Orthogonal Polynomials With Respect To A Nonlinear Form*

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Abstract
In this paper, we study properties of the form $u$ satisfying

$$u = -\lambda \left( x^2 - a^2 \right)^{-1} v + \delta_0,$$

where $v$ is a regular symmetric semi-classical form (linear functional). We give a necessary and sufficient condition for the regularity of the form $u$. The coefficients of the three-term recurrence relation, satisfied by the corresponding sequence of orthogonal polynomials, are given explicitly. A study of the semi-classical character of the founded families is done. An example related to the Generalized Gegenbauer form is worked out.

1 Introduction and Preliminaries

The semi-classical forms are a natural generalization of the classical forms (Hermite, Laguerre, Jacobi, and Bessel). Since the system corresponding to the problem of determining all the semi-classical forms of class $s \geq 1$ becomes non-linear, the problem was only solved when $s = 1$ and for some particular cases [2, 5, 16]. Thus, several authors use different processes in order to obtain semi-classical forms of class $s \geq 1$. For instance, let $v$ be a regular form and let us define a new form $u$ by the relation

$$A(x)u = B(x)v,$$

where $A(x)$ and $B(x)$ are non-zero polynomials. When $A(x) = 1$, $v$ is positive-definite and $B(x)$ is a positive polynomial, Christoffel [8] has proved that $u$ is still a positive-definite form. This result has been generalized in [9]. The cases $B(x) = \lambda \neq 0$ and $A(x) = x - c$, $x^2$, $x^3$, $x^4$ were treated in [15, 17, 18, 22], where it was shown that under certain regularity conditions the form $u$ is still regular. Moreover, if $v$ is semi-classical, then $u$ is also semi-classical; see also [1, 4, 6, 11, 23, 24, 25]. When $A(x) = B(x)$, $u$ is obtained from $v$ by adding finitely mass points and their derivatives [10, 12, 14] and when $A(x)$ and $B(x)$ have no non-trivial common factor, it was found a necessary and sufficient condition for $u$ to be regular in [13]. When $A(x)$ and $B(x)$ are of degree equal to one, an extensive study of the form $u$ has been carried in [27].

In this paper, we consider the situation when $A(x)$ and $B(x)$ are of degree equal to three and one respectively in a particular case. Indeed, we study the form $u$, fulfilling

$$x \left( x^2 - a^2 \right) u = -\lambda xv, \quad (u)_1 = 0, \quad (u)_2 = -\lambda \neq 0,$$

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Dv = 222 cf. [21]. Let us define the operator with respect to polynomials. We also analyze some linear relations linking the polynomials orthogonal with respect to u and v. In the third section, The stability of the semi-classical families is proved. Finally, we apply our result to Generalized Gegenbauer form.

Let $P$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $P'$ be its dual. We denote by $\langle v, f \rangle$ the action of $v \in P'$ on $f \in P$. In particular, we denote by $(v)_n := \langle v, x^n \rangle$, $n \geq 0$, the moments of $v$. For any form $v$ and any polynomial $h$ let $Dv = v'$, $hv$, $\delta_c$, and $(x - c)^{-1} v$ be the forms defined by:

$$\langle v', f \rangle := -\langle v, f' \rangle, \quad \langle hv, f \rangle := \langle v, hf \rangle, \quad \langle \delta_c, f \rangle := f(c), \quad \langle (x - c)^{-1} v, f \rangle := \langle v, \theta_c f \rangle$$

where $(\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}$, $c \in \mathbb{C}$, and $f \in P$.

Then, it is straightforward to prove that for $c, d \in \mathbb{C}$, $c \neq d$, $f, g \in P$ and $v \in P'$, we have

$$(x - c)^{-1} \left( (x - c) v \right) = v - (v)_0 \delta_c,$$

$$(x - c) \left( (x - c)^{-1} v \right) = v,$$

$$(x - d)^{-1} \delta_c = \frac{1}{c - d} (\delta_c - \delta_d).$$

cf. [21]. Let us define the operator $\sigma : P \to P$ by $(\sigma f)(x) = f(x^2)$. Then, we define the even part $\sigma v$ of $v$ by $\langle \sigma v, f \rangle := \langle v, \sigma f \rangle$. Therefore, we have [20]

$$f(x) (\sigma v) = \sigma (f(x^2) v),$$

$$\sigma (v') = 2 (\sigma (xv)).$$

A form $v$ is called regular if there exists a sequence of polynomials $\{S_n\}_{n \geq 0}$ (deg $S_n \leq n$) such that

$$\langle v, S_n S_m \rangle = r_n \delta_{n,m} \quad \text{for } r_n \neq 0 \text{ and } n \geq 0.$$

Then $\deg S_n = n$ for $n \geq 0$ and we can always suppose each $S_n$ is monic. In such a case, the sequence $\{S_n\}_{n \geq 0}$ is unique. It is said to be the sequence of monic orthogonal polynomials with respect to $v$.

It is a very well known fact that the sequence $\{S_n\}_{n \geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [7])

$$\begin{cases}
S_{n+2}(x) = (x - \xi_{n+1}) S_{n+1}(x) - \rho_{n+1} S_n(x) & \text{for } n \geq 0, \\
S_1(x) = x - \xi_0 \text{ and } S_0(x) = 1,
\end{cases}$$

with $(\xi_n, \rho_{n+1}) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $n \geq 0$. By convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n \geq 0}$ be the associated sequence of first order for the sequence $\{S_n\}_{n \geq 0}$ satisfying the recurrence relation

$$\begin{cases}
S_{n+2}^{(1)}(x) = (x - \xi_{n+2}) S_{n+1}^{(1)}(x) - \rho_{n+2} S_n^{(1)}(x) & \text{for } n \geq 0, \\
S_1^{(1)}(x) = x - \xi_1, \quad S_0^{(1)}(x) = 1, \quad \text{and } S_{-1}^{(1)}(x) = 0.
\end{cases}$$
Another important representation of $S_n^{(1)}(x)$ is, (see [7]),

$$S_n^{(1)}(x) := \left\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x - \zeta} \right\rangle. \quad (9)$$

Also, let $\{S_n(\cdot, \mu)\}_{n \geq 0}$ be the co-recursive polynomials for the sequence $\{S_n\}_{n \geq 0}$ satisfying

$$S_n(x, \mu) = S_n(x) - \mu S_{n-1}^{(1)}(x) \quad \text{for } n \geq 0. \quad (10)$$

cf. [7].

We recall that a form $v$ is called symmetric if $(v)_{2n+1} = 0$ for $n \geq 0$. The conditions $(v)_{2n+1} = 0$ for $n \geq 0$ are equivalent to the fact that the corresponding monic orthogonal polynomial sequence $\{S_n\}_{n \geq 0}$ satisfies the recurrence relation (7) with $\xi_n = 0$ for $n \geq 0$. cf. [7].

Throughout this paper, the form $v$ will be supposed normalized, (i.e., $(v)_0 = 1$), symmetric and regular.

Let us consider the decomposition of $\{S_n\}_{n \geq 0}$ and $\{S_n^{(1)}\}_{n \geq 0}$:

$$S_{2n}(x) = P_n(x^2), \quad S_{2n+1}(x) = xR_n(x^2), \quad (11)$$
$$S_{2n}^{(1)}(x) = R_n(x^2, -\rho_1) \quad \text{and} \quad S_{2n+1}^{(1)}(x) = xP_n^{(1)}(x^2). \quad (12)$$

cf. [7, 20]. The sequences $\{P_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ are respectively orthogonal with respective to $\sigma v$ and $x\sigma v$. We also have

$$\left\{ \begin{array}{l} R_{n+2}(x) = (x - \xi_{n+1}^R) R_{n+1}(x) - \rho_{n+1}^R R_n(x) \quad \text{for } n \geq 0, \\ R_1(x) = x - \xi_0^R \quad \text{and} \quad R_0(x) = 1, \end{array} \right. \quad (13)$$

with

$$\xi_0^R = \rho_1 + \rho_2, \quad \xi_{n+1}^R = \rho_{2n+3} + \rho_{2n+4}, \quad \text{and} \quad \rho_{n+1}^R = \rho_{2n+2} \rho_{2n+3} \quad \text{for } n \geq 0. \quad (14)$$

By virtue of (8), with $\xi_n = 0$, we get $\xi_{n+2}^{(1)}(0) = -\rho_{n+2} S_{n+1}^{(1)}(0)$. Consequently,

$$S_{2n}^{(1)}(0) = R_n(0, -\rho_1) = (-1)^n \prod_{\nu=0}^{n} \rho_{2\nu} \quad \text{for } n \geq 0. \quad (15)$$

PROPOSITION 1 ([7, 21]). $v$ is regular if and only if $\sigma v$ and $x\sigma v$ are regular.

2 Algebraic Properties

For fixed $a \in \mathbb{C}$ and $\lambda \in \mathbb{C} - \{0\}$, we can define a new normalized form $u \in P'$ by the relation

$$u = -\lambda(x^2 - a^2)^{-1}v + \delta_0. \quad (16)$$
Equivalently, from (1)-(3) we have
\[ x (x^2 - a^2) u = -\lambda x v, \quad (u)_1 = 0, \quad (u)_2 = -\lambda. \] (17)
The case \( a = 0 \) is treated in [1, 18, 26], so henceforth, we assume \( a \neq 0 \).

**PROPOSITION 2.** \( u \) is regular if and only if
\[ R_n(a^2, -\rho_1) \Delta_n \neq 0 \quad \text{for} \quad n \geq 0, \] (18)
where \( R_n \) is defined by (13), and for \( n \geq 0 \),
\[ \Delta_n = R_{n+1}(a^2, -\rho_1) (\lambda R_n(0, -\rho_1) + a^2 R_n(0)) \]
\[ -R_n(a^2, -\rho_1) (\lambda R_{n+1}(0, -\rho_1) + a^2 R_{n+1}(0)). \] (19)

**PROOF.** Multiplying (17) by \( x \) and applying the operator \( \sigma \) for the obtained equation and using (2), we get
\[ -\lambda^{-1} x \sigma u = \rho_1 (x - a^2)^{-1} (\rho_1^{-1} x \sigma v) + \delta_a^2. \] (20)
From (20) and (3), we get
\[ \sigma u = -\lambda \rho_1 x^{-1} (x - a^2)^{-1} (\rho_1^{-1} x \sigma v) + \left(1 + \frac{\lambda}{a^2}\right) \delta_0 - \frac{\lambda}{a^2} \delta_a^2. \] (21)

From (16), it is plain that \( u \) is a symmetric form. Then, according to Proposition 1, \( u \) is regular if and only if \( x \sigma u \) and \( \sigma u \) are regular. But
\[ -\lambda^{-1} x \sigma u = \rho_1 (x - a^2)^{-1} (\rho_1^{-1} x \sigma v) + \delta_a^2 \]
is regular if and only if \( \lambda \neq 0 \) and \( R_n(a^2, -\rho_1) \neq 0 \) for \( n \geq 0 \) (see [22]). So \( u \) is regular if and only if \( R_n(a^2, -\rho_1) \neq 0 \) and
\[ \sigma u = -\lambda \rho_1 x^{-1} (x - a^2)^{-1} (\rho_1^{-1} x \sigma v) + \left(1 + \frac{\lambda}{a^2}\right) \delta_0 - \frac{\lambda}{a^2} \delta_a^2 \]
is regular. Or, it was shown in [6] that the form
\[ -\lambda \rho_1 x^{-1} (x - a^2)^{-1} (\rho_1^{-1} x \sigma v) + \left(1 + \frac{\lambda}{a^2}\right) \delta_0 - \frac{\lambda}{a^2} \delta_a^2 \]
is regular if and only if \( \Delta_n \neq 0 \) for \( n \geq 0 \). Then, we deduce the desired result.

**REMARK 1.** From (11) and (12), we get
\[ R_n(a^2, -\rho_1) = S_{2n}^{(1)}(a), \quad R_n(0, -\rho_1) = S_{2n}^{(1)}(0), \quad \text{and} \quad R_n(0) = S_{2n+1}^{(1)}(0) \]
for $n \geq 0$. Thus, $u$ is regular if and only if

$$\begin{align*}
S_{2n}^{(1)}(a) \left( S_{2n+2}^{(1)}(a) \left( \lambda S_{2n}^{(1)}(0) + a^2 S_{2n+1}^{(0)}(0) \right) 
- S_{2n}^{(1)}(0) \left( \lambda S_{2n+1}^{(0)}(0) + a^2 S_{2n+3}^{(0)}(0) \right) \right) \\
\neq 0 \quad \text{for } n \geq 0.
\end{align*}$$

(22)

REMARK 2. From (7), we have

$$S_0(1) = 1 \quad \text{and} \quad S_0(2n+3) = S_{2n+2}(0) - \rho_{2n+3} S_{2n+1}(0) \quad \text{for } n \geq 0.$$

Therefore, we can easily prove by induction that

$$S_{2n+1}(0) = (-1)^n \Lambda_n S_{2n}(0) \quad \text{for } n \geq 0,$$

(23) with

$$\Lambda_n = 1 + \sum_{\nu=0}^{n-1} \prod_{k=0}^{\nu} \frac{\rho_{2k+1}}{\rho_{2k+2}} \quad \text{for } n \geq 0 \quad \text{where} \quad \sum_{\nu=0}^{n-1} = 0.$$

(24)

When $u$ is regular, let $\{Z_n\}_{n \geq 0}$ be the corresponding sequence satisfying the recurrence relation

$$\begin{cases}
Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1} Z_n(x) & \text{for } n \geq 0, \\
Z_1(x) = x \quad \text{and} \quad Z_0(x) = 1.
\end{cases}$$

(25)

Let us now consider the quadratic decomposition of the sequence $\{Z_n\}_{n \geq 0}$

$$Z_{2n}(x) = \tilde{P}_n(x^2) \quad \text{and} \quad Z_{2n+1}(x) = x\tilde{R}_n(x^2) \quad \text{for } n \geq 0.$$

(26)

From (20) and (21), we can deduce the following results.

PROPOSITION 3 ([22]). The polynomials of the sequence $\{\tilde{R}_n\}_{n \geq 0}$ satisfy the relation

$$\tilde{R}_{n+1}(x) = R_{n+1}(x) + a_n R_n(x) \quad \text{for } n \geq 0,$$

(27) where

$$a_n = -\frac{S_{2n+2}^{(1)}(a)}{S_{2n}^{(0)}(a)} \quad \text{for } n \geq 0.$$

(28)

PROPOSITION 4 ([6]). The polynomials of the sequence $\{\tilde{P}_n\}_{n \geq 0}$ satisfy the relation

$$\begin{cases}
\tilde{P}_{n+2}(x) = R_{n+2}(x) + c_{n+1} R_{n+1}(x) + b_n R_n(x) & \text{for } n \geq 0, \\
\tilde{P}_1(x) = R_1(x) + c_0,
\end{cases}$$

(29)
where
\[ b_n = -\frac{\Delta_{n+1}}{\Delta_n} \text{ for } n \geq 0, \tag{30} \]
and, for \( n \geq 0, \)
\[
\begin{align*}
    c_{n+1} &= -\Delta_n^{-1} \left\{ S_{2n}^{(l)}(a) \left( \lambda S_{2n+2}^{(l)}(0) + a^2 S_{2n+5}^{(l)}(0) \right) \\
    &\quad - S_{2n+4}^{(l)}(a) \left( \lambda S_{2n}^{(l)}(0) + a^2 S_{2n+1}^{(l)}(0) \right) \right\}, \\
    c_0 &= -\lambda - \rho_1 - \rho_2.
\end{align*} \tag{31} \]

**LEMMA 1.**
\[
x Z_{n+3}(x) = S_{n+4}(x) + \tilde{b}_{n+2} S_{n+2}(x) + \tilde{a}_n S_n(x) \text{ for } n \geq 0, \\
x Z_2(x) = S_3(x) + \tilde{b}_1 S_1(x), \\
x Z_1(x) = S_2(x) + \tilde{b}_0,
\tag{32} \]
with for \( n \geq 0, \)
\[
\begin{align*}
    \tilde{a}_{2n} &= \rho_{2n+1} a_n, \quad \tilde{a}_{2n+1} = b_n, \\
    \tilde{b}_{2n+2} &= \rho_{2n+3} + a_n, \quad \tilde{b}_{2n+3} = c_{n+1}, \\
    \tilde{b}_0 &= \rho_1 \quad \text{and} \quad \tilde{b}_1 = c_0.
\end{align*} \tag{33} \]

**PROOF.** From (26), we have
\[
x Z_{2n+2}(x) = x \tilde{P}_{n+1}(x^2) \quad \text{and} \quad x Z_{2n+1}(x) = x^2 \tilde{R}_n(x^2) \text{ for } n \geq 0.
\]
Then, from the above equation, (11), (27) and (29), we get (32).

**PROPOSITION 5.** We may write
\[
\begin{align*}
    \gamma_1 &= -\lambda, \quad \gamma_{n+2} = \rho_{n+1} \frac{\tilde{a}_{n+1}}{\tilde{a}_n}, \\
    \gamma_{n+3} &= \rho_{n+3} \tilde{b}_{n+2} - \tilde{b}_{n+3},
\end{align*} \tag{34} \]
and
\[
\begin{align*}
    \tilde{a}_{n+1} - \tilde{a}_n &= \rho_{n+2} \tilde{b}_{n+2} - \gamma_{n+3} \tilde{b}_{n+1},
\end{align*} \tag{35} \]
for \( n \geq 0. \)

**PROOF.** After multiplication of (32) by \( x, \) we apply the recurrence relations (7) and (25), we get
\[
x Z_{n+4}(x) + \gamma_{n+3} x Z_{n+2}(x) = S_{n+5}(x) + (\rho_{n+4} + \tilde{b}_{n+2}) S_{n+3}(x) \\
+ (\tilde{a}_n + \rho_{n+2} \tilde{b}_{n+2}) S_{n+1}(x) + \rho_{n} \tilde{a}_n S_{n-1}(x)
\]
for \( n \geq 1 \). Substituting \( xZ_{k+3} \) in the above equation by \( S_{k+4} + \tilde{b}_{k+2}S_{k+2} + \tilde{a}_kS_k \) with \( k = n + 1, n - 1 \), we obtain (34)-(36), after comparing the coefficients of \( S_k \) with \( k = n + 3, n + 1, n - 1 \).

REMARK 3. From (14), (33) and (34), the sequence \( \{ \tilde{R}_n \}_{n \geq 0} \) satisfies the recurrence relation (13) with for \( n \geq 0 \),

\[
\tilde{R}_0 = -\lambda - \frac{b_0}{a_0}, \quad \tilde{R}_{n+1} = \rho_{2n+2}\rho_{2n+3}\frac{a_{n+1}}{b_n} + \frac{b_{n+1}}{a_{n+1}},
\]

and

\[
\gamma_{n+1} = \rho_{2n+2}\rho_{2n+3}\frac{a_{n+1}}{a_n}.
\]

3 The Semi-Classical Case

In this section, we compute the exact class of the semi-classical form \( u \).

DEFINITION 1 ([21]). The form \( v \) is called semi-classical when it is regular and satisfies the Riccati equation

\[
\Phi(z)S'(v)(z) = C(z)S(v)(z) + D(z), \quad (37)
\]

where \( \Phi \) monic, \( C \) and \( D \) are polynomials and \( S(v)(z) \) designes the formal Stieltjes function of the form \( v \) defined by:

\[
S(v)(z) = -\sum_{n \geq 0} \frac{(v)_n}{z^{n+1}}. \quad (38)
\]

It was shown in [21] that equation (37) is equivalent to

\[
(\Phi(x)v)' + \Psi v = 0, \quad (39)
\]

with

\[
\Psi(x) = -\Phi'(x) - C(x). \quad (40)
\]

We also have the following relation :

\[
D(x) = -(v\theta_0\Phi)'(x) - (v\theta_0\Psi)(x).
\]

PROPOSITION 6 ([21]). Define \( r = \deg(\Phi) \) and \( p = \deg(\Psi) \). The semi-classical form \( v \) satisfying (39) is of class \( s = \max (r - 2, p - 1) \) if and only if

\[
\prod_{c \in Z} \{|\Phi'(c) + \Psi(c)| + |(v, \theta^2_c\Phi + \theta_c\Psi)|\} \neq 0, \quad (41)
\]

where \( Z \) denotes the set of zeros of \( \Phi \).
COROLLARY 1 ([19]). The form \( v \) satisfying (37) is of class \( s \) if and only if
\[
\prod_{c \in Z} (|C(c)| + |D(c)|) \neq 0. \tag{42}
\]

PROPOSITION 7. If \( v \) is a semi-classical form and satisfies (37), then for every \( \lambda \in \mathbb{C} \setminus \{0\} \) such that \( R_n(a^2, -\rho_1)\Delta_n \neq 0, n \geq 0 \), the form \( u \) defined by (16) is regular and semi-classical. It satisfies
\[
\tilde{\Phi}(z)S'(u)(z) = \tilde{C}(z)S(u)(z) + \tilde{D}(z), \tag{43}
\]
where
\[
\begin{align*}
\tilde{\Phi}(z) &= z^2 (z^2 - a^2) \Phi(z), \\
\tilde{C}(z) &= z^2 (z^2 - a^2) C(z) - 2z^3 \Phi(z), \\
\tilde{D}(z) &= z (z^2 - a^2) C(z) - (z^2 + a^2) \Phi(z) - \lambda z^2 D(z),
\end{align*} \tag{44}
\]
and \( u \) is of class \( \tilde{s} \) such that \( \tilde{s} \leq s + 4 \).

PROOF. We have [21]
\[
zS(v)(z) = S(\xi v)(z) - (v \theta_0(\xi)) (z) = S(\xi v)(z) - 1.
\]
Using (17), we get
\[
zS(v)(z) = -\frac{1}{\lambda} S(\xi (\xi - a) (\xi - b) u)(z) - 1 = -\frac{1}{\lambda} (z - a) (z - b) (zS(u)(z) - 1). \tag{45}
\]
Multiplying (37) by \( z^2 \) and taking into account (45) we obtain (43)-(44).

From (39) and (43)-(44), the form \( u \) satisfies the distributional equation
\[
\left(\tilde{\Phi}(x)v\right)' + \tilde{\Psi} v = 0, \tag{46}
\]
where \( \tilde{\Phi} \) is the polynomial defined in (44) and
\[
\tilde{\Psi}(x) = -\tilde{\Phi}'(x) - \tilde{C}(x) = x (x^2 - a^2) (x\phi(x) - 2\Phi(x)). \tag{47}
\]
Then \( \deg(\tilde{\Phi}) = \tilde{r} \leq s + 6 \) and \( \deg(\tilde{\Psi}) = \tilde{p} \leq s + 5 \). Thus \( \tilde{s} = \max(\tilde{r} - 2, \tilde{p} - 1) \leq s + 4 \).

PROPOSITION 8. Let \( u \) be a semi-classical form satisfying (43). For every zero of \( \tilde{\Phi} \) different from 0 and \( a \), the equation (43) is irreducible.

PROOF. Since \( v \) is a semi-classical form of class \( s \), \( S(v)(z) \) satisfies (37), where the polynomials \( \Phi, C \) and \( D \) are coprime. Let \( \tilde{\Phi}, \tilde{C} \) and \( \tilde{D} \) be as in Proposition 7. Let \( c \) be a zero of \( \tilde{\Phi} \) different from 0 and \( a \), this implies that \( \Phi(c) = 0 \).

We know that \( |C(c)| + |D(c)| \neq 0 \), (i) if \( C(c) \neq 0 \), then \( \tilde{C}(c) \neq 0 \); and (ii) if \( C(c) = 0 \) and \( D(c) \neq 0 \), then \( \tilde{D}(c) \neq 0 \), whence \( |\tilde{C}(c)| + |\tilde{D}(c)| \neq 0 \).

Concerning the class of \( u \), we have the following result (see Proposition 9). But first, let us recall this technical lemma.

LEMMA 2. We have the following properties:
The equation (43)-(44) is irreducible in 0 if and only if 
\[ \Phi(0) \neq 0. \]

The equation (43)-(44) is divisible by \( z \) but not by \( z^2 \) if and only if 
\[ \Phi(0) = 0 \quad \text{and} \quad C(0) + \Phi'(0) \neq 0. \]

The equation (43)-(44) is irreducible in \( a \) and \(-a\) if and only if 
\[ |\Phi(a)| + |D(a)| \neq 0. \]

The equation (43)-(44) is divisible by \( z^2 - a^2 \) but not by \((z^2 - a^2)^2\) if and only if 
\[ \Phi(a) = D(a) = 0 \quad \text{and} \quad |C(a) - a\Phi''(a)| + |D''(a)| \neq 0. \]

**PROOF.** From (44), we have \( \tilde{C}(0) = 0 \) and \( \tilde{D}(0) = -a^2\Phi(0). \) If \( \Phi(0) \neq 0, \) then \( \tilde{D}(0) \neq 0. \) So, by virtue of (42), we obtain \( P_1. \) Now, if \( \Phi(0) = 0, \) then the equation (43)-(44) is divisible by \( z \) according to (42). Thus \( S(u)(z) \) satisfies (43) with
\[
\begin{align*}
\hat{\Phi}(z) &= z(z^2 - a^2)\Phi(z), \\
\tilde{C}(z) &= z(z^2 - a^2) C(z) - 2z^2\Phi(z), \\
\tilde{D}(z) &= (z^2 - a^2) C(z) - (z^2 + a^2)(\theta_0\Phi)(z) - \lambda zD(z).
\end{align*}
\]
Therefore, \( \tilde{C}(0) = 0 \) and \( \tilde{D}(0) = -a^2(C(0) + \Phi'(0)). \) If \( C(0) + \Phi'(0) \neq 0, \) then the equation (43)-(48) is irreducible in 0. Thus, we deduce \( P_2. \) From (44), we get \( \tilde{C}(a) = -2a^3\Phi(a) \) and \( \tilde{D}(a) = -a^2(\lambda D(a) + 2\Phi(a)). \) Then, we can deduce that \( |\tilde{C}(a)| + |\tilde{D}(a)| \neq 0 \) if and only if \( (\Phi(a), D(a)) \neq (0, 0). \) Thus \( P_3 \) is proved. If \( (\Phi(a), D(a)) = (0, 0), \) then the equation (43)-(44) can be divided by \( z^2 - a^2 \) since \( u \) is symmetric and according to (42). In this case \( S(u)(z) \) satisfies (43) with
\[
\begin{align*}
\hat{\Phi}(z) &= z^2\Phi(z), \\
\tilde{C}(z) &= z C(z) - 2z^2(\theta_a\Phi)\left(\frac{\theta_a}{a}\right)(z), \\
\tilde{D}(z) &= z C(z) - (z^2 + a^2)(\theta_a\Phi)\left(\frac{\theta_a}{a}\right)(z) - \lambda z^2(\theta_a\theta_a D)\left(\frac{\theta_a}{a}\right)(z).
\end{align*}
\]
Substituting \( z \) by \( a \) in (49), we obtain
\[
\begin{align*}
\tilde{C}(a) &= a^2(C(a) - a\Phi''(a)), \\
\tilde{D}(z) &= a\left(C(a) - a\Phi''(a) - \frac{a}{2}D''(a)\right).
\end{align*}
\]
Then (43)-(49) is irreducible in \( a \) and \(-a\) if and only if \( |C(a) - a\Phi''(a)| + |D''(a)| \neq 0. \) Hence \( P_4. \)

**PROPOSITION 9.** Under the conditions of Proposition 7, for the class of \( u, \) we have two different cases:
(A) $\Phi(0) \neq 0$

(i) $\tilde{s} = s + 4$ if $\Phi(a), D(a)) \neq (0, 0)$.

(ii) $\tilde{s} = s + 2$ if $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

(B) $\Phi(0) = 0$ and $C(0) + \Phi'(0) \neq 0$

(i) $\tilde{s} = s + 3$ if $(\Phi(a), D(a)) \neq (0, 0)$.

(ii) $\tilde{s} = s + 1$ if $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

PROOF. From Proposition 8, the class of $P$ depends only on the zeros 0 and $a$. For the zero 0 we consider the following situation:

(A) $\Phi(0) \neq 0$. In this case the equation (43)-(44) is irreducible in 0 according to $P_1$. But what about the zero $a$? We will analyze the following cases:

(i) $(\Phi(a), D(a)) \neq (0, 0)$, the equation (43)-(44) is irreducible in $a$ and $-a$ according to $P_3$. Then $\tilde{s} = s + 4$. Thus we proved (A)(i).

(ii) $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

From $P_3$ and $P_4$, (43)-(44) is divisible by $z^2 - a^2$ but not by $(z^2 - a^2)^2$ and thus the order of the class of $u$ decreases in two units. In fact, $S(u)(z)$ satisfies the irreducible equation (43)-(49) and then $\tilde{s} = s + 2$. Hence (A)(ii).

(B) $\Phi(0) = 0$ and $C(0) + \Phi'(0) \neq 0$.

In this condition, (43)-(44) is divisible by $z$ but not by $z^2$ according to $P_2$. But what about the zero $a$? We have the two following cases:

(i) $(\Phi(a), D(a)) \neq (0, 0)$, the equation (43)-(44) is irreducible in $a$ and $-a$ according to $P_3$. Therefore $S(u)(z)$ satisfies the irreducible equation (43)-(49) and then $\tilde{s} = s + 3$ and (B)(i) is also proved.

(ii) $(\Phi(a), D(a)) = (0, 0)$ and $|C(a) - a\Phi''(a)| + |D''(a)| \neq 0$.

From $P_3$ and $P_4$, (43)-(48) is divisible by $z^2 - a^2$ but not by $(z^2 - a^2)^2$. Then, $S(u)(z)$ satisfies the irreducible equation (43) with

\[
\begin{align*}
\tilde{\Phi}(z) = \Phi(z), \\
\tilde{C}(z) = zC(z) - 2z^2 (\theta_{-a}\theta_a \Phi)(z), \\
\tilde{D}(z) = C(z) - (z^2 + a^2) (\theta_{-a}\theta_a \Phi)(z) - \lambda z (\theta_{-a}\theta_a D)(z),
\end{align*}
\]

and thus $\tilde{s} = s + 1$.

Finally, if we suppose that the form $v$ has the following integral representation:

\[
\langle v, f \rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx \, \text{for} \, f \in \mathcal{P} \, \text{with} \, \langle v \rangle_0 = \int_{-\infty}^{+\infty} V(x)dx = 1,
\]

where $V$ is locally integrable function with rapid decay and continuous at $a$ and $-a$. Then, from (16) the form $u$ is represented by

\[
\langle u, f \rangle = f(0) + \frac{\lambda}{2\pi} \left\{ P \int_{-\infty}^{+\infty} \frac{V(x)}{x+a} f(x)dx - P \int_{-\infty}^{+\infty} \frac{V(x)}{x-a} f(x)dx + \left(f(a) + f(-a)\right) P \int_{-\infty}^{+\infty} \frac{V(x)}{x-a} dx \right\},
\]
where for \( c \in \{a, -a\} \)
\[
P \int_{-\infty}^{+\infty} \frac{V(x)}{x-c} f(x) dx = \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{c-\varepsilon} \frac{V(x)}{x-c} f(x) dx + \int_{c+\varepsilon}^{+\infty} \frac{V(x)}{x-c} f(x) dx \right].
\]

4 Application

Proposition 7 shows that we can generate new semi-classical sequences from well known ones. We apply our results to \( v := G.G(\alpha, \beta) \), where \( G.G(\alpha, \beta) \) is the Generalized Gegenbauer form. In this case, the form \( v \) is symmetric semi-classical of class \( s = 1 \). Thus, we have [7]
\[
\begin{align*}
\rho_{2n+1} &= \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \quad \text{for } n \geq 0, \\
\rho_{2n+2} &= \frac{(n+1)(\alpha+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)} \quad \text{for } n \geq 0.
\end{align*}
\]

The regularity conditions are \( \alpha \neq -n, \beta \neq -n, \alpha + \beta \neq -n, n \geq 1 \). We also have
\[
\begin{align*}
\Phi(x) &= x (x^2 - 1), \quad \Psi(x) = -2 (\alpha + \beta + 1) x^2 + 2 (\beta + 1), \\
C(x) &= (2\alpha + 2\beta + 1) x^2 - (2\beta + 1), \quad D(x) = 2 (\alpha + \beta + 1) x.
\end{align*}
\]

For greater convenience we take \( a = 1 \), and \( \alpha \neq 0 \). From (8) and (52), we can easily obtain by induction
\[
S_{2n}^{(1)}(0) = (-1)^n \frac{\Gamma(\alpha + \beta + 2) \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1) \Gamma(2n + \alpha + \beta + 2)} \quad \text{for } n \geq 0,
\]
and
\[
S_{2n}^{(1)}(1) = \frac{\alpha + \beta + 1}{\alpha \Gamma(2n + \alpha + \beta + 2)} \Omega_n \quad \text{for } n \geq 0,
\]
with, for \( n \geq 0 \),
\[
\Omega_n = \frac{\Gamma(n + \alpha + 1) \Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)} - \frac{\Gamma(\alpha + \beta + 1) \Gamma(n + 1) \Gamma(n + \beta + 2)}{\Gamma(\beta + 1)}.
\]

From (52), we get
\[
\frac{\rho_{2k+1}}{\rho_{2k+2}} = \frac{(k + \beta + 1) (k + \alpha + \beta + 1) (2k + \alpha + \beta + 3)}{(k + 1) (k + \alpha + 1) (2k + \alpha + \beta + 1)}.
\]

Then
\[
\prod_{k=0}^{\nu} \frac{\rho_{2k+1}}{\rho_{2k+2}} = \frac{(2\nu + \alpha + \beta + 3) \Gamma(\alpha + 1) \Gamma(\nu + \beta + 2) \Gamma(\nu + \alpha + \beta + 2)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + 2) \Gamma(\nu + 2) \Gamma(\nu + \alpha + 2)}
\]
\[
= \frac{(2\nu + \alpha + \beta + 3) \Gamma(\alpha + 1)}{\Gamma(\beta + 1) \Gamma(\alpha + \beta + 2) (\nu + 1) (\nu + \alpha + 1)}^h,\]
with
\[ h_n = \frac{\Gamma(n + \beta + 2) \Gamma(n + \alpha + \beta + 2)}{\Gamma(n + 1) \Gamma(n + \alpha + 1)}, \quad n \geq 0, \]

fulfilling
\[ h_{n+1} = \frac{(n + \beta + 2) (n + \alpha + \beta + 2)}{(n + 1) (n + \alpha + 1)} h_n, \quad n \geq 0. \]

Therefore
\[ h_{n+1} - h_n = \frac{(\beta + 1) (2n + \alpha + \beta + 3)}{(n + 1) (n + \alpha + 1)} h_n, \quad n \geq 0, \]

and consequently, from the above results, we obtain that for \( n \geq 1, \)
\[ \sum_{\nu=0}^{n-1} \prod_{k=0}^{\nu} \sum_{k=0}^{\nu} \rho_{2k+1} \rho_{2k+2} = \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 2) \Gamma(\alpha + \beta + 2) \Gamma(n + 1) \Gamma(n + \alpha + 1)} \sum_{\nu=0}^{n-1} (h_{\nu+1} - h_\nu) = \frac{\Gamma(\alpha + 1) \Gamma(n + \beta + 2) \Gamma(n + \alpha + \beta + 2)}{\Gamma(\beta + 2) \Gamma(n + 1) \Gamma(n + \alpha + 1)} - 1. \]

Finally, (24), (23) and (19) become respectively
\[ \Lambda_n = \frac{\Gamma(n + \beta + 2) \Gamma(n + \alpha + \beta + 2)}{\Gamma(\beta + 2) \Gamma(\alpha + \beta + 2) \Gamma(n + 1) \Gamma(n + \alpha + 1)} \quad \text{for } n \geq 0, \quad (56) \]
\[ S'_{2n+1}(0) = \frac{\Gamma(n + \beta + 2) \Gamma(n + \alpha + \beta + 2)}{\Gamma(\beta + 2) \Gamma(n + 1) \Gamma(2n + \alpha + \beta + 2)} \quad \text{for } n \geq 0, \quad (57) \]
and
\[ \Delta_n = \frac{\alpha + \beta + 1}{\alpha \Gamma(2n + \alpha + \beta + 2) \Gamma(2n + \alpha + \beta + 3)} (\Theta_n \lambda + \Upsilon_n) \quad \text{for } n \geq 0, \quad (58) \]
with for \( n \geq 0 \)
\[ \Theta_n = (-1)^n \frac{\Gamma(n + \beta + 2) \Gamma(n + \alpha + 1)}{\Gamma(n + 1) \Gamma(n + \alpha + 1) \Omega_{n+1} + (n + \alpha + 1) \Omega_n}, \]
\[ \Upsilon_n = \frac{\Gamma(n + \beta + 2) \Gamma(n + \alpha + \beta + 2)}{\Gamma(\beta + 2) \Gamma(n + 2)} ((n + 1) \Omega_{n+1} + (n + \beta + 2) (n + \alpha + \beta + 2) \Omega_n). \]

Thus, \( u \) is regular for every \( \lambda \neq 0 \) such that
\[ \Omega_n (\Theta_n \lambda + \Upsilon_n) \neq 0 \quad \text{for } n \geq 0. \]

Using (55) and (58), we obtain for (28) and (30) (for \( n \geq 0 \))
\[ a_n = -\frac{\Omega_{n+1}}{\Omega_n (2n + \alpha + \beta + 2) (2n + \alpha + \beta + 3)}, \]

\[ \vdots \]
\[ b_n = -\frac{\Theta_{n+1} + \Upsilon_{n+1}}{(\Theta_n \lambda + \Upsilon_n) (2n + \alpha + \beta + 2) (2n + \alpha + \beta + 3) (2n + \alpha + \beta + 4) (2n + \alpha + \beta + 5)}. \]

Therefore, we have for (34)

\[ \gamma_1 = -\lambda, \quad \gamma_{2n+2} = \Omega_n \frac{\Theta_{n+1} \lambda + \Upsilon_{n+1}}{(\Theta_n \lambda + \Upsilon_n) (2n + \alpha + \beta + 4) (2n + \alpha + \beta + 5)}. \]

\[ \gamma_{2n+3} = \Omega_{n+1} \frac{\Theta_n \lambda + \Upsilon_n (n+1) (n + \alpha + 1) (n + \beta - 2) (n + \alpha + \beta + 2) (2n + \alpha + \beta + 4)}{(\Theta_{n+1} \lambda + \Upsilon_{n+1}) (2n + \alpha + \beta + 3) (2n + \alpha + \beta + 5)}. \]

Since \( v \) is semi-classical, then according to Proposition 7, (40) and (48), the form \( u \) is also semi-classical of class \( s = 4 \) and fulfills (43) and (47) with

\[ \hat{\Phi}(x) = x^2 (x^2 - 1)^2, \]
\[ \hat{\Psi}(x) = -x (x^2 - 1) ((2\alpha + 2\beta + 5) x^2 - 2\beta - 3), \]
\[ \hat{C}(x) = x (x^2 - 1) ((2\alpha + 2\beta - 1) x^2 - 2\beta - 1), \]
\[ \hat{D}(x) = 2(\beta + 1) x^4 - 2 ((\alpha + \beta + 2) (\alpha + \beta + \beta) x^2 + 2 (\beta + 1). \]

The form \( v \) has the following integral representation [7], for \( \Re \alpha > -1, \Re \beta > -1, \]
\[ \langle v, f \rangle = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(\beta + 1)} \int_{-1}^{1} |x|^{2\beta + 1} (1 - x^2)^{\alpha} f(x) dx. \]

Then, from (51), we obtain

\[ \langle u, f \rangle = f(0) + \lambda \frac{\Gamma(\alpha + \beta + 2)}{2\Gamma(\alpha + 1) \Gamma(\beta + 1)} \left[ \int_{-1}^{1} \frac{|x|^{2\beta + 1} (1 - x^2)^{\alpha}}{x + 1} (f(x) - f(-1)) dx \right. \]
\[ - \left. \int_{-1}^{1} \frac{|x|^{2\beta + 1} (1 - x^2)^{\alpha}}{x - 1} (f(x) - f(1)) dx \right], \]

for \( \Re \alpha > -1 \) and \( \Re \beta > -1 \). But, if \( \Re \alpha > 0 \), we have

\[ \int_{-1}^{1} \frac{|x|^{2\beta + 1} (1 - x^2)^{\alpha}}{x + 1} dx = -\int_{-1}^{1} \frac{|x|^{2\beta + 1} (1 - x^2)^{\alpha}}{x - 1} dx = \frac{\Gamma(\alpha) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)}. \]

Consequently, if \( \Re \alpha > 0, \Re \beta > -1, \) \( f \in \mathcal{P}, \)

\[ \langle u, f \rangle = \lambda \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(\beta + 1)} \left[ \int_{-1}^{1} |x|^{2\beta + 1} (1 - x^2)^{\alpha-1} f(x) dx \right. \]
\[ + f(0) - \lambda \frac{\alpha + \beta + 1}{2\alpha} (f(1) + f(-1)). \]

REMARKS 4. From (59), we have

\[ u = \lambda \frac{\alpha + \beta + 1}{\alpha} G.G(\alpha - 1, \beta) + \delta_0 - \lambda \frac{\alpha + \beta + 1}{2\alpha} (\delta_1 + \delta_{-1}). \]
For more details see \[3\]. Using (59), we get
\[
(u)_{2n+2} = \lambda \frac{\Gamma (\alpha + \beta + 2) \Gamma (n + \beta + 2)}{\alpha \Gamma (\beta + 1) \Gamma (n + \alpha + \beta + 2)} - \frac{\lambda \alpha + \beta + 1}{\alpha}, \quad n \geq 0.
\]

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**References**


