Applied Mathematics E-Notes, 14(2014), 135-143 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On The Solution Sets Of Semicontinuous Quantum Stochastic Differential Inclusions*

Michael Oluniyi Ogundiran†, Victor Folarin Payne‡

Received 1 June 2014

Abstract

The aim of this paper is to provide a unified treatment of the existence of solution of both upper and lower semicontinuous quantum stochastic differential inclusions. The quantum stochastic differential inclusion is driven by operator-valued stochastic processes lying in certain metrizable locally convex space. The unification of solution sets to these two discontinuous non-commutative stochastic differential inclusions is established via the existence of directionally continuous selections.

1 Introduction

Existence results for the solutions of quantum stochastic differential inclusions of Hudson and Parthasarathy quantum stochastic calculus was established in [9]. The Topological properties of solution sets for this Lipschitzian quantum stochastic differential inclusions were established in [2]. The cases of coefficients that are discontinuous multivalued stochastic processes were established in [14, 15, 16]. In [15] the existence of solutions for upper semicontinuous was established via Fixed point theorem while in [14] the multivlaued stochastic processes possess minimal selections. The existence of solution for the case of Lower semicontinuous multivlaued stochastic processes were established in [16] via continuous selection of some predefined integral operators. The extension of quantum stochastic differential inclusions to discontinuous cases was essentially to enhance further applications of the rich quantum stochastic calculus to quantum stochastic control theory and evolutions. The quantum stochastic differential inclusions considered in [9] have Lipschitzian coefficients defined on certain locally convex space and in [10] more locally convex spaces were considered. By employing one of the locally convex spaces defined in [10], a unified treatment of upper and lower semicontinuous cases in this work was established.

For classical differential inclusions, the solution sets of upper and lower semicontinuous differential inclusions were considered via a directionally continuous selection in [4]. This directionally continuous selection which is a non-convex analogue of Michael selection was first considered in [3] for finite dimensional case and for an arbitrary Banach space in [6]. A more general case was established in [5], which shall be employed

*Mathematics Subject Classifications: 81S25, 34A37.

†Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
‡Department of Mathematics, University of Ibadan, Ibadan, Nigeria

in our work. The paracompact property of the locally convex space which is the domain of our multivalued stochastic processes in this work guaranteed the existence of directional continuous selection which provides a link between upper and lower semicontinuous multifunctions. In the sequel, the work shall be as follows; in section 2, preliminaries on quantum stochastic differential inclusions shall be given. In section 3, our main result shall be proved.

2 Preliminaries

In this section, we shall adopt the notations in [10]. Let $\mathbb{D}$ be some pre-Hilbert space whose completion is $\mathcal{R}$; γ is a fixed Hilbert space and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable γ-valued maps on $\mathbb{R}_+$. The inner product of the Hilbert space $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,. \rangle$ and $\| \; . \; \|$ the norm induced by $\langle .,. \rangle$. Let $\mathbb{E}$ be linear space generated by the exponential vectors in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$ and $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ be the set of all sequences $\eta = \{\eta_n\}_{n=1}^\infty$ and $\xi = \{\xi_n\}_{n=1}^\infty$ of members of $\mathbb{D}\underline{\otimes}\mathbb{E}$, such that $\Sigma_{n=1}^\infty \mid \langle \eta_n, x\xi_n \rangle \mid < \infty$, $\forall x \in \mathcal{A}$, where $\mathcal{A} \equiv L^+_w((\mathbb{D}\underline{\otimes}\mathbb{E})_\infty, \mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+)))$. Then the family of seminorms $\{\| \; . \; \|_{\eta\xi}, \eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty\}$, where

$$\|x\|_{\eta\xi} = \sum_{n=1}^{\infty} |\langle \eta_n, x\xi_n \rangle| \text{ for } x \in \mathcal{A},$$

generates a σ-weak topology, denoted by $\tau_{\sigma w}$ [10]. The completion of $(\mathcal{A}, \tau_{\sigma w})$ is denoted by $\widetilde{\mathcal{A}}$. The underlying elements of $\widetilde{\mathcal{A}}$ consist of linear maps from $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ into $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$.

REMARK 1. By Theorem V.5 [18], we remark that the σ-weak topology $\tau_{\sigma w}$ is metrizable since $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ has a countable base, hence $\widetilde{\mathcal{A}}$ is a paracompact space [13].

For a fixed Hilbert space γ, the spaces $L^p_{loc}(\widetilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $L^p_{loc}(I \times \widetilde{\mathcal{A}})$ are adopted as in [10]. For a topological space $\mathcal{N}$, let $clos(\mathcal{N})$ be the collection of all nonempty closed subsets of $\mathcal{N}$; we shall employ the Hausdorff topology on $clos(\widetilde{\mathcal{A}})$ as defined in [9]. Moreover, for $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, we define the Hausdorff distance, $\rho(A,B)$ as

$$\mathbf{d}(x,B) \equiv \inf_{y\in B} |x-y|\,, \quad \delta(A,B) \equiv \sup_{x\in A} \mathbf{d}(x,B), \;\text{ and }\; \rho(A,B) \equiv \max(\delta(A,B), \delta(B,A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.

DEFINITION 1.

(a) By a multivalued stochastic process indexed by $I = [0,T] \subseteq \mathbb{R}_+$, we mean a multifunction on I with values in $clos(\widetilde{\mathcal{A}})$.

(b) If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of Φ is a stochastic process $X : I \to \widetilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

(c) A multivalued stochastic process Φ will be called

(i) adapted if $\Phi(t) \subseteq \widetilde{\mathcal{A}}_t$ for each $t \in \mathbb{R}_+$;

(ii) measurable if $t \mapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$;

(iii) locally absolutely p-integrable if $t \mapsto \| \Phi(t) \|_{\eta\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(\widetilde{\mathcal{A}})$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

(d) The set of all absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\widetilde{\mathcal{A}})_{mvs}$ and for $p \in (0, \infty)$, $L^p_{loc}(I \times \widetilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \widetilde{\mathcal{A}} \to clos(\widetilde{\mathcal{A}})$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$ lies in $L^p_{loc}(\widetilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\widetilde{\mathcal{A}})$.

Consider stochastic processes $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{A}})$ and $(0, x_0)$ be a fixed point in $[0, T] \times \widetilde{\mathcal{A}}$. Then, a relation of the form

$$\begin{aligned} X(t) \in x_0 + \int_0^t &(E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s) \\ &+ G(s, X(s))dA^+_g(s) + H(s, X(s))ds \text{ for } t \in [0, T] \end{aligned}$$

will be called a stochastic integral inclusion with coefficients E, F, and G and H.

The stochastic differential inclusion corresponding to the integral inclusion above is

$$\begin{aligned} dX(t) &\in E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) \\ &\quad + G(t, X(t))dA^+_g(t) + H(t, X(t))dt, \\ X(0) &= x_0 \text{ for almost all } t \in [0, T]. \end{aligned} \tag{1}$$

Let $\mathbb{P} : [0, T] \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ be sesquilinear form valued stochastic process defined in [9] in terms of E, F, G, H by using the matrix elements in Hudson and Parthasarathy quantum stochastic calculus [12], it was established that problem (1) is equivalent to

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in \mathbb{P}(t, X(t))(\eta, \xi), \\ &\qquad X(0) = x_0 \text{ for almost all } t \in [0, T]. \end{aligned} \tag{2}$$

As explained in [9], the map $\mathbb{P}$ is such that:

$$\mathbb{P}(t, x)(\eta, \xi) \neq \widetilde{\mathbb{P}}(t, \langle \eta, x\xi \rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I \times \mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

The notion of solution of (1) or equivalently (2) is defined as follows:

DEFINITION 2. By a solution of (1) or equivalently (2), we mean a stochastic process $\varphi \in \mathrm{Ad}(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}})$ such that

$$\begin{aligned} d\varphi(t) &\in E(t, \varphi(t))d\Lambda_\pi(t) + F(t, \varphi(t))dA_f(t) \\ &\quad + G(t, \varphi(t))dA^+_g(t) + H(t, \varphi(t))dt \text{ for almost all } t \in I, \\ \varphi(t_0) &= \varphi_0, \end{aligned}$$

or equivalently

$$\frac{d}{dt}\langle \eta, \varphi(t)\xi\rangle \in \mathbb{P}(t, \varphi(t))(\eta, \xi).$$
$$\varphi(t_0) = \varphi_0,$$

for arbitrary $\eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$, almost all $t \in I$. A multivalued stochastic process $\Phi : I \times \widetilde{\mathcal{A}} \to 2^{\widetilde{\mathcal{A}}}$ is said to be lower semicontinuous if for every open set $V \subset \widetilde{\mathcal{A}}$, $\Phi^{-1}(V)$ is open. Also, $\Psi : I \times \widetilde{\mathcal{A}} \to 2^{\widetilde{\mathcal{A}}}$ is said to be upper semicontinuous if, for every $x \in \widetilde{\mathcal{A}}$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_{\eta\xi}((t_1, x), (t_2, y)) < \delta \Longrightarrow \Psi(t_2, y) \subseteq B(\Psi(t_1, x), \epsilon),$$

where

$$d_{\eta\xi}((t_1, x), (t_2, y)) = \max\left\{|t_1 - t_2|\,, \|x - y\|_{\eta\xi}\right\}$$

and

$$B(\Psi(t_1, x), \epsilon) = \left\{(t, z) \in I \times \widetilde{\mathcal{A}} : |t - t_1| < \epsilon \text{ and } \|z - x\|_{\eta\xi} < \epsilon\right\}.$$

Let $meas(J)$ be the Lebesgue measure of a set $J \subset \mathbb{R}$, t is a point of density for J if

$$\lim_{\epsilon \to 0} \frac{meas(J \cap [t - \epsilon, t + \epsilon])}{2\epsilon} = 1.$$

It follows from the previous works; [15], [16] and [14], that if $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{A}})$ are upper semicontinuous (resp. lower semicontinuous) then the equivalent sesquilinear form valued stochastic process $\mathbb{P}$ is upper semicontinuous (resp. lower semicontinuous). We consider a topology τ^+ on $I \times \widetilde{\mathcal{A}}$ stronger than the usual metric topology of $I \times \widetilde{\mathcal{A}}$. A topology τ^+ is said to satisfy a property (P):

(P) For every pair of sets $A \subset B$ with A closed and B open (in the original topology) there exists a set C closed-open with respect to the topology τ^+ such that $A \subset C \subset B$

Let $I = [a, b]$ and $\Omega \subset I \times \widetilde{\mathcal{A}}$, the following set defined in [7] is a basis of open neighbourhoods for a topology τ^+ on Ω stronger than the metric one, and satisfies property P. For every $(t, x) \in \Omega$ and $\epsilon > 0$,

$$V(t, x, \epsilon) = \left\{(s, y) \in \Omega : t \leq s < t + \epsilon \text{ and } \|y - x\|_{\eta\xi} \leq M(s - t)\right\}.$$

Moreover, each set $V(t, x, \epsilon)$ is closed-open in the topology τ^+.

3 Main Results

The following Lemma shall be employed in the proof of the main result.

LEMMA 1. Let $X(.)$ be a Caratheodory solution of upper (lower) semicontinuous quantum stochastic differential inclusion

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \Phi(t, X(t))(\eta, \xi) \text{ on } [a, b].$$

Assume that J is the set of times $t \in [a, b]$ such that

(i)
$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \Phi(t, X(t))(\eta, \xi).$$

(ii) If there exists a sequence t_k, strictly decreasing to t, with

$$\frac{d}{dt}\langle \eta, X(t_k)\xi\rangle \in \frac{d}{dt}\langle \eta, X(t)\xi\rangle \text{ and } \frac{d}{dt}\langle \eta, X(t_k)\xi\rangle \in \Phi(t_k, X(t_k))(\eta, \xi)$$

for any k and $\eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$.

Then $meas(J) = b - a$.

PROOF. Let J_1 be the set of times where (i) holds. Since X is a caratheodory solution, then $meas(J_1) = b - a$. Fix any $\epsilon > 0$, since $\frac{d}{dt}\langle \eta, X(t)\xi\rangle$ is measurable, by Lusin's theorem there exists a weakly continuous stochastic process u such that $\langle \eta, u(t)\xi\rangle = \frac{d}{dt}\langle \eta, X(t)\xi\rangle$ for every t in a set $J_2 \subset J_1$ with $meas(J_2) > b - a - \epsilon$. Clearly (ii) holds at every $t \in J_2$ which is a point of density for J_2. Hence $meas(J) \geq meas(J_2) > b - a - \epsilon$, since ϵ was arbitrary, the lemma is proved.

The following is an adaptation of Theorem 1 in [5] to our non commutative setting.

THEOREM 1. Suppose the following hold:

(i) For almost all $t \in I$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the maps $X \to \Psi(t, X)(\eta, \xi)$, $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ are non-empty lower semicontinuous multivalued stochastic processes.

(ii) For almost all $t \in I$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the maps $t \to \Psi(t, X)(\eta, \xi)$ are closed.

(iii) τ^+ is a topology on $I \times \widetilde{\mathcal{A}}$ with property (P).

Then the sesquilinear form valued multifunction, $(t, X(t)) \to \mathbb{P}(t, X(t))(\eta, \xi)$

$$\begin{aligned}\mathbb{P}(t, X(t))(\eta, \xi) = (\mu E)(t, X(t))(\eta, \xi) + (\nu F)(t, X(t))(\eta, \xi) \\ + (\sigma G)(t, X(t))(\eta, \xi) + H(t, X(t))(\eta, \xi)\end{aligned}$$

admits a τ^+-continuous selection.

PROOF. $\mathbb{P}$ is non-empty since each of $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ is non-empty then $\mathbb{P}$ is a non-empty lower semicontinuous sesquilinear form-valued multifunction. We shall employ a similar procedure as in the proof of Theorem 3.2 in [5] to construct a τ^+-continuous ϵ-approximate selections P_ϵ of $\mathbb{P}$, hence by inductive hypothesis we obtain a τ^+-continuous selection P of $\mathbb{P}$. Let $\epsilon > 0$ be fixed, since $X \to \mathbb{P}(t, X)(\eta, \xi)$ is lower semicontinuous, for every $X(t) \in \widetilde{\mathcal{A}}$, we choose point $y_{\eta\xi,X}(t) \in \mathbb{P}(t, X(t))(\eta, \xi)$ and neighbourhood U_X of $X(t)$ such that

$$\inf_{y_{\eta\xi,\mathbb{P}}(t) \in \mathbb{P}(t,X(t'))(\eta,\xi)} |y_{\eta\xi,X}(t) - y_{\eta\xi,\mathbb{P}}(t)| < \epsilon \text{ for } X(t') \in U_X. \tag{3}$$

Now, let $(V_\alpha)_{\alpha\in\beta^\epsilon}$ be a local finite open refinement of $(U_X)_{X(t)\in\widetilde{\mathcal{A}}}$, with $V_\alpha \subset U_{X_\alpha}$, and let $(W_\alpha)_{\alpha\in\beta^\epsilon}$ be another open refinement such that $cl(W_\alpha) \subset V_\alpha$ for all $\alpha \in \beta^\epsilon$. By property (P), for each α, we can choose a set Z_α, clopen w.r.t. τ^+, such that

$$cl(W_\alpha) \subset int(Z_\alpha) \subset cl(Z_\alpha) \subset V_\alpha. \tag{4}$$

Then $(Z_\alpha)_\alpha$ is a local finite τ^+ clopen covering of $\widetilde{\mathcal{A}}$. Let $\preceq$ be a well-ordering of the set β^ϵ, define for each $\alpha \in \beta^\epsilon$,

$$\Omega^\epsilon_\alpha = Z_\alpha \setminus \bigcup_{\lambda<\alpha} Z_\lambda.$$

Set $\mathcal{O}^\epsilon = (\Omega^\epsilon_\alpha)$, $\alpha \in \beta^\epsilon$. By well-ordering, every $x \in \widetilde{\mathcal{A}}$ belongs to exactly one set $\Omega^\epsilon_{\overline{\alpha}}$ where $\overline{\alpha} = \min\{\alpha \in \beta^\epsilon : x \in Z_\alpha\}$. Hence, $\mathcal{O}^\epsilon$ is a partition of $\widetilde{\mathcal{A}}$. Moreover, since Z_α is locally finite(wrt τ and therefore wrt τ^+), the sets $\bigcup_{\lambda<\alpha} Z_\lambda$ are τ^+ clopen. Hence $\mathcal{O}^\epsilon$ is a τ^+ clopen disjoint covering of $\widetilde{\mathcal{A}}$ such that, $\{cl(\Omega^\epsilon_\alpha)\}$ refines $(V_\alpha)_\alpha$. By setting $y^\epsilon_{\eta\xi,\alpha} = y_{\eta\xi,X_\alpha}$ and $P_\epsilon(t, X(t))(\eta, \xi) = y_{\eta\xi,X_\alpha}$, $\forall\alpha \in \beta^\epsilon$, we have τ^+ continuous function P_ϵ, which by (3), satisfies

$$\inf_{y_{\eta\xi,\mathbb{P}}(t) \in \mathbb{P}(t,X(t))(\eta,\xi)} |P_\epsilon(t, X(t))(\eta, \xi) - y_{\eta\xi,\mathbb{P}}(t)| < \epsilon.$$

Therefore, there exists an ϵ-approximate selection P_ϵ of $\mathbb{P}$. Since ϵ was arbitrarily chosen,thus we have a τ^+-continuous selection P of $\mathbb{P}$.

Let $P : I \times \widetilde{\mathcal{A}} \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2_\infty$ be sesquilinear form -valued directionally continuous map as defined above. The upper semicontinuous, convex valued **regularization** of P, corresponding to a given $\eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ is defined as

$$R(t, x)(\eta, \xi) = \bigcap_{\epsilon>0} \overline{co}\left\{P(s, y)(\eta, \xi) : |t - s| < \epsilon \text{ and } \|x - y\|_{\eta\xi} < \epsilon\right\}. \tag{5}$$

THEOREM 2. Let Ω be a closed subset of $I \times \widetilde{\mathcal{A}}$, and let $\mathbb{P} : I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty)}$ be a bounded, lower semicontinuous multifunction. Then there exists an upper semicontinuous map $R : \Omega \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty)}$ with compact convex values such that every Caratheodory solution of

$$\frac{d}{dt}\langle\eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi) \tag{6}$$

is also a solution of

$$\frac{d}{dt}\langle\eta, X(t)\xi\rangle \in R(t, X(t))(\eta, \xi). \tag{7}$$

PROOF. Let $X(t)$ be a Caratheodory solution of $\frac{d}{dt}\langle\eta, X(t)\xi\rangle \in R(t, X(t))(\eta, \xi)$ on $[a, b]$. Define $J \subset [a, b]$ to be the set of times t such that

(i) $\frac{d}{dt}\langle\eta, X(t)\xi\rangle \in R(t, X(t))(\eta, \xi)$.

(ii) There exists a sequence of times t_k strictly decreasing to t such that $\frac{d}{dt}\langle\eta, X(t_k)\xi\rangle \in R(t_k, X(t_k))(\eta, \xi)$ and $\frac{d}{dt}\langle\eta, X(t_k)\xi\rangle \to \frac{d}{dt}\langle\eta, X(t)\xi\rangle$.

By Lemma 1 above, J has a full measure in $[a, b]$. We claim that $\frac{d}{dt}\langle\eta, X(t)\xi\rangle = P(t, X(t))(\eta, \xi)$ for every $t \in J$. Assume on the contrary that $t \in J$ but

$$\left|\frac{d}{dt}\langle\eta, X(t)\xi\rangle - P(t, X(t))(\eta, \xi)\right| = \epsilon > 0. \tag{8}$$

Using the directional continuity of P at the point (t, X), choose $\delta > 0$ such that

$$|P(s, y)(\eta, \xi) - P(t, X(t))(\eta, \xi)| < \frac{\epsilon}{2} \tag{9}$$

whenever $t \leq s < t + \delta$, $\| y - X(t) \|_{\eta\xi} \leq M(s - t)$. Let $t_k \to t$ be a sequence with properties stated in (ii), then there exists k large enough so that $0 < t_k - t < \delta$ and

$$\left|\frac{d}{dt}\langle\eta, X(t_k)\xi\rangle - \frac{d}{dt}\langle\eta, X(t)\xi\rangle\right| < \frac{\epsilon}{2}. \tag{10}$$

The boundedness assumption $| P(t, X)(\eta, \xi) |< L$ implies that $R(t, X)(\eta, \xi) \subseteq \overline{B}(0, L)$ for all (t, X). Our solution $X(t)$ is therefore Lipschitz continuous with constant L. In particular,

$$\|X(t_k) - X(t)\|_{\eta\xi} \leq L(t_k - t) < M(t_k - t).$$

Then we conclude that

$$R(t_k, X(t_k)) \subseteq \overline{B}\left(P(t, X(t))(\eta, \xi), \frac{\epsilon}{2}\right). \tag{11}$$

Hence

$$\left|\frac{d}{dt}\langle\eta, X(t_k)\xi\rangle - P(t, X(t))(\eta, \xi)\right| \leq \frac{\epsilon}{2}. \tag{12}$$

Comparing we obtain a contradiction, which proves that the caratheodory solutions of

$$\frac{d}{dt}\langle\eta, X(t)\xi\rangle = P(t, X(t))(\eta, \xi)$$

and (7) coincide. Now since $\mathbb{P}$ is bounded we can assume $P(t, X)(\eta, \xi) \subset B(0, L)$ for some constant L and all $(t, X) \in \Omega$. Choose $M > L$ and let P be τ^+-continuous selection of $\mathbb{P}$, by Theorem 1 above, such a P exists. Then if R is the regularization multivalued stochastic process as defined above, R is upper semicontinuous compact convex-valued [1]. Let now $X(.)$ be a Caratheodory solution of (7) on $[a, b]$ since P is a selection of $\mathbb{P}$, then $X(.)$ is also a solution of (6).

References

[1] J. P. Aubin and A. Cellina, Differential Inclusions. Set-Valued Maps and Viability Theory. Berlin-Heidelberg-New York-Tokyo, Springer-Verlag 1984.

[2] E. O. Ayoola, Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions, Acta Appl. Math., 100(2008), 15–37.

[3] A. Bressan, Directionally continuous selections and differential inclusions, Funkcial. Ekvac., 31(1988), 459–470.

[4] A. Bressan, Upper and Lower Semicontinuous Differential Inclusions: A Unified Approach. Nonlinear Controllability and Optimal Control, 21–31, Monogr. Textbooks Pure Appl. Math., 133, Dekker, New York, 1990

[5] A. Bressan and G. Colombo, Selections and representations of multifunctions in paracompact spaces, Studia Math., 102(1992), 209–216.

[6] A. Bressan and A. Cortesi, Directionally continuous selections in Banach spaces and differential inclusions, Nonlinear Anal., 13(1989), 987–992.

[7] A. Cernea and V. Staicu, Directionally continuous selections and Nonconvex evolution inclusions, Set-Valued Anal., 11(2003), 9–20.

[8] K. Deimling, Multivalued Differential Equations. de Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter & Co., Berlin, 1992.

[9] G. O. S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions, Internat. J. Theoret. Phys., 31(1992), 2003–2027.

[10] G. O. S. Ekhaguere, Topological solutions of noncommutative stochastic differential equations, Stoch. Anal. Appl., 25(2007), 961–993.

[11] A. Guichardet, Symmetric Hilbert Spaces and Related Topics. Infinitely Divisible Positive Definite Functions. Continuous Products and Tensor Products. Gaussian and Poissonian stochastic processes. Lecture Notes in Mathematics, Vol. 261. Springer-Verlag, Berlin-New York, 1972.

[12] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formula and stochastic evolutions, Comm. Math. Phys., 93(1984), 301–323.

[13] J. L. Kelley, General Topology. D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[14] M. O. Ogundiran, Upper semicontinuous Quantum Stochastic Differential Inclusions, Int. J. Pure and Appl. Math., 85(2013), 609–627.

[15] M. O. Ogundiran and E. O. Ayoola, Upper semicontinuous quantum stochastic differential inclusions via Kakutani-Fan fixed point theorem, Dynam. Systems Appl., 21(2012), 121–132.

[16] M. O. Ogundiran and E. O. Ayoola, Lower semicontinuous Quantum stochastic Differential Inclusions, Eur. J. Math. Sci., 2(2013), 1–16.

[17] K. R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Monographs in Mathematics, 85. Birkhäuser Verlag, Basel, 1992.

[18] M. Reed and B. Simon, Methods of Modern Mathematical Physics. I. Functional analysis. Second edition. Academic Press, Inc., New York, 1980.