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On The Generalized Ulam-Hyers-Rassias Stability For Darboux Problem For Partial Fractional Implicit Differential Equations*

Saïd Abbas†, Mouffak Benchohra‡§

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Abstract

In the present paper we investigate some Ulam's type stability concepts for the Darboux problem of partial fractional implicit differential equations. An example is presented.

1 Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order, it is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [5], Hyers *et al.* [10], Kilbas *et al.* [16], Miller and Ross [17], Podlubny [18], Tarasov [21], the papers of Abbas *et al.* [1, 2, 3, 4, 6, 7], Vityuk *et al.* [23, 24], and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (For more details see [22]). The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces in [11]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [19] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to

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†10–2320, Rue de Salaberry, Montréal, QC H3M 1K9, Canada

‡Laboratory of Mathematics, University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria

§Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs [12, 13, 14] and the papers [8, 15, 20, 25, 26, 27].

In this paper, we discuss the Ulam stabilities for the following fractional partial implicit differential equation

$$\overline{D}_\theta^r u(x,y) = f(x,y,u(x,y),\overline{D}_\theta^r u(x,y)) \text{ if } (x,y) \in J := [0,a]\times[0,b], \tag{1}$$

where $a,b>0,\ \theta=(0,0),\ \overline{D}_\theta^r$ is the mixed regularized derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1],\ f: J\times E\times E\to E$ is a given function, E is a (real or complex) Banach space. This paper initiates the Ulam stabilities of the Darboux problem for hyperbolic implicit differential equations of fractional order.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Denote $L^1(J)$ the space of Bochner-integrable functions $u: J\to E$ with the norm

$$\|u\|_{L^1} = \int_0^a\int_0^b \|u(x,y)\|_E dydx,$$

where $\|.\|_E$ denotes a complete norm on E.

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into E, and $\mathcal{C}:=C(J)$ is the Banach space of all continuous functions from J into E with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J}\|u(x,y)\|_E.$$

Let $\theta=(0,0),\ r_1,\ r_2>0$ and $r=(r_1,r_2)$. For $f\in L^1(J)$, the expression

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dtds,$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi)=\int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi>0$.

In particular,

$$(I_\theta^\theta u)(x,y)=u(x,y),\ (I_\theta^\sigma u)(x,y)=\int_0^x\int_0^y u(s,t)dtds;\ \text{for almost all } (x,y)\in J,$$

where $\sigma=(1,1)$. For instance, $I_\theta^r u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1(J)$. Note also that when $u\in C(J)$, we have $(I_\theta^r u)\in C(J)$; moreover

$$(I_\theta^r u)(x,0)=(I_\theta^r u)(0,y)=0 \text{ for } x\in[0,a],\ y\in[0,b].$$

By $1-r$ we mean $(1-r_1,1-r_2)\in[0,1)\times[0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

DEFINITION 1. [23] Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression $D_\theta^r u(x,y) = (D_{xy}^2 I_\theta^{1-r} u)(x,y)$ and the Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_\theta^r u(x,y) = (I_\theta^{1-r} D_{xy}^2 u)(x,y)$.

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(x,y) = {}^c D_\theta^\sigma u(x,y) = (D_{xy}^2 u)(x,y) \text{ for almost all } (x,y) \in J.$$

DEFINITION 2. [24] For a function $u : J \to E$, we set

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$ of a function $u(x,y)$, we name the function

$$\overline{D}_\theta^r u(x,y) = D_\theta^r q(x,y).$$

Now, we consider the Ulam stability of fractional differential equation (1). Let $\Phi : J \to [0,\infty)$ be a continuous function. We consider the following inequality

$$\|\overline{D}_\theta^r u(x,y) - f(x,y,u(x,y), \overline{D}_\theta^r u(x,y))\|_E \le \Phi(x,y) \text{ if } (x,y) \in J. \tag{2}$$

DEFINITION 3. [20] Problem (1) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality (2), there exists a solution $v \in \mathcal{C}$ of problem (1) with

$$\|u(x,y) - v(x,y)\|_E \le c_{f,\Phi} \Phi(x,y) \text{ for } (x,y) \in J.$$

So, the generalized Ulam-Hyers-Rassias stability of the fractional differential equations is a special type of data dependence of the solutions of fractional differential equations.

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

LEMMA 1. (Gronwall lemma) [9] Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that

$$\upsilon(x,y) \le \omega(x,y) + c \int_0^x \int_0^y \frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dt ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(x,y) \le \omega(x,y) + \delta c \int_0^x \int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dt ds,$$

for every $(x,y) \in J$.

3 Main Results

In this section, we present conditions for the Ulam stability of problem (1). Consider the following Darboux problem of partial differential equations

$$\begin{cases} \overline{D}_{\theta}^{r}u(x,y) = f(x,y,u(x,y),\overline{D}_{\theta}^{r}u(x,y)); \text{ if } (x,y) \in J, \\ u(x,0) = \varphi(x); \ x \in [0,a], \\ u(0,y) = \psi(y); \ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{3}$$

where $\varphi : [0,a] \to E, \ \psi : [0,b] \to E$ are given absolutely continuous functions. In the sequel, we need the following Lemma:

LEMMA 2. [24] Let a function $f(x,y,u,z) : J \times E \times E \to E$ be continuous. Then problem (3) is equivalent to the problem of the solution of the equation

$$g(x,y) = f(x,y,\mu(x,y) + I_{\theta}^{r}g(x,y), g(x,y)), \tag{4}$$

and if $g \in \mathcal{C}$ is the solution of this equation, then $u(x,y) = \mu(x,y) + I_{\theta}^{r}g(x,y)$, where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

THEOREM 1. [7] Assume that the following hypotheses hold:

(H_1) The function $f : J \times E \times E \to E$ is continuous.

(H_2) there exist constants $k_f > 0$ and $0 < l_f < 1$ such that

$$\|f(x,y,u,z) - f(x,y,v,w)\|_E \le k_f\|u-v\|_E + l_f\|z-w\|_E, \ u,v,w,z \in E, \ (x,y) \in J.$$

If

$$\frac{k_f a^{r_1} b^{r_2}}{(1-l_f)\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{5}$$

then there exists a unique solution for problem (3) on J.

THEOREM 2. Assume that the assumptions (H_1), (H_2) and the following hypothesis hold:

(H_3) $\Phi \in L^1(J,[0,\infty))$ and there exists $\lambda_\Phi > 0$ such that, for each $(x,y) \in J$ we have

$$(I_{\theta}^{r}\Phi)(x,y) \le \lambda_\Phi \Phi(x,y).$$

If the condition (5) holds, then equation (1) is generalized Ulam-Hyers-Rassias stable.

PROOF. Let $u \in \mathcal{C}$ be a solution of the inequality (2). By Theorem 1, there v is a unique solution of the problem (3). Then for each $(x,y) \in J$, we have

$$v(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g_v(s,t)dtds;$$

where $g_v \in \mathcal{C}$ such that

$$g_v(x,y) = f(x,y,v(x,y),g_v(x,y)).$$

By differential inequality (2), for each $(x,y) \in J$, we have that

$$\begin{aligned}\|u(x,y) - \mu(x,y) - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)dtds\|_E \\ \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \Phi(s,t)dtds,\end{aligned}$$

where $g \in \mathcal{C}$ such that

$$g(x,y) = f(x,y,u(x,y),g(x,y)).$$

Thus, by (H_3) for each $(x,y) \in J$, we get

$$\begin{aligned}\|u(x,y) - \mu(x,y) - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ \times g(s,t)dtds\|_E \leq \lambda_\Phi \Phi(x,y).\end{aligned}$$

Hence for each $(x,y) \in J$, it follows that

$$\begin{aligned}\|u(x,y) - v(x,y)\|_E \leq & \|u(x,y) - \mu(x,y) \\ & - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)dtds\|_E \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ & \times \|g(s,t) - g_v(s,t)\|_E dtds.\end{aligned}$$

By (H_2), we get

$$\|g(x,y) - g_v(x,y)\|_E \leq k_f \|u(x,y) - v(x,y)\|_E + l_f \|g(x,y) - g_v(x,y)\|_E.$$

Then

$$\|g(x,y) - g_v(x,y)\|_E \leq \frac{k_f}{1-l_f} \|u(x,y) - v(x,y)\|_E.$$

Hence

$$\begin{aligned}\|u(x,y) - v(x,y)\|_E \quad \leq \quad & \lambda_\Phi \Phi(x,y) + \frac{k_f}{(1-l_f)\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ & \times \|u(s,t) - v(s,t)\|_E dtds.\end{aligned}$$

From Lemma 1, there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\begin{aligned}\|u(x,y) - v(x,y)\|_E &\leq \lambda_\Phi \Phi(x,y) + \frac{\delta k_f \lambda_\Phi}{(1-l_f)\Gamma(r_1)\Gamma(r_2)} \\ &\quad \times \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\Phi(s,t)dtds \\ &\leq \left(1 + \frac{\delta k_f \lambda_\Phi}{1-l_f}\right)\lambda_\Phi \Phi(x,y) := c_{f,\Phi}\Phi(x,y).\end{aligned}$$

Finally, equation (1) is generalized Ulam-Hyers-Rassias stable.

4 An Example

Let

$$E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\},$$

be the Banach space with norm

$$\|w\|_E = \sum_{n=1}^{\infty} |w_n|.$$

Consider the following infinite system of partial hyperbolic fractional implicit differential equations of the form

$$\overline{D}_\theta^r u(x,y) = f(x,y,u(x,y), \overline{D}_\theta^r u(x,y)) \text{ if } (x,y) \in [0,1]\times[0,1], \tag{6}$$

where $(r_1, r_2) \in (0,1] \times (0,1]$,

$u = (u_1, u_2, \ldots, u_n, \ldots)$, $\overline{D}_\theta^r u = (\overline{D}_\theta^r u_1, \overline{D}_\theta^r u_2, \ldots, \overline{D}_\theta^r u_n, \ldots)$, $f = (f_1, f_2, \ldots, f_n, \ldots)$,

and

$$f_n(x,y,u_n, \overline{D}_\theta^r u_n) = \frac{1}{10e^{x+y+3}(1 + |u_n| + |\overline{D}_\theta^r u_n|)} \text{ for } (x,y) \in [0,1]\times[0,1] \text{ and } n \in \mathbb{N}.$$

Clearly, the function f is continuous. For each $n \in \mathbb{N}$, $u, v, \overline{u}, \overline{v} \in E$ and $(x,y) \in [0,1]\times[0,1]$ we have that

$$|f_n(x,y,u_n(x,y),v_n(x,y)) - f_n(x,y,\overline{u}_n(x,y),\overline{v}_n(x,y))| \leq \frac{1}{10e^3}(|u_n - \overline{u}_n| + |v_n - \overline{v}_n|).$$

Thus, for each $u, v, \overline{u}, \overline{v} \in E$ and $(x,y) \in [0,1]\times[0,1]$, we get

$$\begin{aligned}&\|f(x,y,u(x,y),v(x,y)) - f(x,y,\overline{u}(x,y),\overline{v}(x,y))\|_E \\ &= \sum_{n=1}^{\infty} |f_n(x,y,u(x,y),v(x,y)) - f_n(x,y,\overline{u}(x,y),\overline{v}(x,y))| \\ &\leq \frac{1}{10e^3}\left(\sum_{n=1}^{\infty}|u_n - \overline{u}_n| + \sum_{n=1}^{\infty}|v_n - \overline{v}_n|\right) \\ &= \frac{1}{10e^3}(\|u - \overline{u}\|_E + \|v - \overline{v}\|_E).\end{aligned}$$

Hence condition (H_2) is satisfied with $k_f = l_f = \frac{1}{10e^3}$. We shall show that condition (5) holds with $a = b = 1$. Indeed,

$$\frac{k_f a^{r_1} b^{r_2}}{(1-l_f)\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{(10e^3-1)\Gamma(1+r_1)\Gamma(1+r_2)} < \frac{4}{10e^3-1} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. The hypothesis (H_3) is satisfied with $\Phi(x,y) = xy^2$ and $\lambda_\Phi = \frac{2}{\Gamma(2+r_1)\Gamma(3+r_2)}$. Indeed, for each $(x,y) \in [0,1] \times [0,1]$ we get

$$(I_\theta^r \Phi)(x,y) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+r_1)\Gamma(3+r_2)} x^{1+r_1} y^{2+r_2} \leq \frac{2}{\Gamma(2+r_1)\Gamma(3+r_2)} xy^2 = \lambda_\Phi \Phi(x,y).$$

Consequently, Theorem 2 implies that equation (6) is generalized Ulam-Hyers-Rassias stable.

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