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A Note On Strongly Quotient Graphs With Harary Energy And Harary Estrada Index*

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Abstract

The main purpose of this paper is to investigate the upper and lower bounds on Harary energy and Harary Estrada index on Strongly Quotient Graphs (SQG). The notion of SQG was introduced by Adiga et al. in 2007. In addition, we have established some relations between the Harary Estrada index and the Harary energy of SQG.

1 Introduction

Let $G(V,E)$ be a finite undirected simple connected (n,m) graph with vertex set $V = V(G)$, edge set $E = E(G)$, $n = |V|$ and $m = |E|$. The vertices of G are labeled by $v_1, v_2, ..., v_n$. The Harary matrix $H(G)$ of a graph G is defined as a square matrix of order n such as $H(G) = H = \left[\frac{1}{d_{ij}}\right]$, where d_{ij} is the distance (i.e. the length of the shortest path) between the vertices v_i and v_j in G in [4, 13]. The eigenvalues of the Harary matrix $H(G)$ are denoted as $\rho_1, \rho_2, \ldots, \rho_n$ and are said to be the H-eigenvalues of G. For more details on H-eigenvalues, especially on the maximum eigenvalue of Harary matrix of a graph G and spectral properties of G refer to [7, 8, 15]. It can be noted that the H-eigenvalues of G are real since the Harary matrix is symmetric. The Harary energy of the graph G, denoted by $HE(G)$, is defined as

$$HE(G) = \sum_{i=1}^{n} |\rho_i|.$$

Recently, there has been tremendous research activity in graph energy, as Harary energy inevitably arouses the interest of chemists. Some lower and upper bounds for Harary energy of connected (n,m)-graphs were obtained in [12]. The diameter of the graph G is the maximum distance between any two vertices of G, denoted by $diam(G)$.

The name Estrada index of the graph G was introduced in [9] which has an important role in Chemistry and Physics and there exists a vast literature that studies

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this special index. Recently, the bounds of distance Estrada index and Harary Estrada index of a graph G are concerned in [11, 12]. The Harary Estrada index is defined as

$$HEE(G) = \sum_{i=1}^{n} e^{\rho_i}.$$

During the past forty years or so, an enormous amount of research work has been done on graph labeling, where the vertices are assigned values subject to certain conditions. These interesting problems have been motivated by practical problems. Recently, Adiga et al., have introduced the notion of strongly quotient graphs and studied these type of graphs [2]. They derived an explicit formula for the maximum number of edges in a strongly quotient graph of order n. In [1], Adiga and Zaferani have established that the clique number and chromatic number are both equal to $1+\pi(n)$, where $\pi(n)$ is the number of primes not exceeding n. Throughout this paper by a labeling f of a graph G of order n, we mean an injective mapping

$$f : V(G) \to \{1, 2, \ldots, n\}.$$

We define the quotient function $f_q : E(G) \to Q$ by

$$f_q(e) = min\left\{\frac{f(v)}{f(w)}, \frac{f(w)}{f(v)}\right\} \text{ if } e \text{ joins } v \text{ and } w.$$

Note that for any $e \in E(G)$, $0 < f_q(e) < 1$.

A graph with n vertices is called a strongly quotient graph if its vertices can be labeled by $1, 2, \ldots, n$ such that the quotient function f_q is injective i.e., the values $f_q(e)$ on the edges are all distinct. For survey and detailed information of graph labeling, strongly quotient graphs, properties of strongly quotient graphs refer to [1, 2, 3, 5, 6, 10, 14]. Throughout this paper, SQG stands for strongly quotient graph of order n with maximum number of edges.

We have organized this paper in the following way. In section 2, two eigen values for Harary matrix of SQG are obtained. In Section 3, a lower bound and two upper bounds are derived for the Harary energy of SQG. In Section 4, the Harary Estrada index of SQG and some better lower and upper bounds are obtained for SQG involving Harary energy and several other graph invariants.

2 Preliminaries

In this section, we have given some lemmas which will be used in our main results and we have obtained two eigen values of SQG with respect to Harary matrix.

LEMMA 2.1 ([12]). Let G be a connected (n, m) graph and let $\rho_1, \rho_2, \ldots, \rho_n$ be its H-eigenvalues. Then

$$\sum_{i=1}^{n} \rho_i = 0 \text{ and } \sum_{i=1}^{n} \rho_i^2 = 2 \sum_{1 \le i < j \le n} \left(\frac{1}{d_{ij}}\right)^2.$$

LEMMA 2.2 ([12]). Let G be a connected (n,m) graph with diameter less than or equal to 2 and let $\rho_1,\rho_2,\ldots,\rho_n$ be its H-eigenvalues. Then

$$\sum_{i=1}^{n}\rho_i^2=\frac{3m}{2}+\frac{n}{4}(n-1).$$

LEMMA 2.3 ([16]). Let $x_1,x_2,\ldots,x_n$ be nonnegative numbers. Then

$$N\le n\sum_{i=1}^{n}x_i-\left(\sum_{i=1}^{n}\sqrt{x_i}\right)^2\le(n-1)N,$$

where

$$N=n\left[\frac{1}{n}\sum_{i=1}^{n}x_i-\left(\prod_{i=1}^{n}x_i\right)^{1/n}\right].$$

The distance matrix $D(G)=[d_{ij}]$ of a graph G is a square matrix of order n in which $d_{ij}=d(v_i,v_j)$. In [14], R. K. Zaferani obtained two eigen values for the distance matrix of SQG. We recall that the SQG is a strongly quotient graph with the maximum number of edges for a fixed order. From Theorems 3.2 and 3.3 in [14], we have obtained the following results for Harary matrix $H(G)$ of SQG. The notation $\lfloor a\rfloor$ denotes the flooring value of a.

RESULT 2.4. If G is a SQG then -1 is the H-eigenvalue of G with multiplicity greater than or equal to $\alpha=|P|$ where

$$P=\left\{p:p\text{ is prime and }\frac{n}{2}<p\le n\right\}. \tag{1}$$

RESULT 2.5. If G is a SQG then $-\frac{1}{2}$ is the H-eigenvalue of G with multiplicity greater than or equal to β where

$$\beta=\sum_{\substack{p\text{-prime}\\ p\le\lfloor\frac{n}{2}\rfloor}}\left(\lfloor\log_p n\rfloor-1\right).$$

3 Bounds on Harary Energy of SQG

In this section, we have presented some upper and lower bounds for Harary Energy $HE(G)$, where G is SQG, with eigen values $\rho_1,\rho_2,\ldots,\rho_n$. For our convenience we have renamed the eigen values as

$$\rho_{n-\alpha-\beta+1}=\rho_{n-\alpha-\beta+2}=\ldots=\rho_{n-\beta}=-1$$

and

$$\rho_{n-\beta+1}=\rho_{n-\beta+2}=\ldots=\rho_n=-\frac{1}{2},$$

where α and β are as defined in results 2.4 and 2.5.

THEOREM 3.1. Let G be a Strongly Quotient Graph (SQG) with $n > 3$ vertices and maximum edges m. Let P be defined by (1) and $\alpha = |P|$. Then

$$HE(G) \leq \alpha + \frac{\beta}{2} + \sqrt{(n-\alpha-\beta)\chi}, \tag{2}$$

where

$$\beta = \sum_{\substack{p\text{-prime} \\ p \leq \lfloor \frac{n}{2} \rfloor}} (\lfloor log_p n \rfloor - 1) \text{ and } \chi = 2 \sum_{1 \leq i < j \leq n} \left(\frac{1}{d_{ij}}\right)^2 - \alpha - \frac{\beta}{4}.$$

PROOF. By Cauchy Schwarz inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are real numbers. Setting $x_i = 1$, $y_i = |\rho_i|$ and replacing n by $n - \alpha - \beta$, we have obtained

$$\left(\sum_{i=1}^{n-\alpha-\beta} |\rho_i|\right)^2 \leq (n-\alpha-\beta) \sum_{i=1}^{n-\alpha-\beta} |\rho_i|^2.$$

By results 2.4 and 2.5, we know that -1 and $-\frac{1}{2}$ are the H-eigenvalues of SQG with multiplicity greater than or equal to α and β respectively and considering Lemma 2.1, we have

$$\left(HE(G) - \alpha - \frac{\beta}{2}\right)^2 \leq (n-\alpha-\beta)\left(2 \sum_{1 \leq i < j \leq n} \left(\frac{1}{d_{ij}}\right)^2 - \alpha - \frac{\beta}{4}\right).$$

or equivalently

$$HE(G) \leq \alpha + \frac{\beta}{2} + \sqrt{(n-\alpha-\beta)\chi},$$

where

$$\chi = 2 \sum_{1 \leq i < j \leq n} \left(\frac{1}{d_{ij}}\right)^2 - \alpha - \frac{\beta}{4}.$$

Hence we get the result.

THEOREM 3.2. Let G be a Strongly Quotient Graph (SQG) with $n > 3$ vertices and maximum edges m. Let P be defined by (1) and $\alpha = |P|$. Then

$$HE(G) \geq \alpha + \frac{\beta}{2} + \sqrt{\chi + (n-\alpha-\beta)(n-\alpha-\beta-1)\zeta}$$

and

$$HE(G) \leq \alpha + \frac{\beta}{2} + \sqrt{(n-\alpha-\beta-1)\chi + (n-\alpha-\beta)\zeta}, \tag{3}$$

where

$$\beta = \sum_{\substack{p\text{-prime} \\ p \leq \lfloor \frac{n}{2} \rfloor}} \left(\lfloor log_p n \rfloor - 1 \right), \ \chi = 2 \sum_{1 \leq i < j \leq n} \left(\frac{1}{d_{ij}} \right)^2 - \alpha - \frac{\beta}{4}$$

and

$$\zeta = \left(2^{\beta} | \det H(G)| \right)^{2/n-\alpha-\beta}.$$

PROOF. Setting $x_i = \rho_i^2$ and replacing n by $n - \alpha - \beta$ in Lemma 2.3 we obtain that

$$N \leq (n-\alpha-\beta) \sum_{i=1}^{n-\alpha-\beta} \rho_i^2 - \left(\sum_{i=1}^{n-\alpha-\beta} |\rho_i| \right)^2 \leq (n-\alpha-\beta-1)N,$$

where

$$N = (n-\alpha-\beta) \left[\frac{1}{n-\alpha-\beta} \sum_{i=1}^{n-\alpha-\beta} \rho_i^2 - \left(\prod_{i=1}^{n-\alpha-\beta} \rho_i^2 \right)^{1/n-\alpha-\beta} \right].$$

By results 2.4 and 2.5, we know that -1 and $-\frac{1}{2}$ are the H-eigenvalues of SQG with multiplicity greater than or equal to α and β respectively. Therefore we have obtained that

$$N \leq (n-\alpha-\beta)\chi - \left(HE(G) - \alpha - \frac{\beta}{2} \right)^2 \leq (n-\alpha-\beta-1)N,$$

where

$$\chi = 2 \sum_{1 \leq i < j \leq n} \left(\frac{1}{d_{ij}} \right)^2 - \alpha - \frac{\beta}{4}.$$

We observe that

$$\begin{aligned} N &= (n-\alpha-\beta) \left[\frac{1}{n-\alpha-\beta} \sum_{i=1}^{n-\alpha-\beta} \rho_i^2 - \left(\prod_{i=1}^{n-\alpha-\beta} \rho_i^2 \right)^{1/n-\alpha-\beta} \right] \\ &= \chi - (n-\alpha-\beta) \left(2^{\beta} \prod_{i=1}^{n} |\rho_i| \right)^{2/n-\alpha-\beta} \\ &= \chi - (n-\alpha-\beta) \left(2^{\beta} | \det H(G)| \right)^{2/n-\alpha-\beta} \\ &= \chi - (n-\alpha-\beta)\zeta, \end{aligned}$$

where $\zeta = \left(2^{\beta} | \det H(G)| \right)^{2/n-\alpha-\beta}$. Hence we get the result.

REMARK 3.3. The upper bound (3) is sharper than the upper bound (2). Using Arithmetic–Geometric mean inequality, we have obtained that

$$\sum_{i=1}^{n-\alpha-\beta} \rho_i^2 \geq (n-\alpha-\beta) \left(\prod_{i=1}^{n-\alpha-\beta} \rho_i^2 \right)^{1/n-\alpha-\beta}.$$

Equivalent to

$$\chi \geq (n-\alpha-\beta)\zeta$$

and considering the upper bound (3) we arrive at

$$HE(G) \leq \alpha + \frac{\beta}{2} + \sqrt{(n-\alpha-\beta-1)\chi + \chi}.$$

Thus we have that

$$HE(G) \leq \alpha + \frac{\beta}{2} + \sqrt{(n-\alpha-\beta)\chi}.$$

Which is the upperbound (2). Using Theorem 3.2 and Lemma 2.2, we can give the following corollary.

COROLLARY 3.4. Let G be a Strongly Quotient graph(SQG) with $n > 3$ vertices and maximum edges m. Let P be defined by (1) and $\alpha = |P|$. Assume that the diameter of G less than or equal to 2. Then

$$HE(G) \geq \alpha + \frac{\beta}{2} + \sqrt{\xi + (n-\alpha-\beta)(n-\alpha-\beta-1)\zeta}$$

and

$$HE(G) \leq \alpha + \frac{\beta}{2} + \sqrt{(n-\alpha-\beta-1)\xi + (n-\alpha-\beta)\zeta},$$

where

$$\beta = \sum_{\substack{p\text{-prime} \\ p \leq \lfloor \frac{n}{2} \rfloor}} \left(\lfloor \log_p n \rfloor - 1\right), \xi = \frac{1}{4}\left[6m + n(n-1) - 4\alpha - \beta\right]$$

and $\zeta = \left(2^{\beta} |\det H(G)|\right)^{2/n-\alpha-\beta}$.

4 Bounds on Harary Estrada Index of SQG

First we recall that the Harary Estrada index [12] of the graph G is equal to $\sum\limits_{i=1}^{n} e^{\rho_i}$ and let n_+ be the number of positive H-eigenvalues of G. In this section we have obtained some upper bound, lower bound for Harary Estrada index $HE(G)$ and the relation between the Harary Estrada index and Harary Energy of SQG with $n > 3$, maximum edges m.

THEOREM 4.1. Let G be a Strongly Quotient graph(SQG) with $n > 3$ vertices and maximum edges m. Let P be defined by (1) and $\alpha = |P|$. Then

$$HEE(G) \geq \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + (n-\alpha-\beta)e^{2\alpha+\beta/2(n-\alpha-\beta)}, \tag{4}$$

where

$$\beta = \sum_{\substack{p\text{-prime} \\ p \leq \lfloor \frac{n}{2} \rfloor}} \left(\lfloor \log_p n \rfloor - 1\right).$$

PROOF. From Lemma 2.1, Results 2.4 and 2.5, we have $\sum_{i=1}^{n-\alpha-\beta} \rho_i = \alpha + \frac{\beta}{2}$. Using Arithmetic -Geometric Mean Inequality, we get

$$
\begin{aligned}
HEE(G) &= \sum_{i=1}^{n-\alpha-\beta} e^{\rho_i} + \alpha e^{-1} + \beta e^{-\frac{1}{2}} \\
&\geq \alpha e^{-1} + \beta e^{-\frac{1}{2}} + (n-\alpha-\beta)\left(\prod_{i=1}^{n-\alpha-\beta} e^{\rho_i}\right)^{1/n-\alpha-\beta} \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + (n-\alpha-\beta)\left(e^{\sum_{i=1}^{n-\alpha-\beta}\rho_i}\right)^{1/n-\alpha-\beta} \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + (n-\alpha-\beta)\left(e^{\alpha+\frac{\beta}{2}}\right)^{1/n-\alpha-\beta} \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + (n-\alpha-\beta)e^{2\alpha+\beta/2(n-\alpha-\beta)}.
\end{aligned}
$$

This completes the proof.

THEOREM 4.2. Let P be defined by (1) and $|P| = \alpha$. The Harary Estrada index $HEE(G)$ and the Harary energy $HE(G)$ of SQG G with $n > 3$ vertices and maximum edges m satisfy the following inequalities

$$HEE(G) \geq \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + \frac{(e-1)HE(G)}{2} + n - n_+ - \frac{\beta}{2} \tag{5}$$

and

$$HEE(G) \leq \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta - 1 + e^{HE(G)/2}, \tag{6}$$

where $\beta = \sum_{\substack{p\text{-prime} \\ p \leq \lfloor \frac{n}{2} \rfloor}} \left(\lfloor \log_p n \rfloor - 1\right)$.

PROOF. **Lower bound:** Using inequalities $e^x \geq xe$ and $e^x \geq 1+x$, we can obtain that

$$
\begin{aligned}
HEE(G) &= \sum_{i=1}^{n-\alpha-\beta} e^{\rho_i} + \alpha e^{-1} + \beta e^{-\frac{1}{2}} \\
&= \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \sum_{\rho_i > 0} e^{\rho_i} + \sum_{\rho_i \leq 0} e^{\rho_i} \\
&\geq \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + e\sum_{\rho_i > 0} \rho_i + \sum_{\substack{i=1 \\ \rho_i \leq 0}}^{n-\alpha-\beta} (1+\rho_i) \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + \frac{eHE(G)}{2} + n - \alpha - \beta - n_+ + \alpha + \frac{\beta}{2} - \frac{HE(G)}{2}
\end{aligned}
$$

$$= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + \frac{(e-1)HE(G)}{2} + n - n_+ - \frac{\beta}{2}.$$

Upper bound: Considering $f(x) = e^x$ which is monotonically increases in the interval $(-\infty, \infty)$, we obtain

$$\begin{aligned}
HEE(G) &= \sum_{i=1}^{n-\alpha-\beta} e^{\rho_i} + \alpha e^{-1} + \beta e^{-\frac{1}{2}} \\
&= \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \sum_{\rho_i<0} e^{\rho_i} + \sum_{\rho_i \geq 0} e^{\rho_i} \\
&\leq \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta - n_+ + \sum_{i=1}^{n_+} e^{\rho_i} \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta - n_+ + \sum_{i=1}^{n_+} \sum_{k \geq 0} \frac{\rho_i^k}{k!} \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta + \sum_{k \geq 1} \frac{1}{k!} \left(\sum_{i=1}^{n_+} \rho_i \right)^k \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta + \sum_{k \geq 1} \frac{(HE(G)/2)^k}{k!} \\
&= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta - 1 + e^{HE(G)/2}. \qquad (7)
\end{aligned}$$

This completes the proof.

We conclude that the upper bound (5) is better than the upper bound (4) for Harary estrada index of SQG with $n > 3$ vertices and maximum edges m.

THEOREM 4.3. Let P be defined by (1) and $|P| = \alpha$. The Harary Estrada index and Harary energy of SQG with $n > 3$ vertices and maximum edges m satisfy the following inequality

$$HEE(G) - HE(G) < \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta - 1 - \sqrt{\chi} + e^{\sqrt{\chi}}, \qquad (8)$$

where

$$\chi = 2 \sum_{1 \leq i < j \leq n} \left(\frac{1}{d_{ij}} \right)^2 - \alpha - \frac{\beta}{4} \text{ and } \beta = \sum_{\substack{p\text{-prime} \\ p \leq \lfloor \frac{n}{2} \rfloor}} \left(\lfloor \log_p n \rfloor - 1 \right).$$

PROOF. By results 2.4 and 2.5, we know that -1 and $-\frac{1}{2}$ are the H-eigenvalues of SQG with multiplicity greater than or equal to α and β respectively. From Lemma 2.1, Results 2.4 and 2.5, we have $\sum\limits_{i=n_+ +1}^{n} \rho_i^2 \geq \alpha + \frac{\beta}{4}$. From (6),

$$HEE(G) \leq \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta + \sum_{i=1}^{n_+} \sum_{k \geq 1} \frac{\rho_i^k}{k!}$$

$$= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta + \sum_{i=1}^{n_+} \rho_i + \sum_{k\geq 2} \frac{1}{k!} \left(\sum_{i=1}^{n_+} \rho_i^2 \right)^{k/2}$$

$$< \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta + HE(G) + \sum_{k\geq 2} \frac{1}{k!} \left[2 \sum_{i<j} \frac{1}{d_{ij}^2} - \sum_{i=n_+ +1}^{n} \rho_i^2 \right]^{k/2}$$

and

$$\begin{aligned} HEE(G) - HE(G) &< \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta + \sum_{k\geq 2} \frac{1}{k!} \left[2 \sum_{i<j} \frac{1}{d_{ij}^2} - \alpha - \frac{\beta}{4} \right]^{k/2} \\ &= \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta - 1 - \sqrt{\chi} + e^{\sqrt{\chi}}, \end{aligned}$$

where $\chi = 2 \sum\limits_{1\leq i<j\leq n} \left(\frac{1}{d_{ij}}\right)^2 - \alpha - \frac{\beta}{4}$. This completes the proof.

From Theorem 4.3 and Lemma 2.2 we can give the following result.

COROLLARY 4.4. Let P be defined by (1) and $|P| = \alpha$. The Harary Estrada index and Harary energy of SQG with $n > 3$ vertices, maximum edges m and let the diameter of G less than or equal to 2 satisfy the following inequality

$$HEE(G) - HE(G) < \frac{\alpha}{e} + \frac{\beta}{\sqrt{e}} + n - \alpha - \beta - 1 - \sqrt{\xi} + e^{\sqrt{\xi}},$$

where

$$\xi = \frac{1}{4} \left[6m + n(n-1) - 4\alpha - \beta \right] \text{ and } \beta = \sum_{\substack{p\text{-prime} \\ p \leq \lfloor \frac{n}{2} \rfloor}} \left(\lfloor \log_p n \rfloor - 1 \right).$$

REMARK 4.5. [12] Let G be a connected (n, m)-graph with diameter less than or equal to 2. Then

$$HEE(G) - HE(G) \leq n - 1 - \sqrt{\frac{3m}{2} + \frac{n(n-1)}{4}} + e^{\sqrt{\frac{3m}{2} + \frac{n(n-1)}{4}}}. \tag{9}$$

Since the functions $f(t) = e^t$ and $f(t) = e^t - t$ are monotonically increase in the intervals $(-\infty, \infty)$ and $(0, \infty)$ respectively, we conclude that the upper bound (8) is better than the upperbound (9) for SQG with $n > 3$ vertices and maximum edges m.

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