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Unicity Of Meromorphic Function That Share A Small Function With Its Derivative*

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Abstract

In this paper, we deal with the problem of uniqueness of a meromorphic function as well as its power which share a small function with its derivative. Basically in the paper we pay our attention to the uniqueness of more generalised form of a function sharing a small function and we obtain two results which improve and generalize the recent results of Zhang and Yang [11].

1 Introduction, Definitions and Results

In this paper, by a meromorphic function we will always mean meromorphic function in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [2]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \longrightarrow \infty$ and $r \notin E$.

Let f and g be two non-constant meromorphic functions and let a be a complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, $r \notin E$.

During the last four decades the uniqueness theory of entire and meromorphic functions has become a prominent branch of the value distribution theory (see [9]). In the direction of the shared value problems concerning the uniqueness of a meromorphic function and its derivative a considerable amount of research work has been obtained by many authors such as Rubel and Yang [4], Gundersen [1], Mues and Steinmetz [3] and Yang [6].

To the knowledge of the author perhaps Yang and Zhang [7] (see also [10]) were the first authors to consider the uniqueness of a power of a meromorphic(entire) function $F = f^n$ and its derivative $F^{'}$.

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Improving all the results obtained in [7], Zhang [10] proved the following theorems.

THEOREM A ([10]). Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and

$$n > k + 4.$$

Then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

THEOREM B ([10]). Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and

$$(n - k - 1)(n - k - 4) > 3k + 6.$$

Then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

In 2009 Zhang and Yang [11] further improved the above results in the following manner.

THEOREM C ([11]). Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and

$$n > k + 1.$$

Then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

THEOREM D ([11]). Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and

$$n > 2k + 3.$$

Then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

THEOREM E ([11]). Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and

$$n > 2k + 3 + \sqrt{(2k+3)(k+3)}.$$

Then $f^n \equiv (f^n)^{(k)}$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

We now explain the following definitions and notations which will be used in the paper.

DEFINITION 1 ([4]). Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. $N(r, a; f \mid\geq p)$ ($\overline{N}(r, a; f \mid\geq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

DEFINITION 2 ([8]). For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p, we denote by $N_p(r, a; f)$ the sum

$$\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq p).$$

Clearly, $N_1(r, a; f) = \overline{N}(r, a; f)$.

It is quite natural to ask the following question:

QUESTION 1. Can the lower bound of n be further reduced in the THEOREMS D and E ?

In this paper, taking the possible answer of the above question into background we obtain the following results which improve and generalize the *THEOREMS D and E.*

THEOREM 1. Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0$ be a nonzero polynomial. Suppose $f^n P(f) - a$ and $[f^n P(f)]^{(k)} - a$ share the value 0 IM and

$$n > 2k + m + 2.$$

Then $P(w)$ reduces to a nonzero monomial, namely $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; and $f^{n+i} \equiv (f^{n+i})^{(k)}$, where f assumes the form

$$f(z) = ce^{\frac{\lambda}{n+i}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

THEOREM 2. Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$ be a nonzero polynomial. Suppose $f^n P(f) - a$ and $[f^n P(f)]^{(k)} - a$ share the value 0 IM and

$$n > k + m + 1.$$

Then $P(w)$ reduces to a nonzero monomial, namely $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; and $f^{n+i} \equiv (f^{n+i})^{(k)}$, where f assumes the form

$$f(z) = c e^{\frac{\lambda}{n+i} z},$$

where c is a nonzero constant and $\lambda^k = 1$.

2 Lemma

In this section we present the lemma which will be needed in the sequel.

LEMMA 1 ([5]). Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$,..., $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, ..., n$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

3 Proofs of the Theorems

In this section, we prove THEOREMS 1 and 2

PROOF OF THEOREM 1. Let

$$F = \frac{f^n P(f)}{a} \text{ and } G = \frac{[f^n P(f)]^{(k)}}{a},$$

where $P(w)$ is defined as in THEOREM 1. Clearly, F and G share 1 IM and so

$$\overline{N}(r, 1; F) = \overline{N}(r, 1; G) + S(r, f).$$

We divide two cases: Case 1. $F \not\equiv G$ and Case 2. $F \equiv G$.

Case 1. Assume that $F \not\equiv G$. Note that

$$\begin{aligned}
\overline{N}(r,1;F) &\leq \overline{N}\left(r,1;\frac{G}{F}\right)+S(r,f) \qquad (1)\\
&\leq T\left(r,\frac{G}{F}\right)+S(r,f)\\
&\leq N\left(r,\infty;\frac{G}{F}\right)+m\left(r,\infty;\frac{G}{F}\right)+S(r,f)\\
&= N\left(r,\infty;\frac{[f^nP(f)]^{(k)}}{f^nP(f)}\right)+m\left(r,\infty;\frac{[f^nP(f)]^{(k)}}{f^nP(f)}\right)+S(r,f)\\
&\leq k\overline{N}(r,\infty;f)+N_k(r,0;f^nP(f))+S(r,f)\\
&\leq k\overline{N}(r,\infty;f)+k\overline{N}(r,0;f)+mT(r,f)+S(r,f).
\end{aligned}$$

Now using (1) and LEMMA 1 we get from the second fundamental theorem that

$$\begin{aligned}
(n+m)\,T(r,f) &= T(r,F)+S(r,f) \qquad (2)\\
&\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)+S(r,F)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f^nP(f))+\overline{N}(r,1;F)+S(r,f)\\
&\leq (k+1)\overline{N}(r,\infty;f)+(k+1)\,\overline{N}(r,0;f)+2mT(r,f)+S(r,f)\\
&\leq (2k+2m+2)T(r,f)+S(r,f).
\end{aligned}$$

Since $n > m+2k+2$, (2) leads to a contradiction.

Case 2. Assume that $F \equiv G$. Then

$$f^nP(f) \equiv [f^nP(f)]^{(k)}. \qquad (3)$$

We now prove that $P(w) = a_iw^i \not\equiv 0$ for some $i \in \{0,1,\ldots,m\}$. If not we may assume that $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ where at least two of $a_0, a_1, \ldots, a_{m-1}, a_m$ are nonzero. Without loss of generality, we assume that $a_s, a_t \neq 0$, where $s \neq t$, $s,t = 0,1,\ldots,m$. From (3) it is clear that f is an entire function. Also since $n > 2k+m+2$, it follows from (3) that 0 is an e.v.P of f. So we can take $f = e^{\alpha}$ where α is a non-constant entire function. Then by induction we get

$$a_i[f^{n+i} - (f^{n+i})^{(k)}] = t_i(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{(n+i)\alpha}, \qquad (4)$$

where $t_i(\alpha^{'}, \alpha^{''}, \ldots, \alpha^{(k)})$ for $i = 0,1,\ldots,m$ are differential polynomials in $\alpha^{'}, \alpha^{''}, ..., \alpha^{(k)}$.

From (3) and (4) we obtain

$$t_m(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{m\alpha} + \cdots + t_1(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{\alpha} + t_0(\alpha', \alpha'', \ldots, \alpha^{(k)}) \equiv 0. \qquad (5)$$

Since $T(r,t_i) = S(r,f)$ for $i = 0,1,\ldots,m$, and by the Borel unicity theorem (see, e.g. [9, Theorem 1.52]), (5) gives $t_i \equiv 0$ for $i = 0,1,\ldots,m$. As $a_s, a_t \neq 0$, from (4) we have

$$f^{n+s} \equiv (f^{n+s})^{(k)} \text{ and } f^{n+t} \equiv (f^{n+t})^{(k)},$$

which is a contradiction. Actually in this case we get two different forms of $f(z)$ simultaneously. Hence $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. So from (3) we get

$$f^{n+i} \equiv [f^{n+i}]^{(k)},$$

where $i \in \{0, 1, \ldots, m\}$. Clearly, f assumes the form

$$f(z) = ce^{\frac{\lambda}{n+i}z},$$

where c is a nonzero constant and $\lambda^k = 1$.

PROOF OF THEOREM 2. Let

$$F = \frac{f^n P(f)}{a} \text{ and } G = \frac{[f^n P(f)]^{(k)}}{a}.$$

Note that $N(r, \infty; F) = N(r, \infty; G) = S(r, f)$. We omit the proof of THEOREM 2 since the proof of the theorem can be carried out in the line of proof of THEOREM 1.

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References

[1] G. G. Gundersen, Meromorphic functions that share finite values with their derivative, J. Math. Anal. Appl., 75(1980), 441–446.

[2] W. K. Hayman, Meromorphic Functions. Oxford Mathematical Monographs Clarendon Press, Oxford 1964.

[3] E. Mues and N. Steinmetz, Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endlichen Wert teilen, Complex Var. Theory Appl., 6(1986), 51–71.

[4] L. A. Rubel and C. C. Yang, Values Shared By An Entire Function and Its Derivative. Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), pp. 101–103. Lecture Notes in Math., Vol. 599, Springer, Berlin, 1977.

[5] C. C. Yang, On Deficiencies of Differential Polynomials, II, Math. Z. Vol., 125(1972), 107–112.

[6] L. Z. Yang, Entire functions that share finite values with their derivatives, Bull. Austral. Math. Soc., 41(1990), 337–342.

[7] L. Z. Yang and J. L. Zhang, Non-existance of meromorphic solutions of Fermat type functional equation, Aequations Math., 76(2008), 140–150.

[8] H. X. Yi, On characteristic function of a meromorphic function and its derivative, Indian J. Math., 33(1991), 119–133.

[9] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[10] J. L. Zhang, Meromorphic functions sharing a small function with their derivatives, Kyungpook Math. J., 49(2009), 143–154.

[11] J. L. Zhang and L. Z. Yang, A power of a meromorphic function sharing a small function with its derivative, Ann. Acad. Sci. Fenn. Math., 34(2009), 249–260.