3,....$$

Choosing $u^{(2)}_{\lambda-\frac{1}{2}}(k_2r)$ to be $\phi_2(\lambda,k_2,r)$ and imposing the continuity at the boundary $r=R_1$ by matching the continuity of ϕ_1 with ϕ_2 and ϕ_1' with ϕ_2', we have

$$\phi_2(\lambda,k_2,r) = r\left[Cj_{\lambda-\frac{1}{2}}(k_2r) + Dy_{\lambda-\frac{1}{2}}(k_2r)\right]$$

and

$$\phi_2^{'}(\lambda,k_2,r) = C\left[j_{\lambda-\frac{1}{2}}(k_2r) + k_2rj^{'}_{\lambda-\frac{1}{2}}(k_2r)\right] + D\left[y_{\lambda-\frac{1}{2}}(k_2r) + k_2ry^{'}_{\lambda-\frac{1}{2}}(k_2r)\right],$$

where

$$C = -\frac{m\left(j_{\lambda-\frac{1}{2}}(k_1R_1)y'_{\lambda-\frac{1}{2}}(k_2R_1) - \frac{k_1}{k_2}j'_{\lambda-\frac{1}{2}}(k_1R_1)y_{\lambda-\frac{1}{2}}(k_2R_1)\right)}{j'_{\lambda-\frac{1}{2}}(k_2R_1)y_{\lambda-\frac{1}{2}}(k_2R_1) - j_{\lambda-\frac{1}{2}}(k_2R_1)y'_{\lambda-\frac{1}{2}}(k_2R_1)},$$

$$D = \frac{m\left(j_{\lambda-\frac{1}{2}}(k_1R_1)j'_{\lambda-\frac{1}{2}}(k_2R_1) - \frac{k_1}{k_2}j'_{\lambda-\frac{1}{2}}(k_1R_1)j_{\lambda-\frac{1}{2}}(k_2R_1)\right)}{j'_{\lambda-\frac{1}{2}}(k_2R_1)y_{\lambda-\frac{1}{2}}(k_2R_1) - j_{\lambda-\frac{1}{2}}(k_2R_1)y'_{\lambda-\frac{1}{2}}(k_2R_1)},$$

and

$$m = 2^{\lambda+\frac{1}{2}}\pi^{-\frac{1}{2}}k_1^{-\lambda+\frac{1}{2}}\Gamma(\lambda+1).$$

Choosing $u^{(3)}_{\lambda-\frac{1}{2}}(k,r)$ to be $f(\lambda,k,r)$ and imposing the Jost solution condition at infinity (see [1, (20)]) we find that $E=0$, $F=ke^{-i\frac{\pi}{2}(\lambda+\frac{1}{2})}$, and hence

$$f(\lambda,k,r)=ke^{-i\frac{\pi}{2}(\lambda+\frac{1}{2})}rh^{(2)}_{\lambda-\frac{1}{2}}(kr)$$

and

$$f^{'}(\lambda,k,r)=ke^{-i\frac{\pi}{2}(\lambda+\frac{1}{2})}\left[h^{(2)}_{\lambda-\frac{1}{2}}(kr)+krh^{(2)^{'}}_{\lambda-\frac{1}{2}}(kr)\right],$$

where we have used the following asymptotic form for $h^{(2)}_{\lambda-\frac{1}{2}}(kr)$:

$$\lim_{kr\to\infty}h^{(2)}_{\lambda-\frac{1}{2}}(kr)=\frac{1}{kr}e^{-i\left[kr-\frac{\pi}{2}(\lambda+\frac{1}{2})\right]}.$$

Since the point $r=R_2$ is the common domain of $\phi_2(\lambda,k_2,r)$ and $f(\lambda,k,r)$, we evaluate the Jost function at $r=R_2$ and thus obtain

$$\begin{aligned}
f(\lambda,k) &= W[f(\lambda,k,r),\phi_2(\lambda,k_2,r)]_{r=R_2}\\
&= f(\lambda,k,r)\phi_2'(\lambda,k_2,r)-f'(\lambda,k,r)\phi_2(\lambda,k_2,r)\\
&= \frac{2^{\lambda+\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\lambda+1)k_1^{-\lambda+\frac{1}{2}}ke^{-i\frac{\pi}{2}(\lambda+\frac{1}{2})}R_2^2}{j'_{\lambda-\frac{1}{2}}(k_2R_1)y_{\lambda-\frac{1}{2}}(k_2R_1)-j_{\lambda-\frac{1}{2}}(k_2R_1)y'_{\lambda-\frac{1}{2}}(k_2R_1)}\\
&\quad\times\left\{h^{(2)}_{\lambda-\frac{1}{2}}(kR_2)k_2\left[a_1j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_3j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]\right.\\
&\quad\left.-h^{(2)^{'}}_{\lambda-\frac{1}{2}}(kR_2)k\left[a_2j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_4j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]\right\}.
\end{aligned}\tag{2}$$

We also have that

$$\begin{aligned}
f(\lambda,-k) &= \frac{2^{\lambda+\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\lambda+1)k_1^{-\lambda+\frac{1}{2}}ke^{-i\frac{\pi}{2}(\lambda+\frac{1}{2})}R_2^2e^{i\pi(\lambda-\frac{1}{2})}}{j'_{\lambda-\frac{1}{2}}(k_2R_1)y_{\lambda-\frac{1}{2}}(k_2R_1)-j_{\lambda-\frac{1}{2}}(k_2R_1)y'_{\lambda-\frac{1}{2}}(k_2R_1)}\\
&\quad\times\left\{-h^{(1)}_{\lambda-\frac{1}{2}}(kR_2)k_2\left[a_1j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_3j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]\right.\\
&\quad\left.+h^{(1)^{'}}_{\lambda-\frac{1}{2}}(kR_2)k\left[a_2j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_4j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]\right\},
\end{aligned}$$

where we have used the following identities:

$$h^{(2)}_{\lambda-\frac{1}{2}}(kre^{i\pi})=h^{(2)}_{\lambda-\frac{1}{2}}(-kr)=(-1)^{\lambda-\frac{1}{2}}h^{(1)}_{\lambda-\frac{1}{2}}(kr)=e^{i\pi(\lambda-\frac{1}{2})}h^{(1)}_{\lambda-\frac{1}{2}}(kr)$$

and

$$h^{(2)^{'}}_{\lambda-\frac{1}{2}}(-kr)=(-1)^{\lambda+\frac{1}{2}}h^{(1)^{'}}_{\lambda-\frac{1}{2}}(kr)=e^{i\pi(\lambda+\frac{1}{2})}h^{(1)^{'}}_{\lambda-\frac{1}{2}}(kr)=-e^{i\pi(\lambda-\frac{1}{2})}h^{(1)^{'}}_{\lambda-\frac{1}{2}}(kr),$$

where $\lambda-\frac{1}{2}=0,1,2,....$ The S-matrix is then given by

$$S(\lambda,k)=-\left\{kh^{(2)^{'}}_{\lambda-\frac{1}{2}}(kR_2)\left[a_2j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_4j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]\right.$$

$$-k_2 h^{(2)}_{\lambda-\frac{1}{2}}(kR_2)\left[a_1 j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_3 j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]\Bigg\}$$
$$\Bigg/\Bigg\{kh^{(1)'}_{\lambda-\frac{1}{2}}(kR_2)\left[a_2 j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_4 j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]$$
$$-k_2 h^{(1)}_{\lambda-\frac{1}{2}}(kR_2)\left[a_1 j_{\lambda-\frac{1}{2}}(k_1R_1)+\frac{k_1}{k_2}a_3 j'_{\lambda-\frac{1}{2}}(k_1R_1)\right]\Bigg\},$$

where

$$\begin{aligned}
a_1 &= j'_{\lambda-\frac{1}{2}}(k_2R_1)y'_{\lambda-\frac{1}{2}}(k_2R_2)-y'_{\lambda-\frac{1}{2}}(k_2R_1)j'_{\lambda-\frac{1}{2}}(k_2R_2),\\
a_2 &= j'_{\lambda-\frac{1}{2}}(k_2R_1)y_{\lambda-\frac{1}{2}}(k_2R_2)-j_{\lambda-\frac{1}{2}}(k_2R_2)y'_{\lambda-\frac{1}{2}}(k_2R_1),\\
a_3 &= y_{\lambda-\frac{1}{2}}(k_2R_1)j'_{\lambda-\frac{1}{2}}(k_2R_2)-j_{\lambda-\frac{1}{2}}(k_2R_1)y'_{\lambda-\frac{1}{2}}(k_2R_2),\\
a_4 &= j_{\lambda-\frac{1}{2}}(k_2R_1)y_{\lambda-\frac{1}{2}}(k_2R_2)-y_{\lambda-\frac{1}{2}}(k_2R_1)j_{\lambda-\frac{1}{2}}(k_2R_2).
\end{aligned}$$

We can calculate the Jost function for $\lambda=\frac{1}{2}$ from (2):

$$\begin{aligned}
f\left(\frac{1}{2},k\right) = \frac{1}{4}e^{-ikR_2}\Bigg\{&\Bigg[\left(\frac{k-ik_2}{k_1}+\frac{ik+k_2}{k_2}\right)e^{k_2(R_2-R_1)}\\
&+\left(\frac{k+ik_2}{k_1}+\frac{k_2-ik}{k_2}\right)e^{-k_2(R_2-R_1)}\Bigg]e^{ik_1R_1}\\
&+\Bigg[\left(\frac{ik+k_2}{k_2}-\frac{k-ik_2}{k_1}\right)e^{k_2(R_2-R_1)}\\
&+\left(\frac{k_2-ik}{k_2}-\frac{k+ik_2}{k_1}\right)e^{-k_2(R_2-R_1)}\Bigg]e^{-ik_1R_1}\Bigg\},
\end{aligned}\tag{3}$$

where we have used the following relations:

$$j_0(kR)=\frac{\sin kR}{kR},\ j_0'(kR)=\frac{\cos kR}{kR}-\frac{\sin kR}{(kR)^2},$$

$$h_0^{(2)}(kR)=\frac{-e^{-ikR}}{ikR},\ \text{and}\ h_0^{(2)'}(kR)=\frac{e^{-ikR}\left(1+\frac{1}{ikR}\right)}{kR}.$$

3 The Jost Integral Equation for $\lambda=\frac{1}{2}$ and Some Approximate Solutions for The Three-Layer Model

We now apply the method in section I [1] to the case of scattering from a piecewise constant by multi-layer spherical inhomogeneity. We have already calculated $f(\frac{1}{2},k)$ exactly in equation (3). Elsewhere we use the exact solution of the Jost function to check for the accuracy of the iteration procedure. If we assume there is an R such that $V(r)=0$ for $r>R$ (certainly true in optics!) and let

$$g\left(\frac{1}{2},k,r\right)=e^{ikr}f\left(\frac{1}{2},k,r\right),\tag{4}$$

then (24) in [1] becomes the Jost integral equation for $\lambda = \frac{1}{2}$:

$$g\left(\frac{1}{2}, k, r\right) = 1 + (2ik)^{-1} \int_r^R [1 - e^{2ik(r-r')}] V(r') g\left(\frac{1}{2}, k, r'\right) dr'$$
$$= 1 - V_1 (2ik)^{-1} \int_r^{R_1} [1 - e^{2ik(r-r')}] g\left(\frac{1}{2}, k, r'\right) dr', \tag{5}$$

for the potential defined by region 1 in equation (1). Next we write the solution of (5) as a perturbation expansion

$$g\left(\frac{1}{2}, k, r\right) = \sum_{n=0}^{\infty} g_n\left(\frac{1}{2}, k, r\right),$$

where

$$g_0\left(\frac{1}{2}, k, r\right) = 1$$

and

$$g_n\left(\frac{1}{2}, k, r\right) = 1 + (2ik)^{-1} \int_r^R [1 - e^{2ik(r-r')}] V(r') g_{n-1}\left(\frac{1}{2}, k, r'\right) dr'.$$

From (13) in [1], we have

$$\lim_{r \to 0} \phi\left(\frac{1}{2}, k, r\right) = 0 \text{ and } \lim_{r \to 0} \frac{\phi'\left(\frac{1}{2}, k, r\right)}{dr} = 1. \tag{6}$$

$f(\frac{1}{2}, k, r)$ and $f'(\frac{1}{2}, k, r)$ are finite and we can evaluate $f(\frac{1}{2}, k)$ at $r = 0$ using (6), thus obtaining the useful relation

$$f\left(\frac{1}{2}, k\right) = f\left(\frac{1}{2}, k, 0\right) = g\left(\frac{1}{2}, k, 0\right).$$

The first iteration $g_I(\frac{1}{2}, k, 0)$ of (5) is

$$\begin{aligned}
&g_I\left(\frac{1}{2}, k, 0\right) \\
&= g_0\left(\frac{1}{2}, k, 0\right) + g_1\left(\frac{1}{2}, k, 0\right) \\
&= 1 - \frac{1}{4}\left\{\left[\left(\frac{k_1}{k}\right)^2 - 1\right](1 - \cos 2kR_1) + \left[1 - \left(\frac{k_2}{k}\right)^2\right](\cos 2kR_2 - \cos 2kR_1)\right\} \\
&+ \frac{i}{2}\Bigg\{\left[\left(\frac{k_1}{k}\right)^2 - 1\right]\left(kR_1 - \frac{1}{2}\sin 2kR_1\right) + \left[1 - \left(\frac{k_2}{k}\right)^2\right]\Bigg[k(R_2 - R_1) \\
&- \frac{1}{2}(\sin 2kR_2 - \sin 2kR_1)\Bigg]\Bigg\}.
\end{aligned}$$

The second iteration $g_{II}(\frac{1}{2}, k, 0)$ is

$$
\begin{aligned}
& g_{II}\left(\frac{1}{2}, k, 0\right) \\
& = g_0\left(\frac{1}{2}, k, 0\right) + g_1\left(\frac{1}{2}, k, 0\right) + g_2\left(\frac{1}{2}, k, 0\right) \\
& = 1 + \frac{1}{4}\left\{\left[\left(\frac{k_1}{k}\right)^2 - 1\right](\cos 2kR_1 - 1) + \left[1 - \left(\frac{k_2}{k}\right)^2\right](\cos 2kR_2 - \cos 2kR_1)\right\} \\
& - \frac{1}{8}\left\{\left[\left(\frac{k_1}{k}\right)^2 - 1\right]^2\left[kR_1\left(kR_1 + \sin 2kR_1\right) + \frac{3}{2}\left(\cos 2kR_1 - 1\right)\right]\right. \\
& + \left[\left(\frac{k_1}{k}\right)^2 - 1\right]\left[1 - \left(\frac{k_2}{k}\right)^2\right]\Big[k(R_2 - R_1)\left(2kR_1 - k(R_2 - R_1) - \sin 2kR_1\right) \\
& - \left(\cos 2k(R_2 - R_1) - 1\right) + \frac{3}{2}\left(\cos 2kR_2 - \cos 2kR_1\right) \\
& + k(R_2 - R_1)\left(\sin 2kR_2 + \sin 2kR_1\right) + kR_1\left(\sin 2kR_2 - \sin 2kR_1\right)\Big] \\
& + \left[1 - \left(\frac{k_2}{k}\right)^2\right]^2\Big[k(R_2 - R_1)[2k(R_2 - R_1) - \sin 2k(R_2 - R_1) \\
& \left.+ \sin 2kR_2 - \sin 2kR_1] + \left(\cos 2k(R_2 - R_1) - 1\right)\Big]\right\} \\
& + i\left\{\frac{1}{2}\left\{\left[\left(\frac{k_1}{k}\right)^2 - 1\right]\left(kR_1 - \frac{1}{2}\sin 2kR_1\right)\right.\right. \\
& \left. + \left[1 - \left(\frac{k_2}{k}\right)^2\right]\left[k(R_2 - R_1) - \frac{1}{2}\left(\sin 2kR_2 - \sin 2kR_1\right)\right]\right\} \\
& - \frac{1}{8}\left\{\left[\left(\frac{k_1}{k}\right)^2 - 1\right]^2\left[kR_1\left(\cos 2kR_1 + 2\right) - \frac{3}{2}\sin 2kR_1\right]\right. \\
& + \left[\left(\frac{k_1}{k}\right)^2 - 1\right]\left[1 - \left(\frac{k_2}{k}\right)^2\right]\Big[kR_1\left(\cos 2kR_2 - \cos 2kR_1\right) \\
& + k(R_2 - R_1)\left(\cos 2kR_2 + \cos 2kR_1\right) \\
& - k(R_2 - R_1)\left(\cos 2kR_1 - 1\right) - \frac{3}{2}\left(\sin 2kR_2 - \sin 2kR_1\right)\Big] \\
& \left.\left. - \left[1 - \left(\frac{k_2}{k}\right)^2\right]^2\left[k(R_2 - R_1)\left(\cos 2k(R_2 - R_1) - \cos 2kR_2 + \cos 2kR_1\right)\right]\right\}\right\}.
\end{aligned}
$$

For real λ and k, we have

$$f(\lambda, -k) = f^*(\lambda, k),$$

and therefore

$$\frac{\sigma_0}{\pi R_1^2} = \frac{\left|1 - e^{2i\delta\left(\frac{1}{2},k\right)}\right|^2}{(kR_1)^2} = \frac{\left|1 - \frac{f\left(\frac{1}{2},k\right)}{f^*\left(\frac{1}{2},k\right)}\right|^2}{(kR_1)^2},$$

where σ_0 is the $l = 0$ total cross section. The accuracy of these approximations as functions of k, k_1 and k_2 will be reported elsewhere.

4 Jost Integral Equation Formulation for Arbitrary λ: 3-Layer Model

In case of scattering from a piecewise constant spherical inhomogeneity, the two integral equations (60) and (61) in [1] become:

$$\begin{aligned}
\phi(\lambda, k, r) = r^{\lambda+\frac{1}{2}} &+ \frac{1}{2}\lambda^{-1} \int_0^{R_1} [(\xi/r)^\lambda - (r/\xi)^\lambda] \times (r\xi)^{\frac{1}{2}} [k^2 + V_1]\phi(\lambda, k, \xi) d\xi \\
&+ \frac{1}{2}\lambda^{-1} \int_{R_1}^{R_2} [(\xi/r)^\lambda - (r/\xi)^\lambda] \times (r\xi)^{\frac{1}{2}} [k^2 + V_2]\phi(\lambda, k, \xi) d\xi \\
&+ \frac{1}{2}\lambda^{-1} \int_{R_2}^{r} [(\xi/r)^\lambda - (r/\xi)^\lambda](r\xi)^{\frac{1}{2}} k^2 \phi(\lambda, k, \xi) d\xi
\end{aligned}$$

and

$$\begin{aligned}
f(\lambda, k, r) = e^{-ikr} &+ k^{-1} \int_r^{R_1} [\sin k(r' - r)] \left[-V_1 + \frac{(\lambda^2 - \frac{1}{4})}{r'^2}\right] f(\lambda, k, r') dr' \\
&+ k^{-1} \int_{R_1}^{R_2} [\sin k(r' - r)] \left[-V_2 + \frac{(\lambda^2 - \frac{1}{4})}{r'^2}\right] f(\lambda, k, r') dr' \\
&+ k^{-1} \int_{R_2}^{\infty} \sin k(r' - r) \left(\frac{\lambda^2 - \frac{1}{4}}{r'^2}\right) f(\lambda, k, r') dr'.
\end{aligned}$$

5 Summary

This iterative technique may be most useful when the scattering system is more complicated than those discussed here. By comparing the present formulation with the numerical results obtained for a constant spherical inhomogeneity [1], it appears that the iteration technique is good for problems with long wavelengths ($kR_1 << 1$) for any k_1/k. For shorter wavelengths, small k_1/k (e.g., $k_1/k = 1.1$) gives a good approximation to σ_0 for the entire range of kR_1 considered ($0 \le R_1 \le 2\pi$); however, large k_1/k (e.g., $k_1/k = 1.5, 2.0$) gives a good approximation to σ_0 in the range of $0 < kR_1 < 3\pi/4$. In case of a piecewise constant spherical inhomogeneity, the iteration procedure gives a better approximation for the problem with long wavelengths ($kR_1 << 1$) only for small ratios of k_1/k and k_2/k (e.g., $k_1/k = 0.7$, $k_2/k = 0.9$; $k_1/k = 1.1$, $k_2/k = 1.3$). For a larger k_1/k and k_2/k (e.g., $k_1/k = 1.5$, $k_2/k = 1.2$), it gives a good approximation when $kR_1 < 2\pi/3$. The approximation for the Jost function becomes less accurate for larger ratios of wavenumber k_1/k and k_2/k (e.g., $k_1/k = 2.0$, $k_2/k = 1.5$). When the ratios of wavenumbers k_1/k is greater than k_2/k, we have a better approximation. However, the approximation for the Jost function is still better than the total cross section for the large wavelengths. For shorter wavelengths, all ratios of the wavenumbers give a better approximation to σ_0 for approximately $kR_1 > 2\pi/3$ [4]. These results will be reported in more detail elsewhere.

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