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Well-Posedness Results For A Third Boundary Value Problem For The Heat Equation In A Disc*

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Abstract

In this work we prove well-posedness results for the following one space linear second order parabolic equation $\partial_t u - \partial_x^2 u = f$, set in a domain

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : -r < t < r; \varphi_1(t) < x < \varphi_2(t)\right\}$$

of $\mathbb{R}^2$, where $\varphi_i(t) = (-1)^i \left(r^2 - t^2\right)^{\frac{1}{2}}$, $i = 1, 2$ and with lateral boundary conditions of Robin type. The right-hand side f of the equation is taken in $L^2(\Omega)$. The method used is based on the approximation of the domain Ω by a sequence of subdomains $(\Omega_n)_n$ which can be transformed into regular domains.

1 Introduction

Let $\Omega = D(0,r)$ be the open disc centred at the origin of $\mathbb{R}^2$ and with radius $r > 0$, characterized by $\Omega = \left\{(t,x) \in \mathbb{R}^2 : -r < t < r; \varphi_1(t) < x < \varphi_2(t)\right\}$, where φ_1 and φ_2 are defined on $[-r,r]$ by $\varphi_k(t) = (-1)^k \left(r^2 - t^2\right)^{\frac{1}{2}}$, $k = 1,2$. The lateral boundary of Ω is defined by $\Gamma_k = \left\{(t, \varphi_k(t)) \in \mathbb{R}^2 : -r < t < r\right\}$, $k = 1, 2$. In Ω, we consider the Robin type boundary value problem

$$\begin{cases} \partial_t u - \partial_x^2 u = f \text{ in } \Omega, \\ \partial_x u + \beta_k u|_{\Gamma_k} = 0, \ k = 1, 2, \end{cases} \tag{1}$$

where the coefficients β_k, $k = 1$,2 are real numbers satisfying non-degeneracy assumptions (to be made more precise later) and the right-hand side term f of the equation lies in $L^2(\Omega)$, the space of square-integrable functions on Ω with the measure $dtdx$.

The main difficulty related to this kind of problems is due to the fact that φ_1 coincides with φ_2 for $t = -r$ and for $t = r$, which prevents the domain Ω to be transformed into a regular domain by means of a smooth transformation.

The case $\beta_k = \infty$, $k = 1, 2$, corresponding to Dirichlet boundary conditions is considered in [19]. We can find in [6] a study of the case $\beta_k = 0$, $k = 1, 2$, corresponding to Neumann boundary conditions and in [23] an abstract study in the case

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$(\beta_1, \beta_2) = (\infty, 0)$, corresponding to mixed (Dirichlet-Neumann) lateral boundary conditions. However, the boundary assumptions dealt with by the authors exclude our domain. Further references on the analysis of parabolic problems in non-cylindrical domains are: Labbas et al. [13, 14, 15], Kheloufi et al. [8, 9, 10, 12], Degtyarev [5], Aref'ev and Bagirov [3, 4], Sadallah [20, 21, 22], Alkhutov [1, 2] and Paronetto [17].

In this work, we consider the case of Robin type boundary condition, namely, the case where $\beta_k \neq 0$, $k = 1,2$, and we look for sufficient conditions (as weak as possible) on the lateral boundary of the domain and on the coefficients β_k, $k = 1$, 2 in order to obtain the maximal regularity of the solution in an anisotropic Hilbertian Sobolev space.

In previous works (see [7, 11]), we have studied the case where

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T; \psi_1(t) < x < \psi_2(t)\right\}$$

with the fundamental hypothesis $\psi_1(0) = \psi_2(0)$ and we have proved that the solution u of Problem (1) is unique and has the optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_\gamma^{1,2}(\Omega) := \left\{u \in H^{1,2}(\Omega) : \partial_x u + \beta_k u|_{\Gamma_k} = 0,\ k = 1{,}2\right\}$$

with

$$H^{1,2}(\Omega) = \left\{u \in L^2(\Omega) : \partial_t u,\ \partial_x u,\ \partial_x^2 u \in L^2(\Omega)\right\},$$

under sufficient conditions on ψ_k, $k = 1, 2$, that are

$$\psi_k'(t)\left(\psi_2(t) - \psi_1(t)\right) \quad \longrightarrow \quad 0 \quad \text{as } t \longrightarrow 0, \quad k = 1, 2.$$

Examples of functions satisfying this last condition are $\psi_k(t) = (-1)^k \left(r^2 - t^2\right)^{\frac{1}{2+\epsilon}}$, $k = 1, 2$ for all $\epsilon < 0$. However, the above condition is false in the case $\epsilon = 0$ corresponding to the class of domains considered in this article. So, the well-posedeness result which we will prove here can not be derived from [7] and [11]. In order to overcome this difficulty, we impose sufficient conditions on the lateral boundary of the domain and on the coefficients β_k, $k = 1$, 2, that are,

$$\beta_1 < 0,\ \beta_2 > 0, \tag{2}$$

$$(-1)^k \left(\beta_k - \frac{t}{2\sqrt{r^2 - t^2}}\right) \geq 0 \text{ a.e. } t \in \left]-r, r\right[,\ k = 1, 2, \tag{3}$$

and

$$1 - \left[(16 + 4\beta_1^2 + 4\beta_2^2)r + (4\left|\beta_1\right| + 4\left|\beta_2\right|)r^2 + (8 + 4\beta_1^2 + 4\beta_2^2)r^3\right] > 0. \tag{4}$$

Then, our main result is following:

THEOREM 1. Under the hypothesis (2), (3) and (4), the heat operator $L = \partial_t - \partial_x^2$ is an isomorphism from $H_\gamma^{1,2}(\Omega)$ into $L^2(\Omega)$.

It is not difficult to prove the injectivity of the operator L. Indeed, If u is a solution of Problem (1) with a null right-hand side, the calculations show that the inner product $\langle Lu, u\rangle$ in $L^2(\Omega)$ gives

$$0 = \sum_{k=1}^{2} \int_{\Gamma_k} (-1)^k \left(\beta_k - \frac{t}{2\sqrt{r^2 - t^2}}\right) u^2 (t, \varphi_k(t))\, dt + \int_{\Omega} (\partial_x u)^2\, dtdx.$$

The hypothesis (3) implies that $\partial_x u = 0$ and consequently $\partial_x^2 u = 0$. Then, the equation of Problem (1) gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions and the fact that $\beta_k \neq 0$, $k = 1, 2$ imply that $u = 0$ in Ω. So, in the sequel, we will be interested only by the question of the surjectivity of the operator L.

The method used here is the domain decomposition method. More precisely, we divide Ω into two parts

$$\Omega_1 = \{(t, x) \in \Omega : -r < t < 0\} \text{ and } \Omega_2 = \{(t, x) \in \Omega : 0 < t < r\}.$$

So, we obtain two solutions $u_k \in H^{1,2}(\Omega_k)$ in Ω_k, $k = 1, 2$. Finally, we prove that the function u defined by

$$u := \begin{cases} u_1 \text{ in } \Omega_1, \\ u_2 \text{ in } \Omega_2, \end{cases}$$

is the solution of problem (1) and has the optimal regularity, that is $u \in H^{1,2}(\Omega)$. The plan of this paper is as follows. In Section 2, we prove that Problem (1) admits a (unique) solution in the case of a "truncated" domain. Then, in Section 3, we approximate Ω by a sequence (Ω_n) of such truncated domains and we establish an energy estimate which will allow us to pass to the limit and complete the proof of our main result.

2 Resolution of Problem (1) in a Truncated Disc Ω_n

For each $n \in \mathbb{N}^*$, we define

$$\Omega_n := \left\{(t, x) \in \mathbb{R}^2 : -r < t < r - \frac{1}{n}; \varphi_1(t) < x < \varphi_2(t)\right\}.$$

THEOREM 2. Assume that β_k and φ_k, $k = 1, 2$ verify assumptions (2) and (3) and let $f_n = f|_{\Omega_n}$ and

$$\Gamma_{n,k} = \left\{(t, \varphi_k(t)) \in \mathbb{R}^2 : -r < t < r - \frac{1}{n}\right\} \text{ for } k = 1,\ 2.$$

Then, for each $n \in \mathbb{N}^*$, the problem

$$\begin{cases} \partial_t u_n - \partial_x^2 u_n = f_n \in L^2(\Omega_n), \\ \partial_x u_n + \beta_k u_n|_{\Gamma_{n,k}} = 0,\ k = 1,\ 2, \end{cases} \tag{5}$$

admits a (unique) solution $u_n \in H^{1,2}(\Omega_n)$.

PROOF. We divide Ω_n, $n \in \mathbb{N}^*$ into two parts

$$\Omega^- = \{(t,x) \in \Omega : -r < t < 0\} \text{ and } \Omega_n^+ = \left\{(t,x) \in \Omega : 0 < t < r - \frac{1}{n}\right\}.$$

So, we have $\Omega_n = \Omega^- \cup \Omega_n^+ \cup (\{0\} \times]\varphi_1(0), \varphi_2(0)[)$.

LEMMA 1. Let $f^- = f|_{\Omega^-}$ and

$$\Gamma_k^- = \{(t, \varphi_k(t)) \in \mathbb{R}^2 : -r < t < 0\} \text{ for } k = 1,\ 2.$$

Then, the problem

$$\begin{cases} \partial_t u^- - \partial_x^2 u^- = f^- \in L^2(\Omega^-), \\ \partial_x u^- + \beta_k u^-|_{\Gamma_k^-} = 0,\ k = 1,2, \end{cases}$$

admits a (unique) solution $u^- \in H^{1,2}(\Omega^-)$.

PROOF. Since φ_1 is a decreasing function on $]-r, 0[$ and φ_2 is an increasing function on $]-r, 0[$, then the result follows from [18].

Hereafter, we denote the trace $u^-|_{\{0\}\times]\varphi_1(0),\varphi_2(0)[}$ by ψ, which is in the Sobolev space $H^1(\{0\} \times]\varphi_1(0), \varphi_2(0)[)$ because $u^- \in H^{1,2}(\Omega^-)$ (see [16]). Now, consider the following problem on Ω_n^+, $n \in \mathbb{N}^*$

$$\begin{cases} \partial_t u_n^+ - \partial_x^2 u_n^+ = f_n^+ \in L^2(\Omega_n^+), \\ u_n^+|_{\{0\}\times]\varphi_1(0),\varphi_2(0)[} = \psi \in H^1(\{0\} \times]\varphi_1(0), \varphi_2(0)[), \\ \partial_x u_n^+ + \beta_k u_n^+|_{\Gamma_{n,k}^+} = 0,\ k = 1,\ 2, \end{cases} \tag{6}$$

where $\Gamma_{n,k}^+ = \left\{(t, \varphi_k(t)) \in \mathbb{R}^2 : 0 < t < r - \frac{1}{n}\right\}$, $k = 1,\ 2$.

We use the following result, which is a consequence of Theorem 4.3 in [16] to solve Problem (6).

PROPOSITION 1. Let Q be the rectangle $]0, T[\times]0, 1[$, $f \in L^2(Q)$ and $\psi \in H^1(\gamma_0)$ with $\gamma_0 = \{0\} \times]0, 1[$. Then, the problem

$$\begin{cases} \partial_t u - \partial_x^2 u = f \in L^2(Q), \\ u|_{\gamma_0} = \psi, \\ \partial_x u + \beta_k u|_{\gamma_k} = 0,\ k = 1,\ 2, \end{cases}$$

where $\gamma_1 =]0, T[\times \{0\}$ and $\gamma_2 =]0, T[\times \{1\}$ admits a (unique) solution $u \in H^{1,2}(Q)$.

REMARK 1. We have ψ lies in $H^1(\{0\} \times]\varphi_1(0), \varphi_2(0)[)$, then $\partial_x \psi$ is (only) in $L^2(\{0\} \times]\varphi_1(0), \varphi_2(0)[)$ and its pointwise values should not make sense. So in the application of [[16] Theorem 4.3, Vol. 2], there are no compatibility conditions to satisfy.

Thanks to the transformation $(t,x) \mapsto (t,y) = (t, (\varphi_2(t) - \varphi_1(t))x + \varphi_1(t))$, we deduce the following result:

PROPOSITION 2. For each $n \in \mathbb{N}^*$, Problem (6) admits a unique solution $u_n^+ \in H^{1,2}(\Omega_n^+)$.

So, the function $u_n \in H^{1,2}(\Omega_n)$, $n \in \mathbb{N}^*$ defined by

$$u_n := \begin{cases} u^- \text{ in } \Omega^-, \\ u_n^+ \text{ in } \Omega_n^+, \end{cases}$$

is the (unique) solution of Problem (5). This completes the proof of Theorem 2.

3 Resolution of Problem (1) in the Half Disc Ω^+

In this section, we define

$$\Omega^+ := \left\{(t,x) \in \mathbb{R}^2 : 0 < t < r; \varphi_1(t) < x < \varphi_2(t)\right\}$$

and consider the following problem in Ω^+

$$\begin{cases} \partial_t u^+ - \partial_x^2 u^+ = f^{\ +} \in L^2(\Omega^+), \\ u^+|_{\{0\}\times]\varphi_1(0),\varphi_2(0)[} = 0, \\ \partial_x u^+ + \beta_k u^+|_{\Gamma_k^+} = 0,\ k = 1,\ 2, \end{cases} \tag{7}$$

where $f^+ = f|_{\Omega^+}$ and

$$\Gamma_k^+ = \left\{(t, \varphi_k(t)) \in \mathbb{R}^2 : 0 < t < r\right\} \text{ for } k = 1,\ 2.$$

We assume that β_k and φ_k, $k = 1, 2$ verify assumptions (2), (3) and (4) and we denote $f_n^+ = f^+|_{\Omega_n^+}$ and $u_n^+ \in H^{1,2}(\Omega_n^+)$ the solution of Problem (7) in Ω_n^+. Such a solution exists by Proposition 2.

PROPOSITION 3. There exists a constant $K > 0$ independent of n such that

$$\left\|u_n^+\right\|_{H^{1,2}\left(\Omega_n^+\right)} \leq K \left\|f_n^+\right\|_{L^2\left(\Omega_n^+\right)} \leq K \left\|f^+\right\|_{L^2(\Omega^+)},$$

where

$$\left\|u_n^+\right\|_{H^{1,2}\left(\Omega_n^+\right)} = \sqrt{\left\|u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} + \left\|\partial_t u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} + \left\|\partial_x u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} + \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}}.$$

In order to prove Proposition 3, we need the following result

LEMMA 2. We have the following estimations

(i) $|\varphi_k'(t)|(\varphi_2(t) - \varphi_1(t)) \leq 2r$ for $t \in]-r, r[$ and $k = 1,\ 2$.

(ii) $\int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_x^j u_n^+(s,x)]^2 ds \leq [\varphi_2(t) - \varphi_1(t)]^2 \int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_x^{j+1} u_n^+(s,x)]^2 ds$ for $j = 0, 1$.

(iii) $\left\|\partial_x u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} \leq 4r^2 \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}$.

PROOF OF PROPOSITION 3. We have

$$\begin{aligned}\left\|f_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} &= \left\langle \partial_t u_n^+ - \partial_x^2 u_n^+, \partial_t u_n^+ - \partial_x^2 u_n^+ \right\rangle \\ &= \left\|\partial_t u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} + \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} - 2\int_{\Omega_n^+} \partial_t u_n^+ .\partial_x^2 u_n^+ dtdx.\end{aligned}$$

Let us consider the term $-2\int_{\Omega_n^+} \partial_t u_n^+ .\partial_x^2 u_n^+ dtdx$. We have

$$\partial_t u_n^+ .\partial_x^2 u_n^+ = \partial_x \left(\partial_t u_n^+ .\partial_x u_n^+\right) - \frac{1}{2}\partial_t \left(\partial_x u_n^+\right)^2.$$

Then

$$\begin{aligned}-2\int_{\Omega_n^+} \partial_t u_n^+ .\partial_x^2 u_n^+ dtdx &= -2\int_{\Omega_n^+} \partial_x \left(\partial_t u_n^+ \partial_x u_n^+\right) dt\ dx + \int_{\Omega_n^+} \partial_t \left(\partial_x u_n^+\right)^2 dt\ dx \\ &= \int_{\partial\Omega_n^+} \left[\left(\partial_x u_n^+\right)^2 \nu_t - 2\partial_t u_n^+ \partial_x u_n^+ \nu_x\right] d\sigma,\end{aligned}$$

with ν_t, ν_x are the components of the unit outward normal vector at $\partial\Omega_n^+$. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Ω_n^+ where $t = 0$, we have $u_n^+ = 0$ and consequently $\partial_x u_n^+ = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = r - \frac{1}{n}$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral $\int_{\varphi_1\left(r-\frac{1}{n}\right)}^{\varphi_2\left(r-\frac{1}{n}\right)} \left(\partial_x u_n^+\right)^2\ dx$ is nonnegative. On the parts of the boundary where $x = \varphi_k(t)$, $k = 1, 2$, we have

$$\nu_x = \frac{(-1)^k}{\sqrt{1 + \left(\varphi_k'\right)^2(t)}},\ \ \nu_t = \frac{(-1)^{k+1}\varphi_k'(t)}{\sqrt{1 + \left(\varphi_k'\right)^2(t)}} \text{ and } \partial_x u_n^+\left(t, \varphi_k(t)\right) + \beta_k u_n^+\left(t, \varphi_k(t)\right) = 0.$$

Consequently, the corresponding boundary integrals $I_{n,k}$ and $J_{n,k}$, $k = 1, 2$ are the following:

$$\begin{aligned}I_{n,k} &= (-1)^{k+1}\int_0^{r-\frac{1}{n}} \varphi_k'(t)\left[\partial_x u_n^+\left(t, \varphi_k(t)\right)\right]^2 dt,\ k = 1, 2, \\ J_{n,k} &= (-1)^k\, 2\int_0^{r-\frac{1}{n}} \beta_k \partial_t u_n^+\left(t, \varphi_k(t)\right).u_n^+\left(t, \varphi_k(t)\right) dt,\ k = 1, 2.\end{aligned}$$

We have

$$-2\int_{\Omega_n^+} \partial_t u_n^+ .\partial_x^2 u_n^+ dtdx \geq -\left|I_{n,1}\right| - \left|I_{n,2}\right| - \left|J_{n,1}\right| - \left|J_{n,2}\right|. \tag{8}$$

It is the reason for which we look for an estimate of the type

$$\left|I_{n,1}\right| + \left|I_{n,2}\right| + \left|J_{n,1}\right| + \left|J_{n,2}\right| \leq \delta \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)},$$

where δ is a positive constant independent of n belonging to the interval $]0,1[$. By introducing the function $\phi(t,x) = \frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}$ like in [18], we write for $I_{n,1}$

$$\begin{aligned}
|I_{n,1}| &= \left| \int_0^{r-\frac{1}{n}} \left\{ \int_{\varphi_1(t)}^{\varphi_2(t)} \varphi_1'(t)\, \partial_x \left(\phi(t,x) \left[\partial_x u_n^+(t,x)\right]^2 \right) dx \right\} dt \right| \\
&\leq \int_{\Omega_n^+} \varphi_1'(t) \left(\varphi_2(t) - \varphi_1(t)\right) \left[\partial_x^2 u_n^+\right]^2 dtdx + 2\int_{\Omega_n^+} |\varphi_1'| \left|\partial_x u_n^+\right| \left|\partial_x^2 u_n^+\right| dtdx \\
&\leq 2r \left\|\partial_x^2 u_n^+\right\|^2 + \epsilon \left\|\partial_x^2 u_n^+\right\|^2 + \frac{1}{\epsilon} \int_{\Omega_n^+} |\varphi_1'|^2 \left[\partial_x u_n^+\right]^2 dtdx \\
&\leq \left[2r + \epsilon + \frac{4r^2}{\epsilon}\right] \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} \\
&\leq 7r \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}.
\end{aligned}$$

The last inequality is obtained by choosing $\epsilon = r$. Similarly, we have

$$|I_{n,2}| \leq 7r \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}.$$

Let us now consider the terms $J_{n,k}$, $k = 1, 2$. By setting $h(t) = \left(u_n^+\right)^2 (t, \varphi_k(t))$, we obtain

$$\begin{aligned}
J_{n,k} &= (-1)^k \int_0^{r-\frac{1}{n}} \beta_k. \left[h'(t) - \varphi_k'(t)\, \partial_x \left(u_n^+\right)^2 (t, \varphi_k(t))\right] dt \\
&= (-1)^k \beta_k.h(t)\Big|_0^{r-\frac{1}{n}} + (-1)^{k+1} \int_0^{r-\frac{1}{n}} \beta_k.\varphi_k'(t)\, \partial_x \left(u_n^+\right)^2 (t, \varphi_k(t))\, dt.
\end{aligned}$$

Condition (2) and the fact that $\left(u_n^+\right)^2 (0, \varphi_k(0)) = 0$ give $(-1)^k \beta_k.h(t)\Big|_0^{r-\frac{1}{n}} \geq 0$. In the sequel, we estimate the last boundary integral in the expression of $J_{n,k}$, namely

$$L_{n,k} = (-1)^{k+1} \int_0^{r-\frac{1}{n}} \beta_k.\varphi_k'(t)\, \partial_x \left(u_n^+\right)^2 (t, \varphi_k(t))\, dt.$$

We have

$$\begin{aligned}
& \partial_x \left(u_n^+\right)^2 (t, \varphi_1(t)) \\
=& -\frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} \partial_x \left(u_n^+\right)^2 (t,x) \Big|_{x=\varphi_1(t)}^{x=\varphi_2(t)} \\
=& -\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x \left\{ \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} \partial_x \left(u_n^+\right)^2 (t,x) \right\} dx \\
=& \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\frac{1}{\varphi_2(t) - \varphi_1(t)} \partial_x \left(u_n^+\right)^2 (t,x) - \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} \partial_x^2 \left(u_n^+\right)^2 (t,x) \right] dx.
\end{aligned}$$

So,

$$L_{n,1} = \int_{\Omega_n^+} \left[\frac{\beta_1 . \varphi_1'(t)}{\varphi_2(t) - \varphi_1(t)} \partial_x \left(u_n^+\right)^2 (t,x) - \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} \beta_1 . \varphi_1'(t) \partial_x^2 \left(u_n^+\right)^2 (t,x) \right] dtdx.$$

By using the equalities

$$\partial_x \left(u_n^+\right)^2 (t,x) = 2\partial_x u_n^+(t,x)\, u_n^+(t,x)$$

and

$$\partial_x^2 \left(u_n^+\right)^2 (t,x) = 2\partial_x^2 u_n^+(t,x)\, u_n^+(t,x) + 2\left[\partial_x u_n^+(t,x)\right]^2,$$

we obtain

$$\begin{aligned} L_{n,1} &= \int_{\Omega_n^+} \frac{2\beta_1 . \varphi_1'(t)}{\varphi_2(t) - \varphi_1(t)} \partial_x u_n^+(t,x)\, u_n^+(t,x)\, dtdx \\ &\quad - \int_{\Omega_n^+} \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} 2\beta_1 . \varphi_1'(t)\, \partial_x^2 u_n^+(t,x)\, u_n^+(t,x)\, dtdx \\ &\quad - \int_{\Omega_n^+} \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} 2\beta_1 . \varphi_1'(t) \left[\partial_x u_n^+(t,x)\right]^2 dtdx \\ &= A_{n,1} + B_{n,1} + C_{n,1}. \end{aligned}$$

Estimation of $A_{n,1}$, $B_{n,1}$ and $C_{n,1}$

a) We have

$$A_{n,1} = \int_{\Omega_n^+} \frac{2\beta_1 . \varphi_1'(t)}{\varphi_2(t) - \varphi_1(t)} \partial_x u_n^+(t,x)\, u_n^+(t,x)\, dtdx,$$

then

$$\begin{aligned} |A_{n,1}| &\leq \int_{\Omega_n^+} \frac{1}{\epsilon} \left[\beta_1 . \varphi_1'(t)\right]^2 \left[\partial_x u_n^+(t,x)\right]^2 dtdx \\ &\quad + \epsilon \int_{\Omega_n^+} \frac{1}{\left[\varphi_2(t) - \varphi_1(t)\right]^2} \left[u_n^+(t,x)\right]^2 dtdx \\ &\leq \int_{\Omega_n^+} \frac{1}{\epsilon} \left[\beta_1 . \varphi_1'(t)\right]^2 \left[\varphi_2(t) - \varphi_1(t)\right]^2 \left[\partial_x^2 u_n^+(t,x)\right]^2 dtdx \\ &\quad + \epsilon \int_{\Omega_n^+} \left[\varphi_2(t) - \varphi_1(t)\right]^2 \left[\partial_x^2 u_n^+(t,x)\right]^2 dtdx \\ &\leq \left[\frac{\beta_1^2}{\epsilon} 4r^2 + 4r^2\epsilon\right] \left\|\partial_x^2 u_n^+\right\|_{L^2(\Omega_n^+)}^2 \leq \left[4r^3 + 4\beta_1^2 r\right] \left\|\partial_x^2 u_n^+\right\|_{L^2(\Omega_n^+)}^2. \end{aligned}$$

The last inequality is obtained by choosing $\epsilon = r$.

b) We have

$$B_{n,1} = -\int_{\Omega_n^+} \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)} 2\beta_1 . \varphi_1'(t)\, \partial_x^2 u_n^+(t,x)\, u_n^+(t,x)\, dtdx,$$

then

$$\begin{aligned}|B_{n,1}| &\leq \frac{\beta_1^2}{\epsilon}\int_{\Omega_n^+}|\varphi_1'(t)|^2\left[u_n^+(t,x)\right]^2 dtdx+\epsilon\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}\\ &\leq \frac{\beta_1^2}{\epsilon}\sup_{t\in[0,r]}\left(|\varphi_1'(t)|^2\left[\varphi_2(t)-\varphi_1(t)\right]^4\right)\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}\\ &\quad +\epsilon\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}\\ &\leq \left(\frac{4\beta_1^2 r^4}{\epsilon}+\epsilon\right)\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}\leq\left(4\beta_1^2 r^3+r\right)\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}.\end{aligned}$$

The last inequality is obtained by choosing $\epsilon = r$.

c) We have

$$C_{n,1}=-\int_{\Omega_n^+}\frac{\varphi_2(t)-x}{\varphi_2(t)-\varphi_1(t)}2\beta_1.\varphi_1'(t)\left[\partial_x u_n^+(t,x)\right]^2 dtdx$$

then

$$\begin{aligned}|C_{n,1}| &\leq 2|\beta_1|\int_{\Omega_n^+}|\varphi_1'(t)|\,|\varphi_2(t)-\varphi_1(t)|^2\left[\partial_x^2 u_n^+(t,x)\right]^2 dtdx\\ &\leq 4|\beta_1|\,r^2\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}.\end{aligned}$$

Consequently,

$$|L_{n,1}|\leq\left[(4+4\beta_1^2)r^3+4|\beta_1|\,r^2+(1+4\beta_1^2)r\right]\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}.$$

Similarly, we can obtain

$$|L_{n,2}|\leq\left[(4+4\beta_2^2)r^3+4|\beta_2|\,r^2+(1+4\beta_2^2)r\right]\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}.$$

Summing up the above estimates, we obtain

$$\begin{aligned}\left\|f_n^+\right\|^2_{L^2\left(\Omega_n^+\right)} &\geq \left\|\partial_t u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}+\left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}-|I_{n,1}|-|I_{n,2}|-|L_{n,1}|-|L_{n,2}|\\ &\geq \left\|\partial_x^2 u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}\Big\{1-\Big[(16+4\beta_1^2+4\beta_2^2)r+(4|\beta_1|+4|\beta_2|)\,r^2\\ &\quad +(8+4\beta_1^2+4\beta_2^2)r^3\Big]\Big\}+\left\|\partial_t u_n^+\right\|^2_{L^2\left(\Omega_n^+\right)}.\end{aligned}$$

Using the condition (4) and since $\|f_n^+\|^2_{L^2\left(\Omega_n^+\right)}\leq\|f^+\|^2_{L^2(\Omega^+)}$, then Proposition 3 is proved.

THEOREM 3. Problem (7) admits a (unique) solution $u^+\in H^{1,2}(\Omega^+)$.

PROOF. The estimation of Proposition 3 shows that

$$\left\|\widetilde{u_n^+}\right\|_{L^2(\Omega^+)} + \left\|\widetilde{\partial_t u_n^+}\right\|_{L^2(\Omega^+)} + \sum_{i=1}^{2} \left\|\widetilde{\partial_x^i u_n^+}\right\|_{L^2(\Omega^+)} \leq C \left\|f^+\right\|_{L^2(\Omega^+)},$$

where $\widetilde{.}$ denotes the $0-$extension of u_n^+ to Ω^+. This means that $\widetilde{u_n^+}$, $\widetilde{\partial_t u_n^+}$, $\widetilde{\partial_x^i u_n^+}$, $i = 1, 2$ are bounded functions in $L^2(\Omega^+)$. The following compactness result is well known: A bounded sequence in a reflexive Banach space (and in particular in a Hilbert space) is weakly convergent. So for a suitable increasing sequence of integers n_k, $k = 1, 2, ...$, there exists functions $u^+, v^+, v_i^+, i = 1, 2$ in $L^2(\Omega^+)$ such that

$$\widetilde{u_{n_k}^+} \rightharpoonup u^+, \widetilde{\partial_t u_{n_k}^+} \rightharpoonup v^+, \widetilde{\partial_x^i u_{n_k}^+} \rightharpoonup v_i^+, i = 1, 2$$

weakly in $L^2(\Omega^+)$ as $k \to \infty$. Clearly, $v^+ = \partial_t u^+, v_i^+ = \partial_x^i u^+, i = 1, 2$ in the sense of distributions in Ω^+ and so in $L^2(\Omega^+)$. Finally, $u^+ \in H^{1,2}(\Omega^+)$ and

$$\partial_t u^+ - \partial_x^2 u^+ = f \text{ in } \Omega^+.$$

On the other hand, the solution u^+ satisfies the boundary conditions, since

$$\forall n \in \mathbb{N}^*, \quad u^+\big|_{\Omega_n^+} = u_n^+.$$

REMARK 2. The function $u \in H^{1,2}(\Omega)$ defined by

$$u := \begin{cases} u^- \text{ in } \Omega^-, \\ u^+ \text{ in } \Omega^+, \end{cases}$$

is the (unique) solution of Problem (1).

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