Further Solutions Of The General Abel Equation Of 
The Second Kind: Use Of Julia’s Condition*

Lazhar Bougoffa†

Received 25 October 2014

Abstract

In this paper, we propose a direct method to obtain an implicit solution of
the Abel equation of the second kind

\[ [g_0(x) + g_1(x)u] u' = f_0(x) + f_1(x)u + f_2(x)u^2. \]

We first reduce it into an equivalent equation, and assume that the coefficient
functions \( f_i(x), i = 0, 1, 2 \) and \( g_i(x), i = 0, 1 \) satisfy the well-known Julin’s
condition. Therefore the given Abel equation can be transformed into a first-
order linear differential equation, which can be easily solved, and then the implicit
solutions of this equation are obtained.

1 Introduction

The Abel equation of the second kind has the general form

\[ [g_0(x) + g_1(x)u] u' = f_0(x) + f_1(x)u + f_2(x)u^2. \] (1)

This equation was derived in 1829 in the context of the studies of N.H. Abel [1] on
the theory of elliptic functions. The first important well-known result in the analysis of the
Abel equation is that: If \( g_0, g_1 \in C^1(a, b) \), \( g_1(x) \neq 0 \) and \( g_0(x) + g_1(x) \neq 0 \), then Abel’s
differential equation of the second kind can be reduced to Abel’s differential equation of
the first kind by substituting \( g_0(x) + g_1(x)u = \frac{1}{z} \). The second important result is that:
Eq.(1) can be reduced to the canonical form [2], by using various admissable functional
transformations,

\[ uu' - u = \Phi(x), \]

where the function \( \Phi(x) \) is defined parametrically. It is often very difficult, if not impossible,
to find explicit solutions of such nonlinear differential equations. But a number
of solutions of the Abel equation of the second kind can be obtained by assuming that
the coefficients \( f_i(x), i = 0, 1, 2 \) and \( g_i(x), i = 0, 1 \) satisfy some particular constraints.
In 1933, the French mathematician Gaston Julia [3] proved that the equation

\[ d\frac{Au^2 + Bu + C}{Du + E} dx = 0, \]

*Mathematics Subject Classifications: 35F20, 34L30.
†Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Faculty of Science, Department of
Mathematics, P.O. Box 90950, Riyadh 11623, Saudi Arabia
for $A$, $B$, $C$, $D$ and $E$ functions of $x$, has an implicit solution if the condition

$$E(2A - D') = D(B - E'), \quad D \neq 0$$

is satisfied. Then the solution is implicitly given by

$$D \frac{u^2}{2} e^{\int \frac{2A - D'}{D} dx} + Eue^{\int \frac{2A - D'}{D} dx} + \int ce^{\int \frac{2A - D'}{D} dx} dx = \lambda,$$

where $\lambda$ is any constant. The Julia’s result can be summarized by the following theorem:

**THEOREM 1.** For the general form of the Abel equation of the second kind. If the coefficients of Eq.(1) satisfy the functional relation

$$g_0(x)(2f_2(x) + g_1(x)) = g_1(x)(f_1(x) + g_0(x)), \quad (2)$$

where $g_1(x) \neq 0$, then the implicit solutions of Eq.(1) are given by

$$\frac{2g_0(x)u + g_1(x)u^2}{2g_1(x)J(x)} = \int \frac{f_0(x)}{g_1(x)J(x)} dx + c, \quad (3)$$

where $c$ is an integration constant and

$$J(x) = \exp(\int \frac{2f_2(x)}{g_1(x)} dx).$$

A new functional relation between the variable coefficients that can lead to the general solutions of Eq.(1) is presented in [4] as follows:

**THEOREM 2.** For the general form of the Abel equation of the second kind Eq.(1). If there exists a constant $\lambda$ such that

$$2B_1(x)g_0(x) = \lambda B_2(x)g_1(x), \quad g_i(x) \neq 0, \quad i = 0, 1,$$

then Eq.(1) admits the general solution

$$B_1(x)u^2 + \lambda B_2(x)u = 2 \int \frac{f_0(x)}{g_1(x)} B_1(x) dx + c,$$

where $B_1(x) = \exp(-2 \int \frac{f_2(x)}{g_1(x)} dx)$ and $B_2(x) = \exp(- \int \frac{f_1(x)}{g_0(x)} dx)$.

In this note, a new technique is analyzed to establish new different solutions of the general Abel equation of the second kind, we first reduce it into an equivalent equation, and then we formulate the relations between the coefficient functions $f_i(x), \quad i = 0, 1, 2$ and $g_i(x), \quad i = 0, 1$ to obtain the well-known Julia’s condition. This leads to a first-order linear differential equation, which can be solved in a closed form. Therefore the given Abel equation can be solved implicitly.
2 Main Result

Here, we prove the following result

THEOREM 3. For the general form of the Abel equation of the second kind. If the coefficients of Eq.(1) satisfy the Julia’s condition (2), where $g_0, g_1 \in C^1(a, b)$, $g_0(x) \neq 0$ and $f_i(x) \in C(a, b)$, $i = 0, 1, 2$. Then the general solution of Eq.(1) can be exactly obtained by

$$
\frac{2g_0(x)u + g_1(x)u^2}{2g_0(x)L(x)} = \int \frac{f_0(x)}{g_0(x)L(x)} dx + c, \tag{4}
$$

where $c$ is a constant and

$$
L(x) = e^{\int \frac{f_1(x)}{g_0(x)} dx}.
$$

REMARK. Clearly, this solution is completely different from the one obtained by Julia [3], which can be regarded as a new implicit form of the Abel equation.

PROOF. First of all, we begin our approach by writing Eq.(1) in an equivalent form as

$$
u' + \frac{g_1}{g_0} uu' = \frac{f_0}{g_0} + \frac{f_1}{g_0} u + \frac{f_2}{g_0} u^2 \tag{5}\$$

in view of

$$
\left( \frac{g_1}{g_0} u^2 \right)' = 2\frac{g_1}{g_0} uu' + \left( \frac{g_1}{g_0} \right)^2.
$$

Thus Eq.(5) can be written as

$$
u' + \left( \frac{g_1}{2g_0} u^2 \right)' = \frac{f_0}{g_0} + \frac{f_1}{g_0} u + \left[ \frac{f_2}{g_0} + \left( \frac{g_1}{2g_0} \right)' \right] u^2.
$$

It follows

$$
\left( u + \frac{g_1}{2g_0} u^2 \right)' = \frac{f_0}{g_0} + \frac{f_1}{g_0} \left[ u + \frac{f_2}{g_0} + \left( \frac{g_1}{2g_0} \right)' \right] u^2. \tag{6}
$$

We assume now that

$$
\frac{f_2}{f_1} + \left( \frac{g_1}{2g_0} \right)' = \frac{g_1}{2g_0},
$$

or

$$
\frac{f_2}{f_1} + \frac{g_1'}{2f_1} - \frac{g_1g_0'}{2f_1g_0} = \frac{g_1}{2g_0}.
$$

A direct calculation produces the following equation

$$
g_0 (2f_2 + g_1'(x)) = g_1 \left( f_1 + g_0' \right),
$$
which is indeed the Julia’s condition (2). Eq.(6) is then easily integrated to give a solution of Eq.(1). The substitution of Eq. (2) into Eq. (6) leads to the following equation

$$\left(u + \frac{g_1}{2g_0}u^2\right)' = \frac{f_0}{g_0} + \frac{f_1}{g_0} \left[u + \frac{g_1}{2g_0}u^2\right].$$

(7)

Let $\psi = u + \frac{g_1}{2g_0}u^2$. Thus Eq.(7) becomes

$$\psi' = \frac{f_0(x)}{g_0(x)} + \frac{f_1(x)}{g_0(x)} \psi,$$

(8)

which is a first-order linear differential equation, and we can obtain its explicit solution form

$$\psi(x) = \frac{\int f_0(x) \exp\left(-\int \frac{f_1(x)}{g_0(x)} dx\right) dx + c}{e^{-\int \frac{f_1(x)}{g_0(x)} dx}}.$$ 

Hence

$$u + \frac{g_1(x)}{2g_0(x)} u^2 = \frac{\int f_0(x) \exp\left(-\int \frac{f_1(x)}{g_0(x)} dx\right) dx + c}{e^{-\int \frac{f_1(x)}{g_0(x)} dx}}.$$ 

This completes the proof of the theorem.

References


