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Some Remarks On Block Group Circulant Matrices*

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Abstract

Let C denote a block group circulant matrix over a finite non-Abelian group G. We prove results concerning the spectral properties of the matrix C. We give an example of the spectral decomposition of a block group circulant matrix over the symmetric group S_3.

1 Introduction

Block circulant matrices over the cyclic group $\mathbf{Z}_n$ have been well studied, see [11] for example. In our paper we will consider the setting where the cyclic group $\mathbf{Z}_n$ is replaced by a non-Abelian finite group G. Some of the framework needed for the block group circulant case needs to be taken from the group circulant matrix case that was discussed in [13]. Let $l^2(G)$ denote the finite-dimensional Hilbert space of all complex-valued functions, with the usual inner product, for which the elements of G form the (standard) basis. We assume that this basis (G) is ordered and make the natural identification with $\mathbf{C}^n$, where $|G| = n$, as a linear space.

Let $\mathbf{C}[G]$ be the group algebra of complex-valued functions on G. Consider $\psi = (c_0, c_1, \ldots, c_{n-1}) \in \mathbf{C}^n$ and identify the function ψ with its symbol $\Psi = c_0\mathbf{1} + c_1 g_1 + \cdots c_{n-1} g_{n-1} \in \mathbf{C}[G]$.

DEFINITION. Let $\widehat{G}$ be the set of all (equivalence classes) of irreducible representations of the group G and let r denote the cardinality of $\widehat{G}$. Let $\rho \in \widehat{G}$ denote an irreducible representation of G of degree j and let $\phi \in \mathbf{C}^n$. Then the Fourier transform of ϕ at ρ is the $j \times j$ matrix

$$\widehat{\phi}(\rho) = \sum_{s \in G} \phi(s)\rho(s^{-1}).$$

Let ψ and ϕ be two elements in $\mathbf{C}^n$. A G-convolution of ψ and ϕ is defined by the following action

$$(\psi * \phi)(\sigma) = \sum_{\tau \in G} \psi(\tau)\phi(\tau^{-1}\sigma) \text{ for } \sigma \in G.$$

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We have a natural identification $\psi * \phi \mapsto \Psi\Phi$ understood with respect to the induced group algebra multiplication. Moreover, the Fourier transform turns convolution into (matrix) multiplication $\widehat{\psi * \phi} = \widehat{\psi}\widehat{\phi}$. Thus we have a non-Abelian version of the classical z transform. For further references on this subject we refer the reader to [1, 2, 3, 4, 5, 6, 7, 9, 10, 12].

The Fourier transform gives us a natural isomorphism $\mathbf{C}[G] \Rightarrow M(\widehat{G})$ where

$$M(\widehat{G}) = M_{d_1 \times d_1}(\mathbf{C}) \oplus M_{d_2 \times d_2}(\mathbf{C}) \oplus \cdots \oplus M_{d_r \times d_r}(\mathbf{C})$$

with $d_1^2 + d_2^2 + \cdots + d_r^2 = n$. A typical element of $\mathbf{C}^n$ is a complex-valued function $\psi = (c_0, c_1, \ldots, c_{n-1})$ and the typical element of $M(\widehat{G})$ is the direct sum of Fourier transforms

$$\widehat{\phi}(\rho_1) \oplus \widehat{\phi}(\rho_2) \oplus \cdots \oplus \widehat{\phi}(\rho_r).$$

Cyclic circulant matrices are normal (hence diagonalizable) and the Fourier basis of eigenvectors, the complex exponentials, are fixed and independent of the function ψ. In the Abelian setting the Fourier transform is a unitary linear transformation (proper scaling required). In the non-Abelian setting we recapture this property if we define the right inner product on the space $M(\widehat{G})$. Let $\phi \in \mathbf{C}^n$ and define a function ϕ_j by the following action

$$\phi_j(s) = \frac{d_j}{|G|} \operatorname{tr}\left(\rho_j(s)\widehat{\phi}(\rho_j)\right) \text{ for } s \in G.$$

Note $\phi = \sum_{j=1}^{r} \phi_j$ which constitutes the inverse Fourier transform. We are able to decompose a function ϕ into a sum of r functions which is the number of conjugacy classes of G.

DEFINITION. Let $A = (a_{i,j})$ be a $m \times n$ matrix. The Frobenius norm of A is given by

$$||A||_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2.$$

If we let ϕ_j be given as above, then we have

$$\langle \phi_i, \phi_j \rangle = \frac{d_j}{|G|} ||\widehat{\phi}(\rho_j)||_F^2 \delta_{ij}.$$

and if we let $\phi \in \mathbf{C}^n$ then

$$||\phi||^2 = \frac{1}{|G|} \sum_{j=1}^{r} d_j ||\widehat{\phi}(\rho_j)||_F^2.$$

Thus, with proper scaling, the Fourier transform is a unitary transformation from $\mathbf{C}^n$ onto $\left(M(\widehat{G}), \bullet_F\right)$.

In the case of a group circulant matrix $C = C_G(\psi)$ over a non-Abelian group G its eigenvectors need not be orthogonal nor are ψ independent in general. Moreover, the matrix $C_G(\psi)$ need not be diagonalizable, an example was given in [13] with $G = D_4$, the dihedral group of order 4. The group D_4 is a semi-direct product of the cyclic group $\mathbf{Z}_4$ and the cyclic group $\mathbf{Z}_2$. Let $\mathbf{Z}_n =< r >$ and $\mathbf{Z}_2 =< s >$. We have $r^n = s^2 = \mathbf{1}$ and $r^j s = s r^{-j}$ for all $j \in \{0, 1, \ldots, n-1\}$. The matrix C corresponding to the convolution operator induced by the symbol $\Psi = r + rs$ is not diagonalizable.

Eigenvalue analysis for group circulant matrices was studied in [8] and the eigenvector decomposition in [13]. In the Fourier domain the eigenvalue problem for a group circulant matrix translates to $AB = \lambda B$ where λ is an eigenvalue of $A = \widehat{\psi}(\rho_j)$ and the columns of B are the corresponding eigenvectors (any collection including the zero vector).

Assume the matrix $\widehat{\psi}(\rho_j)$ is diagonalizable for each j with d_j eigenvalues (possibly counting multiplicities). Let $\sigma\left(\widehat{\psi}(\rho_j)\right) = \{\lambda_{1,j}, \ldots, \lambda_{d_j,j}\}$. Consider an (unital) eigenvector $\mathbf{v}_{\lambda_{s,j}}$ of $\widehat{\psi}(\rho_j)$ corresponding to the eigenvalue $\lambda_{s,j}$ with $s \in \{1, \ldots, d_j\}$. In the case of a multiple eigenvalue we choose linearly-independent (preferably orthogonal) unital eigenvectors. To obtain the eigenvector decompositon of C we review some of the developments in [13].

Define a sequence of Fourier (orthogonal) eigenvectors in $M(\widehat{G})$

$$\widehat{\mathbf{v}}_p(\lambda_{s,j}) = (\mathbf{0})_{d_1 \times d_1} \oplus \cdots \oplus \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{v}_{\lambda_{s,j}} & \mathbf{0} \cdots & \mathbf{0} \\ & & & & & \end{pmatrix} \oplus \cdots \oplus (\mathbf{0})_{d_r \times d_r}$$

where the unital eigenvector $\mathbf{v}_{\lambda_{s,j}}$ is located in the p-th column with $p \in \{1, \ldots, d_j\}$.

The orthogonality properties are respected in the space $l^2(\mathbf{C}^n)$ upon taking the inverse Fourier Trasform which is unitary. Namely, upon taking the inverse Fourier transform of the vectors $\{\widehat{\mathbf{v}}_p(\lambda_{s,j})\}$ we obtain eigenvectors

$$\{\mathbf{v}_p(\lambda_{s,j})\} \text{ for } p, s \in \{1, 2, \ldots, d_j\} \text{ and } j \in \{1, 2, \ldots, r\}.$$

For a given $\lambda_{s,j}$ the eigenvectors $\{\mathbf{v}_p(\lambda_{s,j}) \,|\, p \in \{1, 2, \ldots, d_j\}\}$ are pairwise mutually-orthogonal. Moreover $\mathbf{v}_p(\lambda_{s,i}) \perp \mathbf{v}_q(\lambda_{t,j})$ for $i \neq j$ and any choice of p, q and s, t.

The group circulant matrix $C_G(\psi)$ admits pairwise mutually-orthogonal, ψ independent, d_j^2-dimensional, C-invariant subspaces

$$\begin{aligned} V_j &= \operatorname{span}\{\mathbf{v}_p(\lambda_{s,j}) \,|\, p \in \{1, \ldots, d_j\}, s \in \{1, \ldots, d_j\}\} \\ &= \operatorname{span}\{\rho_j(k,l) \,|\, k, l \in \{1, \ldots, d_j\}\} \end{aligned}$$

where $j \in \{1, 2, \ldots, r\}$ and the action of the function $\rho_j(k,l)$ is seen as $\rho_j(k,l)(s) = \rho_j(s)(k,l)$ for $s \in G$. This decomposition could be sufficient as far as the response of G-convolution by ψ on functions in $\mathbf{C}^n$ is concerned. The functions $\{\rho_j(k,l)\}$ are the generalizations of the complex exponentials (cyclic case) as they are mutually-orthogonal though not necessarily eigenvectors. The values $\{||\widehat{\psi}(\rho_j)||\}$ could act as frequency responses.

2 Main Results: Block Group Circulant Matrices

Define a vector space $\mathbf{C}^k[G]$ consisting of elements

$$\mathbf{v} = v_0 \mathbf{1} + v_1 g_1 + \cdots v_{n-1} g_{n-1}$$

where $v_i = (v_{1i}, v_{1i}, \ldots, v_{ki})^T \in \mathbf{C}^k$. Note that $\mathbf{C}^k[G]$ is not an algebra as we do not have multiplication defined. We can identify the element $\mathbf{v}$ with

$$(v_{10}g_1 + \cdots + v_{1n-1}g_{n-1}) \oplus \cdots \oplus (v_{k0}g_1 + \cdots + v_{kn-1}g_{n-1})$$

so that $\mathbf{v} = \mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_k$ with $\mathbf{v}_s \in \mathbf{C}[G]$. Each $\mathbf{v}_s$ results from collecting the same entries in $\mathbf{v}$, ranging from 1 to k. Thus $\mathbf{C}^k[G]$ can be identified with k copies of $\mathbf{C}[G]$. We refer to this as the block stacking (with respect to entry position). Undoing this operation is referred to as block merging. To give an example we let $G = \mathbf{Z}_2$ with the elements $\{g_0, g_1\}$ where g_0 is the identity element and $g_1^2 = g_0$. Consider

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} g_0 + \begin{pmatrix} 3 \\ 4 \end{pmatrix} g_1 = (1\,,2\,,3\,,4)^T.$$

Then $\mathbf{v}_0 = g_0 + 3g_1$ and $\mathbf{v}_1 = 2g_0 + 4g_1$. Now define a group algebra $\mathbf{C}^{k\times k}[G]$ over the group G with coefficients $k \times k$ matrices over the complex numbers. The group algebra $\mathbf{C}^{k\times k}[G]$ consists of elements

$$\Psi = \mathbf{c}_0\mathbf{1} + \mathbf{c}_1 g_1 + \cdots \mathbf{c}_{n-1}g_{n-1}$$

where $\mathbf{c}_i$ are $k\times k$ matrices over the complex numbers. The element Ψ can be identified with a $k \times k$ matrix $[\psi_{ts}]_{t,s=1}^k$ where the entry ψ_{ts} is an element of the group algebra $\mathbf{C}[G]$. The matrix Ψ is obtained by collecting likewise entries in the symbol Ψ similar to the vector block stacking.

Let $\Psi \in \mathbf{C}^{k\times k}[G]$ and $\mathbf{v} \in \mathbf{C}^k[G]$. The $kn \times kn$ block group circulant matrix C is induced by the following action

$$\mathbf{w} = \Psi\mathbf{v}$$

where $\mathbf{v} \in \mathbf{C}^k[G]$, $\mathbf{w} \in \mathbf{C}^k[G]$ and $\Psi \in \mathbf{C}^{k\times k}[G]$. To give an example let $G = \mathbf{Z}_2$ as before. Consider

$$\mathbf{\Psi} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ & \end{pmatrix} g_0 + \begin{pmatrix} 5 & 6 \\ 7 & 8 \\ & \end{pmatrix} g_1.$$

Then $\psi_{1,1} = g_0 + 5g_1$, $\psi_{1,2} = 2g_0 + 6g_1$, $\psi_{2,1} = 3g_0 + 7g_1$ and $\psi_{2,2} = 4g_0 + 8g_1$. Recall $\{\rho_j\}_{j=1}^r$ denote the irreducible representations of G. Let $\widehat{\psi}_{ts}(j)$ be the Fourier transform ($d_j \times d_j$ matrix) of ψ_{ts} evaluated at ρ_j, where ρ_j is the irreducible representation of the group G. Similarly, $\widehat{\mathbf{v}}_i(j)$ is the Fourier transform ($d_j \times d_j$ matrix) of $\mathbf{v}_i$ evaluated at ρ_j.

The action of the block group circulant matrix C can now be lifted to the Fourier domain and can be seen as the following action

$$\oplus_{j=1}^r \begin{pmatrix} \widehat{\psi}_{11}(j) & \widehat{\psi}_{12}(j) & \cdots & \widehat{\psi}_{1k}(j) \\ \widehat{\psi}_{21}(j) & \widehat{\psi}_{22}(j) & \cdots & \widehat{\psi}_{2k}(j) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{\psi}_{k1}(j) & \widehat{\psi}_{k2}(j) & \cdots & \widehat{\psi}_{kk}(j) \\ & & & \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{v}}_1(j) \\ \widehat{\mathbf{v}}_2(j) \\ \vdots \\ \widehat{\mathbf{v}}_k(j) \end{pmatrix} = \oplus_{j=1}^r \widehat{\Psi}_j \widehat{\mathbf{v}}.$$

We will assume the matrix C is diagonalizable. This assumption is made for simplicity reasons namely a notational one, as an extension to non-diagonalizable case can be readily accomplished.

THEOREM 1. Let C be a diagonalizable block group circulant matrix over a finite non-Abelian group G. Then the eigenvalues of C are the eigenvalues $\{\lambda_{m,j}\}$, each with multiplicity d_j, with $m \in \{1, 2 \ldots, kd_j\}$ and $j \in \{1, 2, \ldots, r\}$, of the matrices $\{\Psi_j\}$. Let $\lambda_{m,j}$ be given. Then we have d_j corresponding (linearly-independent though not necessarily orthogonal) eigenvectors $\mathbf{u}_p(\lambda_{m,j})$ for $p \in \{1, 2, \ldots, d_j\}$. These eigenvectors have the following properties. Let p be given. Perform the block stacking of $\mathbf{u}_p(\lambda_{m,j})$. Then the Fourier transform of each block $\mathbf{u}_p^s(\lambda_{m,j})$, $s \in \{1, 2, \ldots, k\}$ is given by

$$\widehat{\mathbf{u}^s}_p(\lambda_{m,j}) = (\mathbf{0})_{d_1 \times d_1} \oplus \cdots \oplus \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{u}^s_{\lambda_{m,j}} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \oplus \cdots \oplus (\mathbf{0})_{d_r \times d_r}$$

where $\mathbf{u}^s_{\lambda_{m,j}}$ is the s^{th} block (top to bottom) of the eigenvector $\mathbf{u}_{\lambda_{m,j}}$ of the matrix Ψ_j with eigenvalue $\lambda_{m,j}$. The vector $\mathbf{u}^s_{\lambda_{m,j}}$ is in the p^{th} column of the $d_j \times d_j$ matrix above.

PROOF. It is clear from the preceding discussion that the eigenvalues of the matrix C are the eigenvalues of the matrices $\{\Psi_j\}_{j=1}^r$ counting multiplicities. We list these as $\{\lambda_{m,j}\}$ with $m \in \{1, 2 \ldots, kd_j\}$ and $j \in \{1, 2, \ldots, r\}$. The eigenvectors of the matrix C can be obtained as follows. Let j be fixed. Let $\mathbf{u}_{\lambda_{m,j}}$ be an unital eigenvector of the matrix $\widehat{\Psi}_j$. Split the vector $\mathbf{u}_{\lambda_{m,j}}$ into k parts (top to bottom) and consider a block $\mathbf{u}^s_{\lambda_{m,j}}$, a $d_j \times 1$ vector. Define a vector

$$\widehat{\mathbf{u}^s}_p(\lambda_{m,j}) = (\mathbf{0})_{d_1 \times d_1} \oplus \cdots \oplus \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{u}^s_{\lambda_{m,j}} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \oplus \cdots \oplus (\mathbf{0})_{d_r \times d_r}$$

where $\mathbf{u}^s_{\lambda_{m,j}}$ is located in the pth column with some fixed choice of $p \in \{1, \ldots, d_j\}$ the same for all the k blocks. Let $\{\mathbf{u}^s_p(\lambda_{m,j})\}$ be the vectors in $\mathbf{C}^n$ whose Fourier transform is the given Fourier sequence $\{\widehat{\mathbf{u}^s}_p(\lambda_{s,j})\}$ for each $s \in \{1, 2, \ldots, k\}$. We form the eigenvector $\mathbf{u}_p(\lambda_{m,j})$ of C for the eigenvalue $\lambda_{m,j}$ via block merging using the blocks $\{\mathbf{u}^s_p(\lambda_{m,j})\}$.

For $i \neq j$ we have

$$\mathbf{u}_p(\lambda_{m,j}) \perp \mathbf{u}_q(\lambda_{t,i}) \text{ for all } p, q, m, t.$$

However, unlike the group circulant case, the eigenvector $\mathbf{u}_p(\lambda_{m,j})$ need not be orthogonal to $\mathbf{u}_q(\lambda_{m,j})$ for $p \neq q$. The group circulant matrix $C_G(\psi)$ admits mutually-orthogonal, ψ independent, kd_j^2-dimensional, C-invariant subspaces

$$\begin{aligned} U_j &= \operatorname{span}\{\mathbf{u}_p(\lambda_{m,j}) \,|\, p \in \{1, \ldots, d_j\}, m \in \{1, \ldots, kd_j\}\} \\ &= \operatorname{span}\{\rho^i_j(k,l) \,|\, k, l \in \{1, \ldots, d_j\}, i \in \{1, 2, \ldots, k\}\} \end{aligned}$$

for $j \in \{1, 2, \ldots, r\}$. The function $\rho^i_j(k,l)$ is created as follows. Consider a function $\rho_j(k,l)$ acting as $\rho_j(k,l)(g) = \rho_j(g)(k,l)$ for $g \in G$. Then choose a block location

$i \in \{1, 2, \ldots, k\}$ and merge $\rho_j(k,l)$ from the location i with the $k-1$ blocks of zeros of size $n \times 1$ to create a vector of size $kn \times 1$. Note that the functions $\rho_j^i(k,l)$ are mutually-orthogonal though not necessarily eigenvectors.

3 Example

We now consider an example of a block group circulant matrix C over the symmetric group S_3. The group S_3 consists of elements

$$g_0 = (1)\ ;\ g_1 = (12)\ ;\ g_2 = (13)\ ;\ g_3 = (23)\ ;\ g_4 = (123)\ ;\ g_5 = (132).$$

We have three irreducible representations, two of which are one-dimensional, ρ_1 is the identity map, ρ_2 is the map that assigns the value of 1 if the permutation is even and the value of -1 if the permutation is odd. Finally, we have ρ_3 defined by the following assignment

$$g_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ ;\ g_1 \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\ ;\ g_2 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\ ;\ g_3 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$g_4 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\ ;\ g_5 \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider the block group circulant matrix induced by the symbol

$$\Psi = \mathbf{c}_0 g_0 + \mathbf{c}_1 g_1 \text{ with } \mathbf{c}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \mathbf{c}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The induced block circulant matrix is given by (respecting the order of elements)

$$\begin{pmatrix} \mathbf{c}_0 & \mathbf{c}_1 & 0 & 0 & 0 & 0 \\ \mathbf{c}_1 & \mathbf{c}_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{c}_0 & 0 & \mathbf{c}_1 & 0 \\ 0 & 0 & 0 & \mathbf{c}_0 & 0 & \mathbf{c}_1 \\ 0 & 0 & \mathbf{c}_1 & 0 & \mathbf{c}_0 & 0 \\ 0 & 0 & 0 & \mathbf{c}_1 & 0 & \mathbf{c}_0 \end{pmatrix}.$$

The matrices $\widehat{\Psi}(j)$ are given by

$$\widehat{\Psi}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\ ;\ \widehat{\Psi}(2) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\ ;\ \widehat{\Psi}(3) = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of $\widehat{\Psi}(1)$ are $\lambda_{1,1} = 1$ and $\lambda_{2,1} = 2$ with $u_{\lambda_{1,1}} = (1,0)^T$ and $u_{\lambda_{2,1}} = (1,1)^T$. We collect 2 corresponding eigenvectors of C (non-normalized)

$$\begin{aligned} \mathbf{u}_1(\lambda_{1,1}) &= (1,0,1,0,1,0,1,0,1,0,1,0)^T \\ \mathbf{u}_1(\lambda_{2,1}) &= (1,1,1,1,1,1,1,1,1,1,1,1)^T. \end{aligned}$$

Note that the above eigenvectors span the C-invariant subspace U_1. The eigenvalues of $\widehat{\Psi}(2)$ are $\lambda_{1,2} = 1$ and $\lambda_{2,2} = 2$ with $u_{\lambda_{1,2}} = (1,0)^T$ and $u_{\lambda_{2,2}} = (1,-1)^T$. We collect 2 corresponding eigenvectors of C (non-normalized)

$$\begin{aligned} \mathbf{u}_1(\lambda_{1,2}) &= (1,0,-1,0,-1,0,-1,0,1,0,1,0)^T \\ \mathbf{u}_1(\lambda_{2,2}) &= (1,-1,-1,1,-1,1,-1,1,1,-1,1,-1)^T. \end{aligned}$$

Note that the above eigenvectors span the C-invariant subspace U_2. The eigenvalues of $\widehat{\Psi}(3)$ are $\lambda_{1,3} = \lambda_{2,3} = 1$ with multiplicity 2, and $\lambda_{3,3} = \lambda_{4,3} = 2$ with multiplicity 2 as well. We have $u_{\lambda_{1,3}} = (1,0,0,0)^T$, $u_{\lambda_{2,3}} = (0,1,0,0)^T$, $u_{\lambda_{3,3}} = (1,0,-1,0)^T$ and $u_{\lambda_{4,3}} = (1,1,0,1)^T$. As a result we collect 8 eigenvectors (non-normalized) of C corresponding to these eigenvalues

$$\begin{aligned} \mathbf{u}_1(\lambda_{1,3}) &= (1,0,-1,0,0,0,1,0,0,0,-1,0)^T \\ \mathbf{u}_2(\lambda_{1,3}) &= (0,0,0,0,-1,0,1,0,1,0,-1,0)^T \\ \mathbf{u}_1(\lambda_{2,3}) &= (0,0,1,0,-1,0,0,0,-1,0,1,0)^T \\ \mathbf{u}_2(\lambda_{2,3}) &= (1,0,1,0,0,0,-1,0,-1,0,0,0)^T \\ \mathbf{u}_1(\lambda_{3,3}) &= (1,-1,-1,1,0,0,1,-1,0,0,-1,1)^T \\ \mathbf{u}_2(\lambda_{3,3}) &= (0,0,0,0,-1,1,1,-1,1,-1,-1,1)^T \\ \mathbf{u}_1(\lambda_{4,3}) &= (1,0,0,1,-1,-1,1,0,-1,-1,0,1)^T \\ \mathbf{u}_2(\lambda_{4,3}) &= (1,1,1,1,-1,0,0,-1,0,-1,-1,0)^T. \end{aligned}$$

Note that the above eigenvectors span the C-invariant subspace U_3. We will explain how we obtained $u_2(\lambda_{4,3})$. Consider $\lambda_{4,3} = 2$ and $u_{\lambda_{4,3}} = (1,1,0,1)^T$, the corresponding eigenvector of $\widehat{\Psi}(3)$. Form $\widehat{\mathbf{u}^1}_2(\lambda_{4,3})$ by positioning $(1,1)^T$ in the second column and zero columns elsewhere. The inverse Fourier transform of $\widehat{\mathbf{u}^1}_2(\lambda_{4,3})$ is given by $(1,1,-1,0,0,-1)$. Next, form $\widehat{\mathbf{u}^2}_2(\lambda_{4,3})$ by positioning $(0,1)^T$ in the second column and zeros elsewhere. The inverse Fourier transform of $\widehat{\mathbf{u}^2}_2(\lambda_{4,3})$ is given by $(1,1,0,-1,-1,0)$. Now we merge and obtain

$$u_2(\lambda_{4,3}) = (1,1,1,1,-1,0,0,-1,0,-1,-1,0)^T.$$

Observe that for $i \neq j$ we have $u_p(\lambda_{m,j}) \perp u_q(\lambda_{t,i})$ for all choices of p, q, m, t, but for $p \neq q$ $u_p(\lambda_{m,j})$ need not be orthogonal to $u_q(\lambda_{m,j})$.

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