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Some L_r Inequalities Involving The Polar Derivative Of A Polynomial*

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Abstract

Let $P(z)$ be a polynomial of degree n and for $\alpha \in \mathbb{C}$, let $D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to α. In this paper, we obtain L_r mean extension of some inequalities concerning the polar derivative of a polynomial having all zeros inside a circle. Our results generalize and sharpen some well-known polynomial inequalities.

1 Introduction

Let $P(z)$ be a polynomial of degree n, then concerning the estimate for the upper bound of the maximum modulus of $|P'(z)|$ in terms of the maximum modulus of $|P(z)|$ on the unit circle $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| .$$

Inequality (1) is a famous result known as Bernstein's Inequality (for reference see [10]). Equality in (1) holds if and only if $P(z)$ has all its zeros at the origin. For the polynomials having all their zeros in the disk $|z| \leq 1$, Paul Turán [13] estimated the lower bound for the maximum modulus of $|P'(z)|$ on $|z| = 1$ by showing that if $P(z)$ is a polynomial of degree n and has all its zeros in $|z| \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq 2\max_{|z|=1} |P'(z)| . \tag{1}$$

Inequality (1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| \neq 0$.

As an extension of (1), Malik [7] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq (1 + k)\max_{|z|=1} |P'(z)| . \tag{2}$$

Equality in (2) holds for $P(z) = (z + k)^n$ where $k \leq 1$.

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On the other hand, for the class of polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, of degree n having all their zeros in $|z| \leq k$, $k \leq 1$, Aziz and Shah [5] proved that

$$n \max_{|z|=1} |P(z)| \leq (1+k^{\mu}) \max_{|z|=1} |P'(z)| - \frac{n}{k^{n-\mu}} \min_{|z|=k} |P(z)|. \tag{3}$$

Malik [8] obtained a generalization of (1) in the sense that the left-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^q d\theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q d\theta \right\}^{\frac{1}{q}} \max_{|z|=1} |P'(z)|. \tag{4}$$

The corresponding extension of (2), which is a generalization of (4), was obtained by Aziz [1] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for each $q \geq 0$

$$n \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^q d\theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} \left| 1 + k e^{i\theta} \right|^q d\theta \right\}^{\frac{1}{q}} \max_{|z|=1} |P'(z)|. \tag{5}$$

Inequality (5) reduces to the inequality (2) by letting $q \to \infty$, .

As a generalization of (5), Aziz and Ahemad [2] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for each $r > 0$, $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} \left| 1 + k e^{i\theta} \right|^{qr} d\theta \right\}^{\frac{1}{qr}} \left\{ \int_0^{2\pi} |P'(e^{i\theta})|^{pr} d\theta \right\}^{\frac{1}{pr}} \tag{6}$$

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$, then (see [9])

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R$ and $R > 0$.

As an extension of (2) to the polar derivative, Aziz and Rather [3] proved that if all the zeros of $P(z)$ lie in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$n(|\alpha| - k) \max_{|z|=1} |P(z)| \leq (1+k) \max_{|z|=1} |D_\alpha P(z)|. \tag{7}$$

For the class of lacunary type polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, of degree n having all their zeros in $|z| \leq k$ where $k \leq 1$, Aziz and Rather [4] also proved that if for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^{\mu}$,

$$n(|\alpha| - k^{\mu}) \max_{|z|=1} |P(z)| \leq (1 + k^{\mu}) \max_{|z|=1} |D_{\alpha} P(z)| . \tag{8}$$

As a refinement of inequality (8), and an extension of inequality (3) to polar derivative, Rather and Mir [12] proved that if $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^{\mu}$,

$$\max_{|z|=1} |D_{\alpha} P(z)| \geq \frac{n\,(|\alpha| - k^{\mu})}{1 + k^{\mu}} \max_{|z|=1} |P(z)| + \frac{n\,(|\alpha| + 1)}{k^{n-\mu}\,(1 + k^{\mu})} \min_{|z|=k} |P(z)|. \tag{9}$$

2 Main Results

In this paper, we first extend inequality (6) to the polar derivative and prove the following result.

THEOREM 1. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and for each $r > 0$, $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$\begin{aligned} &n(|\alpha| - k) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\beta m}{k^{n-1}} \right|^r d\theta \right\}^{\frac{1}{r}} \\ &\leq \left\{ \int_0^{2\pi} |1 + k e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} \left(\left| D_{\alpha} P(e^{i\theta}) \right| - \frac{mn}{k^{n-1}} \right)^{qr} d\theta \right\}^{\frac{1}{qr}} \end{aligned} \tag{10}$$

where $m = \min_{|z|=k} |P(z)|$.

REMARK 1. By letting $r \to \infty$ and choosing the argument of β in the left side of inequality (10) suitably, we obtain a result due to Aziz and Rather [3]. Instead of proving Thereon 1, we prove the following more general result which is also L_r mean extension of (9).

THEOREM 2. If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k^{\mu}$, $|\beta| \leq 1$ and for each $r > 0$, $p > 1, q > 1$ with $p^{-1} + q^{-1} = 1$,

$$\begin{aligned} &n(|\alpha| - k^{\mu}) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \frac{\beta m}{k^{n-\mu}} \right|^r d\theta \right\}^{\frac{1}{r}} \\ &\leq \left\{ \int_0^{2\pi} |1 + k^{\mu} e^{i\theta}|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} \left(\left| D_{\alpha} P(e^{i\theta}) \right| - \frac{mn}{k^{n-\mu}} \right)^{qr} d\theta \right\}^{\frac{1}{qr}} \end{aligned} \tag{11}$$

where $m = \min_{|z|=k} |P(z)|$.

If we let $q \to \infty$, in (11) so that $p \to 1$, we obtain the following result.

COROLLARY 1. If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k^\mu$, $|\beta| \leq 1$ and for each $r > 0$,

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \frac{\beta m}{k^{n-\mu}}\right|^r d\theta\right\}^{\frac{1}{r}}$$
$$\leq \left\{\int_0^{2\pi}|1 + k^\mu e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}\left\{\max_{|z|=1}\left|D_\alpha P(z)\right| - \frac{mn}{k^{n-\mu}}\right\} \tag{12}$$

where $m = \min_{|z|=k} |P(z)|$.

REMARK 2. Again, letting $r \to \infty$ and choosing the argument of β in the left side of inequality (12) suitably, we obtain inequality (9).

For the proof of Theorem 2, we need the following Lemma.

3 Lemma

The following Lemma holds due to N. A. Rather [11].

LEMMA 1. If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree almost n having all its zeros in in $|z| \leq k$ where $k \leq 1$, then for $|z| = 1$,

$$|Q'(z)| + \frac{nm}{k^{n-\mu}} \leq k^\mu |P'(z)| \tag{13}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$ and $m = \min_{|z|=k}|P(z)|$.

4 Proof of Theorem 2

In this section, we prove Theorem 2.

Let $Q(z) = z^n \overline{P(1/\overline{z})}$, then $P(z) = z^n \overline{Q(1/\overline{z})}$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \text{ and } |P'(z)| = |nQ(z) - zQ'(z)|. \tag{14}$$

By Lemma 1, we have for every β with $|\beta| \leq 1$ and $|z| = 1$,

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-\mu}}\right| \leq |Q'(z)| + \frac{nm}{k^{n-\mu}} \leq k^\mu |P'(z)|. \tag{15}$$

Using (14) in (15), we get for $|z| = 1$,

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-\mu}}\right| \leq k^\mu |nQ(z) - zQ'(z)|. \tag{16}$$

Again, by Lemma 1 for every real or complex number α with $|\alpha| \geq k$ and $|z| = 1$, we have

$$|D_\alpha P(z)| \geq |\alpha|\,|P'(z)| - |Q'(z)| \geq (|\alpha| - k^\mu)\,|P'(z)| + \frac{mn}{k^{n-\mu}},$$

so that

$$|D_\alpha P(z)| - \frac{mn}{k^{n-\mu}} \geq (|\alpha| - k^\mu)|P'(z)|. \tag{17}$$

Since $P(z)$ has all its zeros in $|z| \leq k \leq 1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z| \leq k \leq 1$. This implies that the polynomial

$$z^{n-1}\overline{P'(1/\overline{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (16) that the function

$$w(z) = \frac{z\left(Q'(z) + \bar{\beta}\dfrac{nmz^{n-1}}{k^{n-\mu}}\right)}{k^\mu\left(nQ(z) - zQ'(z)\right)}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + k^\mu w(z)$ is subordinate to the function $1 + k^\mu z$ for $|z| \leq 1$. Hence by a well known property of subordination [6], we have

$$\int_0^{2\pi} \left|1 + k^\mu w(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^r d\theta,\ r > 0. \tag{18}$$

Now

$$1 + k^\mu w(z) = \frac{n\left(Q(z) + \bar{\beta}\dfrac{mz^n}{k^{n-\mu}}\right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1}\overline{P'(1/\overline{z})}| = |nQ(z) - zQ'(z)| \text{ for } |z| = 1,$$

therefore for $|z| = 1$,

$$n\left|Q(z) + \bar{\beta}\frac{mz^n}{k^{n-\mu}}\right| = |1 + k^\mu w(z)|\,|nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|.$$

Equivalently,

$$n\left|z^n\overline{P(1/\overline{z})} + \bar{\beta}\frac{mz^n}{k^{n-\mu}}\right| = |1 + k^\mu w(z)||P'(z)|.$$

This implies

$$n\left|P(z) + \beta\frac{m}{k^{n-\mu}}\right| = |1 + k^\mu w(z)||P'(z)| \text{ for } |z| = 1. \tag{19}$$

From (17) and (19), we deduce that for $r > 0$,

$$n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-\mu}}\right|^r d\theta \leq \int_0^{2\pi} \left|1 + k^\mu w(e^{i\theta})\right|^r \left(\left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-\mu}}\right)^r d\theta.$$

This gives with the help of Hölder's inequality and (18), for $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$,

$$n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} \left|P(e^{i\theta}) + \beta \frac{m}{k^{n-\mu}}\right|^r d\theta$$
$$\leq \left(\int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^{pr} d\theta \right)^{1/p} \left(\int_0^{2\pi} \left\{ \left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-\mu}} \right\}^{qr} d\theta \right)^{1/q},$$

equivalently,

$$n(|\alpha| - k^\mu) \left\{ \int_0^{2\pi} \left|P(e^{i\theta}) + \beta \frac{m}{k^{n-\mu}}\right|^r d\theta \right\}^{\frac{1}{r}}$$
$$\leq \left\{ \int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^{pr} d\theta \right\}^{\frac{1}{pr}} \left\{ \int_0^{2\pi} \left(\left|D_\alpha P(e^{i\theta})\right| - \frac{mn}{k^{n-\mu}} \right)^{qr} d\theta \right\}^{\frac{1}{qr}}$$

which proves the desired result.

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