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A Note On The Norm Of Oblique Projections*

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Abstract

The purpose of this note is to give an alternative version of T. Katos proof on the norm of oblique projections in a Hilbert space.

1 Introduction and Proof of Lemma

D. B. Szyld collected in [8] several proofs of the identity $\|P\| = \|I - P\|$ for nontrivial linear projections P on a Hilbert space, see also [7] and [2, Example 5.8]. It has found numerous applications, see for instance [5, 3, 9, 1, 6]. We provide here a somewhat simplified version of the proof given by T. Kato in [4, Lemma 4]. The difference is in the choice of the vector y.

LEMMA. Let H be a Hilbert space. Let $P : H \to H$ be a linear idempotent operator such that $0 \neq P^2 = P \neq I$. Then $\|P\| = \|I - P\|$.

PROOF. Since $P^2 = P$ and $(I-P)^2 = I - P$, both norms are no less than one. If $\|P\| = 1 = \|I - P\|$, there is nothing to prove, so let $x \in H$ be nonzero with, say, $\alpha := \|Px\| / \|x\| > 1$. Then $y := \alpha^2 x - Px$ is nonzero due to $Py \neq 0$. Moreover, the identity $\alpha \|(I-P)\, x\| = \|y\|$ is easily seen by expanding the square of the norms. The definition of y, the fact that $P^2 = P$, and this identity together yield

$$\|(I-P)\, y\| = \left\|\alpha^2 x - \alpha^2 Px\right\| = \alpha^2 \|(I-P)\, x\| = \alpha \|y\|.$$

Since $x \neq 0$ was arbitrary, as long as $\alpha > 1$, we divide the latter identity by $\|y\|$, and take the supremum over all such x to obtain

$$\|I - P\| \geq \sup_x \frac{\|(I-P)\, y\|}{\|y\|} = \sup_x \alpha = \|P\|,$$

where y and α depend on x. Therefore, $\|I - P\| \geq \|P\| > 1$. Swapping the roles of P and $I - P$ concludes the proof.

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