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Maximal Mean Exit Time Related To The p-Laplace Operator*

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Abstract

In this paper we introduce a boundary value problem involving powers of the p-Laplace operator. We will then prove a variant of Talenti inequality which shows that the Schwarz symmetrization of the solution of the boundary value problem is majorized by the solution of the appropriately symmetrized version of the problem. The case of equality is also investigated. Finally, as an application, we will consider an optimization problem related to the mean exit time of a Wiener process and derive a symmetry result.

1 Introduction

Partial differential equations involving the p-Laplace operator Δ_p have become important subjects of research for a wide spectrum of scientists, e. g. mathematicians, engineers, biologists, economists, etc. The reason is obvious as this operator can be used to model a range of physical phenomena. To name a few areas of applications we mention non-Newtonian fluids, logistic equations, flows through porous media, nonlinear elasticity, glaciology (see e. g. [1, 2, 5]). However, equations involving powers of Δ_p have not been sufficiently investigated. In this paper we introduce one such case, and look at some applications related to the mean exit time of a Wiener process. This paper is mainly motivated by [8, 9]. The former discusses the Talenti inequality for the case of the biharmonic operator Δ^2 whilst the latter focuses on radial symmetry problems and, amongst other results, draws a rather nice conclusion pertaining to the connection between radial symmetry and the maximal mean exit time of a Wiener process.

Let us now expand further on the motivation behind the present paper. Consider the boundary value problem

$$\begin{cases} -\Delta_p u = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

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where D is a bounded smooth domain in $\mathbb{R}^n$, $f : D \to \mathbb{R}$ a bounded positive function, and Δ_p stands for the usual p-Laplace operator

$$\Delta_p u = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) \quad \text{(where } p > 1\text{)}$$

The symmetrized problem associated with (1) is as follows

$$\begin{cases} -\Delta_p v = f^* & \text{in } B, \\ v = 0 & \text{on } \partial B, \end{cases} \tag{2}$$

where f^* is the Schwarz symmetrization of f (see e.g. [8]) and B is the ball in $\mathbb{R}^n$ centred at the origin such that $|B| = |D|$. Henceforth, for a measurable set $E \subseteq \mathbb{R}^n$, $|E|$ denotes the Lebesgue measure of E. The following pointwise inequality is attributed to G. Talenti [12]

$$u^*(x) \leq v(x) \tag{3}$$

almost everywhere in B. Actually, the inequality (3) is a special case of Theorem 1 in [12]. In Section 2 we will present a proof of (3) for the convenience of the reader.[1] As a consequence of (3), we will see that if $u^*(0) = v(0)$, then $u^*(x) = v(x)$ almost everywhere in B. When $f = 1$ this will lend itself to drawing an interesting conclusion pertaining to an optimization problem involving the mean exit time of a Wiener process starting at a point within D.

Let us describe the setting. First, we introduce a system as follows. Let $f \in L^\infty(D)$ be a non-negative function. Consider the system

$$(S) \quad \begin{cases} -\Delta_p v_j = v_{j-1} & \text{in } D, \\ v_j = 0 & \text{on } \partial D, \end{cases}$$

for $j = 1, \ldots, N$ in which $v_0 = f$. Note that for every j the function $v_j \in W_0^{1,p}(D)$ is the unique minimizer of the functional

$$F_j(w) = \frac{1}{p} \int_D |\nabla w|^p \, dx - \int_D v_{j-1} w \, dx$$

relative to $w \in W_0^{1,p}(D)$. Clearly the system (S) has a unique solution $(v_1, \ldots, v_N) \in W_0^{1,p}(D) \times \cdots \times W_0^{1,p}(D)$. In addition each v_j is in $H^2(D) \cap C^{1,\alpha}(\overline{D})$ and is positive in D. For $1 \leq k \leq N$, we define the projection operator $\mathcal{P}_k : W_0^{1,p}(D) \times \cdots \times W_0^{1,p}(D) \to W_0^{1,p}(D)$ by $\mathcal{P}_k(w_1, \ldots, w_N) = w_k$. Henceforth, we set $u_f := \mathcal{P}_N(v_1, \ldots, v_N)$, where $(v_1, \ldots, v_N)$ is the solution of the system (S). We also prefer to identify the system (S) with the following *nonstandard* boundary value problem

$$\begin{cases} (-\Delta_p)^N u = f & \text{in } D, \\ (-\Delta_p)^k u = 0 & \text{on } \partial D \quad (0 \leq k \leq N-1), \end{cases} \tag{4}$$

where $N \in \mathbb{N}$.

REMARK 1.1. The reader should distinguish (4) from the standard boundary value problems where solutions lie in the Sobolev space $W^{2N,p}(D)$. The notation used in (4)

[1] See Lemma 2.2.

is merely symbolic. So, by a solution to (4) we mean the function u_f which is the solution of (S) corresponding to the input f.

Note that the level sets of u_f

$$\{u = c\} := \{x \in D : u(x) = c\} \quad (c \geq 0)$$

have zero measure. This follows from the positiveness of f and Lemma 7.7 in [6]. We now state a comparison result whose proof follows readily from the standard comparison results for the p-Laplace operator.

LEMMA 1.1. Suppose

$$\begin{cases} (-\Delta_p)^J u_1 \geq (-\Delta_p)^J u_2 & \text{in } D, \\ (-\Delta_p)^j u_1 = (-\Delta_p)^j u_2 = 0 & \text{on } \partial D, \quad (0 \leq j \leq J-1). \end{cases} \tag{5}$$

Then $u_1 \geq u_2$ in D.

REMARK 1.2. The meaning of (5) is that there are two nonstandard boundary value problems

$$\begin{cases} (-\Delta_p)^J u_1 = f_1 & \text{in } D, \\ (-\Delta_p)^j u_1 = 0 & \text{on } \partial D, \quad (0 \leq j \leq J-1), \end{cases}$$

and

$$\begin{cases} (-\Delta_p)^J u_2 = f_2 & \text{in } D, \\ (-\Delta_p)^j u_2 = 0 & \text{on } \partial D, \quad (0 \leq j \leq J-1), \end{cases}$$

where $f_1 \geq f_2$.

In order to state our main result we first need to introduce the symmetrized problem associated with (4)

$$\begin{cases} (-\Delta_p)^N V = f^* & \text{in } B, \\ (-\Delta_p)^k V = 0 & \text{on } \partial B, \quad (0 \leq k \leq N-1). \end{cases} \tag{6}$$

Our main result is the following theorem

THEOREM 1.2. Let u and V be the solutions of (4) and (6), respectively. Then

$$u^*(x) \leq V(x) \tag{7}$$

almost everywhere in B.

2 Preliminaries

Let us recall the definition of the distribution function. For a bounded non-negative function $h : D \to \mathbb{R}$, the function $\lambda_h : [0, \|h\|_\infty] \to [0, |D|]$—called the distribution function of h—is defined by

$$\lambda_h(t) = |\{x \in D : \ h(x) \geq t\}| \equiv |\{h \geq t\}|.$$

The decreasing rearrangement of h, denoted by $h^{\Delta} : [0, |D|] \to [0, \|h\|_{\infty}]$, is defined as

$$h^{\Delta}(s) = \inf\{t : \lambda_h(t) \leq s\}.$$

It is well known that if h is continuous, and its graph has no significant flat sections (i.e. the sets $\{h = c\}$ all have zero measure), then h^{Δ} is the inverse of λ_h:

$$\forall t \in [0, \|h\|_{\infty}],\ s \in [0, |D|] : \quad (h^{\Delta} \circ \lambda_h(t) = t) \quad \wedge \quad (\lambda_h \circ h^{\Delta}(s) = s).$$

For a non-negative $u : D \to \mathbb{R}$, the function $u^* : B \to \mathbb{R}$ denotes the Schwarz symmetrization of u which is defined as follows

$$u^*(x) = u^{\Delta}(\omega_n |x|^n)$$

in which

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

is the measure of the unit ball in $\mathbb{R}^n$, and B is the ball in $\mathbb{R}^n$ centred at the origin satisfying $|B| = |D|$. In Section 4, we will use the following well known result (see e. g. [4])

LEMMA 2.1. Suppose $u \in W_0^{1,p}(\mathbb{R}^n)$ is non-negative. Then the following two statements hold.

(i) $u^* \in W_0^{1,p}(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} |\nabla u^*|^p dx \leq \int_{\mathbb{R}^n} |\nabla u|^p dx.$$

(ii) If u has compact support and

$$\int_{\mathbb{R}^n} |\nabla u^*|^p dx = \int_{\mathbb{R}^n} |\nabla u|^p dx,$$

then for all $0 \leq \alpha < M \equiv \text{ess sup } u$, the set $u^{-1}(\alpha, \infty)$ is a translate of the ball $(u^*)^{-1}(\alpha, \infty)$, apart from a set of measure zero.

As promised in Section 1, we present a proof of inequality (3).

LEMMA 2.2. Let u and v be the solutions of (1) and (2), respectively. Then $u^*(x) \leq v(x)$ almost everywhere in B.

PROOF. First we observe that

$$v(x) = \frac{1}{n^q \omega_n^{\frac{q}{n}}} \int_{\omega_n |x|^n}^{|D|} s^{q(\frac{1}{n} - 1)} \left(\int_0^s f^{\Delta}(\tau) d\tau \right)^{\frac{1}{p-1}} ds,$$

where $q = \frac{p}{p-1}$. From the differential equation in (1) and Divergence theorem we obtain

$$\begin{aligned}\int_{\{u>t\}} f\,dx &= -\int_{\{u>t\}} \nabla\cdot(|\nabla u|^{p-2}\nabla u)\,dx\\ &= -\int_{\{u=t\}} |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\,d\mathcal{H}^{n-1}\\ &= \int_{\{u=t\}} |\nabla u|^{p-1}\,d\mathcal{H}^{n-1}, \end{aligned}\tag{8}$$

where $d\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. The following formula is well known and it is a consequence of the coarea formula (see [10])

$$-\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^p\,dx = \int_{\{u=t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}. \tag{9}$$

On the other hand,

$$\begin{aligned}&-\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^p\,dx\\ =\;& -\lim_{h\to 0}\frac{1}{h}\left(\int_{\{u>t+h\}} |\nabla u|^p\,dx - \int_{\{u>t\}} |\nabla u|^p\,dx\right)\\ =\;& \left(\lim_{h\to 0}\frac{|\{t<u<t+h\}|}{h}\right)\left(\lim_{h\to 0}\frac{1}{|\{t<u<t+h\}|}\int_{\{t<u<t+h\}} |\nabla u|^p dx\right)\\ \geq\;& (-\lambda_u'(t))\lim_{h\to 0}\frac{1}{|\{t<u<t+h\}|}\int_{\{t<u<t+h\}} |\nabla u|^p dx. \end{aligned}\tag{10}$$

By applying the Jensen inequality to the last limit in (10) we get

$$\begin{aligned}-\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^p\,dx &\geq (-\lambda_u'(t))\left(\lim_{h\to 0}\frac{1}{|\{t<u<t+h\}|}\int_{\{t<u<t+h\}} |\nabla u|\,dx\right)^p\\ &= (-\lambda_u'(t))\left(\frac{1}{(-\lambda_u'(t))}\int_{\{u=t\}} d\mathcal{H}^{n-1}\right)^p, \end{aligned}\tag{11}$$

where in the last equality in (11) we have used the coarea formula. An application of the isoperimetric inequality (see e.g. [3])

$$P(\{u>t\}) \geq n\omega_n^{\frac{1}{n}}\lambda_u^{1-\frac{1}{n}}(t),$$

to (11) yields

$$-\frac{d}{dt}\int_{\{u>t\}} |\nabla u|^p\,dx \geq (-\lambda_u'(t))^{1-p} n^p \omega_n^{\frac{p}{n}}\lambda_u^{p-\frac{p}{n}}(t). \tag{12}$$

From (8), (9) and (12) we deduce

$$1 \leq \frac{1}{n^q\omega_n^{\frac{q}{n}}}(-\lambda_u'(t))\lambda_u^{q(\frac{1}{n}-1)}(t)\left(\int_{\{u>t\}} f\,dx\right)^{\frac{1}{p-1}}. \tag{13}$$

From the Hardy-Littlewood inequality [7], we infer

$$\int_{\{u>t\}} f\,dx \leq \int_0^{\lambda_u(t)} f^{\Delta}(s)ds. \tag{14}$$

So, the combination of (13) and (14) leads to

$$1 \leq \frac{1}{n^q \omega_n^{\frac{q}{n}}}(-\lambda_u'(t))\lambda_u^{q(\frac{1}{n}-1)}(t)\left(\int_0^{\lambda_u(t)} f^{\Delta}(s)ds\right)^{\frac{1}{p-1}} \equiv H(t). \tag{15}$$

Integrating (15) from 0 to t, and changing variables give us

$$t \leq \frac{1}{n^q \omega_n^{\frac{q}{n}}}\int_{\lambda_u(t)}^{|D|} s^{q(\frac{1}{n}-1)}\left(\int_0^s f^{\Delta}(\tau)d\tau\right)^{\frac{1}{p-1}} ds. \tag{16}$$

By setting $t = u^*(x) = u^{\Delta}(\omega_n|x|^n)$ in (16), we obtain

$$u^*(x) \leq \frac{1}{n^q \omega_n^{\frac{q}{n}}}\int_{\omega_n|x|^n}^{|D|} s^{q(\frac{1}{n}-1)}\left(\int_0^s f^{\Delta}(\tau)d\tau\right)^{\frac{1}{p-1}} ds \equiv v(x),$$

as desired. The proof is complete.

LEMMA 2.3. Let the hypotheses of Lemma 2.2 hold. Suppose $u^*(0) = v(0)$. Then

$$u^*(x) = v(x) \tag{17}$$

almost everywhere in B.

PROOF. We begin by recalling the function $H(t)$ from (15). One can readily verify the following identity

$$\int_0^{u^*(x)} H(t)dt = v(x). \tag{18}$$

Since $H(t) \geq 1$ in $[0, \max_{\overline{D}}(u)] = [0, u^*(0)]$, we infer that $H(t) = 1$ throughout the interval $[0, \max_{\overline{D}}(u)]$, whence (17) follows from (18). The proof is complete.

3 Proof of the Main Theorem

In this section, we prove THEOREM 1.2. We apply the method of induction. Let $P(l)$ stand for the following claim

$P(l)$: Every pair of systems of the form

$$\begin{cases} (-\Delta_p)^{\hat{l}} w = g & \text{in } D, \\ (-\Delta_p)^k w = 0 & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta_p)^{\hat{l}} W = g^* & \text{in } B, \\ (-\Delta_p)^k W = 0 & \text{on } \partial B, \end{cases}$$

in which $\hat{l} \in \{1, \ldots, l\}$ and k ranges over all values in $\{0, \ldots, \hat{l}-1\}$ satisfies

$$w^*(x) \leq W(x)$$

almost everywhere in B.

For the base case, from Lemma 2.2 we see that $P(1)$ is a valid statement. Then, assuming the validity of $P(l)$ we will prove that of $P(l+1)$. To this end, consider an arbitrary pair of systems

$$\begin{cases} (-\Delta_p)^{l+1}w = g & \text{in } D, \\ (-\Delta_p)^k w = 0 & \text{on } \partial D. \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta_p)^{l+1}W = g^* & \text{in } B, \\ (-\Delta_p)^k W = 0 & \text{on } \partial B, \end{cases}$$

with $0 \le k \le l$. Next, set $\tilde{w} = (-\Delta_p)^l w$. Thus, we obtain

$$\begin{cases} (-\Delta_p)^l w = \tilde{w} & \text{in } D, \\ (-\Delta_p)^k w = 0 & \text{on } \partial D, \quad (0 \le k \le l-1). \end{cases} \tag{19}$$

The symmetric problem associated with (19) is

$$\begin{cases} (-\Delta_p)^l Z = \tilde{w}^* & \text{in } B, \\ (-\Delta_p)^k Z = 0 & \text{on } \partial B, \quad (0 \le k \le l-1). \end{cases} \tag{20}$$

By hypothesis, $w^* \le Z$ almost everywhere in B. On the other hand, we have

$$\begin{cases} (-\Delta_p)\tilde{w} = g & \text{in } D, \\ -\Delta_p \tilde{w} = 0 & \text{on } \partial D. \end{cases} \tag{21}$$

The symmetric problem associated with (21) is

$$\begin{cases} (-\Delta_p)\hat{w} = g^* & \text{in } B, \\ -\Delta_p \hat{w} = 0 & \text{on } \partial B. \end{cases} \tag{22}$$

Since $P(1)$ holds, we have $\tilde{w}^* \le \hat{w} = (-\Delta_p)^l W$. Thus, we derive

$$\begin{cases} (-\Delta_p)^l Z \le (-\Delta_p)^l W & \text{in } B, \\ (-\Delta_p)^k Z = (-\Delta_p)^k W = 0 & \text{on } \partial B, \quad (0 \le k \le l-1). \end{cases} \tag{23}$$

By applying Lemma 1.1 to (23), we deduce $Z \le W$ almost everywhere in B. So, we obtain the desired result

$$w^*(x) \le Z(x) \le W(x)$$

almost everywhere in B. The proof is complete.

COROLLARY 3.1. Suppose the hypotheses of Theorem 1.2 hold. In addition, suppose $u^*(0) = V(0)$. Then

$$u^*(x) = V(x)$$

almost everywhere in B.

PROOF. Let $w_1 = (-\Delta_p)^{N-1}u$. Then

$$\begin{cases} (-\Delta_p)^{N-1}u = w_1 & \text{in } D, \\ (-\Delta_p)^k u = 0 & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta_p)^{N-1}Z_1 = {w_1}^* & \text{in } B, \\ (-\Delta_p)^k Z_1 = 0 & \text{on } \partial B. \end{cases} \tag{24}$$

for $k = 0, \ldots, N-2$. The problem on the right hand side in (24) is the symmetrized problem associated with the one on the left hand side, a convention which will be followed throughout the proof, and thereafter. So by Theorem 1.2 we infer that $u^* \leq Z_1$ almost everywhere in B. On the other hand we have

$$\begin{cases} -\Delta_p w_1 = f & \text{in } D, \\ w_1 = 0 & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_p W_1 = f^* & \text{in } B, \\ -\Delta_p W_1 = 0 & \text{on } \partial B. \end{cases} \tag{25}$$

So, by applying Lemma 2.2 to (25), we get ${w_1}^* \leq W_1 = (-\Delta_p)^{N-1} V$. This, in turn, yields

$$\begin{cases} (-\Delta_p)^{N-1} Z_1 \leq (-\Delta_p)^{N-1} V & \text{in } B, \\ (-\Delta_p)^k Z_1 = (-\Delta_p)^k V = 0 & \text{on } \partial B, \end{cases} \tag{26}$$

for $k = 0, \ldots, N-2$. We can now apply Lemma 1.1 to (26) to obtain $Z_1 \leq V$, almost everywhere in B. Whence, we derive $u^* \leq Z_1 \leq V$, almost everywhere in B.

Next, we set $w_j = (-\Delta_p)^{N-j} u$, where $2 \leq j \leq N-1$. Hence, we have

$$\begin{cases} (-\Delta_p)^{N-j} u = w_j & \text{in } D, \\ (-\Delta_p)^k u = 0 & \text{on } \partial D, \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta_p)^{N-j} Z_j = {w_j}^* & \text{in } B, \\ (-\Delta_p)^k Z_j = 0 & \text{on } \partial B, \end{cases} \tag{27}$$

for $k = 0, \ldots, N-j-1$. We can apply the above arguments to the pairs in (27) and obtain

$$u^*(x) \leq Z_{N-1}(x) \leq \cdots \leq Z_1(x) \leq V(x), \tag{28}$$

almost everywhere in B. From the hypothesis, $u^*(0) = V(0)$, and (28), we obtain

$$u^*(0) = Z_{N-1}(0) = \cdots = Z_1(0) = V(0). \tag{29}$$

By applying Lemma 2.2 to each pair $(u^*, Z_{N-1}), (Z_{N-1}, Z_{N-2}), \ldots, (Z_2, Z_1), (Z_1, V)$ and taking into account (29), we derive

$$u^*(x) = Z_{N-1}(x) = \cdots = V(x),$$

almost everywhere in B. This completes the proof.

4 Maximal Mean Exit Time

This section is motivated by [9] in which the author discusses an optimization problem related to the mean exit time of a Wiener process. Let us briefly review this problem as described in [9]. Consider the classical Poisson problem

$$\begin{cases} -\Delta w &= 2 \quad \text{in } D, \\ w &= 0 \quad \text{on } \partial D. \end{cases} \tag{30}$$

Fix $\hat{x} \in D$ and let $\{X(t)\}$ be a Wiener process starting at $\hat{x}$, i. e. $X(0) = \hat{x}$. Define

$$\tau = \inf\{t \mid X(t) \in \partial D\}.$$

That is, τ is the first exit time and as a result $\forall t < \tau : X(t) \in D$.

By a straightforward application of Dynkin's formula (see e. g. [11]) one would get

$$\mathbf{E}\left(w(X(0))\right) - \mathbf{E}\left(w(X(\tau))\right) = \mathbf{E}\left(\int_0^\tau dt\right) = \mathbf{E}(\tau). \tag{31}$$

From the boundary condition in (30), we see that $w(X(\tau)) = 0$, hence (31) reduces to

$$w(\hat{x}) = \mathbf{E}(w(\hat{x})) = \mathbf{E}(\tau). \tag{32}$$

Now, assume that $\hat{x}$ is a point where w attains its maximum, i. e. $w(\hat{x}) = \max_{\overline{D}} w$. Note that by the Maximum Principle $w(\hat{x})$ is positive, hence $\hat{x}$ is an interior point of $\overline{D}$. Clearly w depends on the domain D, so let us stress this point by writing w_D instead of w. Next, we define a quantity $\Phi(D)$ as follows

$$\Phi(D) = \max_{\overline{D}} w_D = \mathbf{E}(\tau).$$

We can state the following

LEMMA 4.1 ([9]). The maximization problem

$$\sup_{D \in \mathcal{A}_\alpha} \Phi(D)$$

in which $\alpha > 0$ is some prescribed positive constant and $\mathcal{A}_\alpha = \{\Omega \subseteq \mathbb{R}^n \mid \Omega \text{ is open}, |\Omega| = \alpha\}$ has the unique solution B modulo translations. Here, B is the ball in $\mathbb{R}^n$ with $|B| = \alpha$.

So the interpretation of Lemma 4.1 is that amongst all bounded open sets with given measure it is the ball B that attains the maximal mean exit time (at its center).

We will prove a result similar to Lemma 4.1, in the framework of the previous sections. Prior to this, we need some notation. For $D \in \mathcal{A}_\alpha$, we let u_D denote the unique solution of the boundary value problem

$$\begin{cases} (-\Delta_p)^N u = 1 & \text{in } D, \\ (-\Delta_p)^k u = 0 & \text{on } \partial D, \quad (0 \le k \le N-1). \end{cases} \tag{33}$$

We introduce a quantity $\Psi(D)$ associated with (33) as follows

$$\Psi(D) = \max_{\overline{D}} u_D.$$

We can now state the main result of this section.

THEOREM 4.2. The maximization problem

$$\sup_{D \in \mathcal{A}_\alpha} \Psi(D) \tag{34}$$

has the unique solution B modulo translations.

PROOF. Consider $D \in \mathcal{A}_\alpha$. Following the convention mentioned in the previous section, we have the pair

$$\left\{\begin{array}{ll}(-\Delta_p)^N u_D = 1 & \text{in } D\\ (-\Delta_p)^k u_D = 0 & \text{on } \partial D\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll}(-\Delta_p)^N V = 1 & \text{in } B\\ (-\Delta_p)^k V = 0 & \text{on } \partial B\end{array}\right. \tag{35}$$

in which $0 \leq k \leq N-1$. By applying Theorem 1.2 to (35) we obtain $u_D^* \leq V$, which can be assumed to hold everywhere in B. So, in particular, we have $u_D^*(0) \leq V(0)$. Note that $u_D^*(0) = \max_{\overline{D}} u_D$ and $V(0) = \max_{\overline{D}} V$. Since $\Psi(D) = u_D^*(0)$ and $\Psi(B) = V(0)$, we deduce that $\Psi(D) \leq \Psi(B)$ and B is a solution of the maximization problem (34).

It remains to prove the uniqueness part. So let us assume that $\Psi(D) = \Psi(B)$ for some $D \in \mathcal{A}_\alpha$. In what follows, for simplicity we write u instead of u_D. As in Corollary 3.1, there exist functions $Z_1, \ldots, Z_{N-1}$ such that

$$\forall x \in B : u^*(x) \leq Z_{N-1}(x) \leq \cdots \leq Z_1(x) \leq V(x).$$

From the hypothesis we infer $u^*(0) = V(0)$. Hence

$$u^*(0) = Z_{N-1}(0) = \cdots = Z_1(0) = V(0).$$

Thus, from Corollary 3.1 we obtain

$$\forall x \in B : u^*(x) = Z_{N-1}(x) = \cdots = Z_1(x) = V(x). \tag{36}$$

Now we focus on the last equality in (36), that is $Z_1(x) = V(x)$, which holds in B. Let us recall from the proof of Corollary 3.1 the following pairing

$$\left\{\begin{array}{ll}(-\Delta_p)^{N-1} u = w_1 & \text{in } D,\\ (-\Delta_p)^k u = 0 & \text{on } \partial D,\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll}(-\Delta_p)^{N-1} Z_1 = {w_1}^* & \text{in } B,\\ (-\Delta_p)^k Z_1 = 0 & \text{on } \partial B,\end{array}\right. \tag{37}$$

in which $0 \leq k \leq N-2$. We also have the following pairing

$$\left\{\begin{array}{ll}-\Delta_p w_1 = 1 & \text{in } D,\\ w_1 = 0 & \text{on } \partial D,\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll}-\Delta_p W_1 = 1 & \text{in } B,\\ -\Delta_p W_1 = 0 & \text{on } \partial B,\end{array}\right. \tag{38}$$

Lemma 2.2 together with (37) and (38) would imply that

$$(-\Delta_p)^{N-1} Z_1 = w_1^* \leq W_1 = (-\Delta_p)^{N-1} V. \tag{39}$$

From (39), recalling that $Z_1 = V$, we obtain $w_1^* = W_1$ in B.

The combination of (38) and Lemma 2.1 yields

$$\begin{aligned}\int_D w_1\,dx = \int_D |\nabla w_1|^p\,dx &\geq \int_B |\nabla w_1^*|^p\,dx = \int_B |\nabla W_1|^p dx\\ &= \int_B W_1\,dx = \int_B w_1^*\,dx = \int_D w_1\,dx,\end{aligned}$$

which in turn implies

$$\int_D |\nabla w_1|^p\,dx = \int_B |\nabla w_1^*|^p dx.$$

By Lemma 2.1, we deduce that $\{w_1 \geq \delta\}$ is a translation of $\{w_1^* \geq \delta\}$, for all $0 \leq \delta < \max_{\overline{D}} w_1$. So, if we choose $\delta = 0$ we will see that $\{w_1 \geq 0\} = \overline{D}$ must be a ball. This completes the proof of the theorem.

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