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Oscillatory Behavior Of A Higher-Order Nonlinear Neutral Type Functional Difference Equation With Oscillating Coefficients*

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Abstract

In this work, we shall consider oscillation of bounded solutions of higher-order nonlinear neutral delay difference equations of the following type

$$\Delta^n \left[y\left(t\right)+p\left(t\right)f\left(y\left(\tau\left(t\right)\right)\right)\right]+q\left(t\right)h\left(y\left(\sigma\left(t\right)\right)\right)=0, t\in\mathbb{N},$$

where $n\in\{2,3,\ldots\}$ is fixed and can take both odd and even values, $\{p(t)\}_{t=1}^{\infty}$ is a sequence of reals such that $\lim_{t\to\infty}p(t)=0$, $\{q(t)\}_{t=1}^{\infty}$ is a nonnegative sequence of reals, and $\{\tau(t)\}_{t=1}^{\infty}$ and $\{\sigma(t)\}_{t=1}^{\infty}$ are sequences of integers tending to infinity asymptotically and bounded above by $\{t\}_{t=1}^{\infty}$, and $f,h\in C(\mathbb{R},\mathbb{R})$.

1 Introduction

We consider the higher-order nonlinear difference equation of the form

$$\Delta^n \left[y\left(t\right)+p\left(t\right)f\left(y\left(\tau\left(t\right)\right)\right)\right]+q\left(t\right)h\left(y\left(\sigma\left(t\right)\right)\right)=0 \text{ for } t\in\mathbb{N}, \tag{1}$$

where $n\in\{2,3,\ldots\}$ is fixed, $\mathbb{N}=\{0,1,2,\ldots\}$, $p:\mathbb{N}\to\mathbb{R}=(-\infty,\infty)$, $\{p(t)\}_{t=1}^{\infty}$ is a sequence of real such that $\lim_{t\to\infty}p(t)=0$, and it is an oscillating function; $q:\mathbb{N}\to[0,\infty)$, $\tau(t):\mathbb{N}\to\mathbb{Z}$ ($\mathbb{Z}$ denotes the set of integers) with $\tau(t)\le t$, and $\tau\left(t\right)\to\infty$ as $t\to\infty$, $\sigma(t):\mathbb{N}\to\mathbb{Z}$($\mathbb{Z}$ denotes the set of integers) with $\sigma(t)\le t$, for all $t\in\mathbb{N}$ and $\sigma(t)\to\infty$ as $t\to\infty$, $f(u),h(u)\in C(\mathbb{R},\mathbb{R})$ are nondecreasing functions (1), $uf(u)>0$ and $uh(u)>0$, for all $u\neq0$, we mean any function $y(t):\mathbb{Z}\to\mathbb{R}$, which is defined for all $t\ge\min_{i\ge0}\{\tau\left(i\right),\sigma(i)\}$, and satisfies equation (1) for sufficiently large t. As it is customary, a solution $\{y\left(t\right)\}$ is said to be oscillatory if the terms $y(t)$ of the sequence are not eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory. In this paper, we restrict our attention to real valued solutions y.

Recently, much research has been done on the oscillatory and asymptotic behavior of solutions of higher-order delay and neutral type difference equations. The results

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obtained here are an extension of work in [7]. Most of the known results are for special cases of equation (1) and related equations; see, for example, [1, 2, 3, 16].

The purpose of this paper is to study oscillatory behavior of bounded solutions of solutions of equation (1). For the general theory of difference equations, one can refer to [1, 2, 3, 10, 11, 12, 15]. Many references to applications of the difference equations can be found in [10, 11, 12].

For the sake of convenience, we let $\mathbb{N}(a) = \{a, a+1, \ldots\}$, $\mathbb{N}(a, b) = \{a, a+1, \ldots, b\}$, and the function $z(t)$ is defined by

$$z(t) = y(t) + p(t)f\left(y\left(\tau\left(t\right)\right)\right). \tag{2}$$

2 Some Auxiliary Lemmas

In this section, we present the known results.

LEMMA 1 ([2]). Let $y(t)$ be defined for $t \geq t_0 \in \mathbb{N}$, and $y(t) > 0$ with $\Delta^n y(t)$ of constant sign for $t \geq t_0$, $n \in \mathbb{N}(1)$, and not identically zero. Then there exists an integer $m \in [0, n]$ satisfying either $(n+m)$ is even for $\Delta^n y(t) \geq 0$ or $(n+m)$ is odd for $\Delta^n y(t) \leq 0$ such that

(i) if $m \leq n-1$ implies $(-1)^{m+i}\Delta^i y(t) > 0$ for all $t \geq t_0$ and $m \leq i \leq n-1$,

(ii) if $m \geq 1$ implies $\Delta^i y(t) > 0$ for all large $t \geq t_0$ and $1 \leq i \leq m-1$.

LEMMA 2 ([2]). Let $y(t)$ be defined for $t \geq t_0$, and $y(t) > 0$ with $\Delta^n y(t) \leq 0$ for $t \geq t_0$ and not identically zero. Then there exists a large $t_1 \geq t_0$, such that

$$y\left(t\right) \geq \frac{1}{(n-1)!}\left(t-t_1\right)^{n-1}\Delta^{n-1}y(2^{n-m-1}t), \quad t \geq t_1,$$

where m is defined as in Lemma 2. Furthermore, if $y(t)$ is increasing, then

$$y\left(t\right) \geq \frac{1}{(n-1)!}\left(\frac{t}{2^{n-1}}\right)^{n-1}\Delta^{n-1}y(t), \quad t \geq 2^{n-1}t_1.$$

3 Main Results

In this section, we present main results and give some examples.

THEOREM 1. Assume than n is odd and the following assertions (C_1)–(C_2) hold:

(C_1) $\lim_{t\to\infty} p(t) = 0$,

(C_2) $\sum_{s=t_0}^{\infty} s^{n-1}q\left(s\right) = \infty$.

Then every bounded solution of equation (1) either is oscillatory or tends to zero as $t \to \infty$.

PROOF. Assume that equation (1) has a bounded nonoscillatory solution y. Without loss of generality, assume that y is eventually positive (the proof is similar when y is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Furthermore, we assume that $y(t)$ does not to zero as $t \to \infty$. By (1) and (2), we have that

$$\Delta^n z(t) = -q(t)\, h(y(\sigma(t))) \leq 0 \text{ for } t \geq t_1. \tag{3}$$

That is, $\Delta^n z(t) \leq 0$. It follows that $\Delta^\alpha z(t)$ for $\alpha = 0, 1, 2, \ldots, n-1$ is strictly monotone and eventually of constant sign. Since $\lim_{t\to\infty} p(t) = 0$, there exists $t_2 \geq t_1$ such that $z(t) > 0$ for $t \geq t_2$. Since y is bounded, and by virtue of (C_1) and (2), there exists $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Because n is odd, $z(t)$ is bounded and $m = 0$ (otherwise, $z(t)$ is not bounded by Lemma 1), there exists $t_4 \geq t_3$ such that for $t \geq t_4$, we have $(-1)^i \Delta^i z(t) > 0$ for $i = 0, 1, 2, \ldots, n-1$. In particular, since $\Delta z(t) < 0$ for $t \geq t_4$, z is decreasing. Since z is bounded, we obtain that $\lim_{t\to\infty} z(t) = L$ where $-\infty < L < \infty$. Assume that $0 \leq L < \infty$. Let $L > 0$. Then there exist a constant $c > 0$ and t_5 with $t_5 \geq t_4$ such that $z(t) > c > 0$ for $t \geq t_5$. Since y is bounded, $\lim_{t\to\infty} p(t) f(y(\tau(t))) = 0$ by (C_1). Therefore, there exist a constant $c_1 > 0$ and t_6 with $t_6 \geq t_5$ such that

$$y(t) = z(t) - p(t) f(y(\tau(t))) > c_1 > 0 \text{ for } t \geq t_6.$$

So we may find t_7 with $t_7 \geq t_6$ such that $y(\sigma(t)) > c_1 > 0$ for $t \geq t_7$. From (3), we have

$$\Delta^n z(t) \leq -q(t)\, h(c_1) \ \text{ for } t \geq t_7. \tag{4}$$

If we multiply (4) by t^{n-1}, and summing it from t_7 to $t-1$, we obtain

$$F(t) - F(t_7) \leq -h(c_1) \sum_{s=t_7}^{t-1} q(s)\, s^{n-1}, \tag{5}$$

where

$$F(t) = \sum_{\gamma=2}^{n-1} (-1)^\gamma \Delta^\gamma t^{n-1} \Delta^{n-\gamma-1} z(t+\gamma).$$

Since $(-1)^i \Delta^i z(t) > 0$ for $i = 0, 1, 2, \ldots, n-1$ and $t \geq t_4$, we have $F(t) > 0$ for $t \geq t_7$. From (5), we have

$$-F(t_7) \leq -h(c_1) \sum_{s=t_7}^{t-1} q(s)\, s^{n-1}.$$

By (C_2), we obtain

$$-F(t_7) \leq -h(c_1) \sum_{s=t_7}^{t-1} q(s)\, s^{n-1} = -\infty \text{ as } t \to \infty.$$

This is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{t\to\infty} z(t) = 0$. Since y is bounded, and by virtue of (C_1) and (2), we obtain

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) - \lim_{t\to\infty} p(t) f(y(\tau(t))) = 0.$$

Now, let us consider the case of $y(t) < 0$ for $t \geq t_1$. By (1) and (2),

$$\Delta^n z(t) = -q(t) h(y(\sigma(t))) \geq 0 \text{ for } t \geq t_1.$$

That is, $\Delta^n z(t) \geq 0$. It follow that $\Delta^\alpha z(t)$ for $\alpha = 0, 1, 2, \ldots, n-1$ is strictly monotone and eventually constant sign. Since $\lim_{t\to\infty} p(t) = 0$, there exists $t_2 \geq t_1$, such that $z(t) < 0$ for $t \geq t_2$. Since $y(t)$ is bounded, by virtue of (C_1) and (2), there exists $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Assume that $x(t) = -z(t)$. Then $\Delta^n x(t) = -\Delta^n z(t)$. Therefore, $x(t) > 0$ and $\Delta^n x(t) \leq 0$ for $t \geq t_3$. From this, we observe that $x(t)$ is bounded. Because n is odd, $x(t)$ is bounded and $m = 0$ (otherwise, $x(t)$ is not bounded by Lemma 1) there exists $t_4 \geq t_3$ such that $(-1)^i \Delta^i x(t) > 0$ for $i = 0, 1, 2, \ldots, n-1$ and $t \geq t_4$. That is, $(-1)^i \Delta^i z(t) < 0$ for $i = 0, 1, 2, \ldots, n-1$ and $t \geq t_4$. In particular, we have $\Delta z(t) > 0$ for $t \geq t_4$. Therefore, $z(t)$ is increasing. So, we can assume that $\lim_{t\to\infty} z(t) = L$ where $-\infty < L \leq 0$. As in the proof of $y(t) > 0$, we may prove that $L = 0$. As for the rest, it is similar to the case $y(t) > 0$. That is, $\lim_{t\to\infty} y(t) = 0$. This contradicts our assumption. Hence, the proof is completed.

THEOREM 2. Assume that n is even and the following condition (C_3) holds:

(C_3) there exists a function $H : \mathbb{R} \to \mathbb{R}$ such that H is continuous and nondecreasing, and satisfies the inequality

$$-H(-uv) \geq H(uv) \geq KH(u)H(v) \quad \text{for } u, v > 0,$$

where K is a positive constant, and

$$|h(u)| \geq |H(u)|, \quad \frac{H(u)}{u} \geq \gamma > 0 \quad \text{and} \quad H(u) > 0 \quad \text{for } u \neq 0.$$

and every bounded solution of the first-order delay difference equation

$$\Delta w(t) + q(t) K\gamma H\left(\frac{1}{2}\frac{1}{(n-1)!}\left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right) w(\sigma(t)) = 0 \tag{6}$$

is oscillatory.

Then every bounded solution of equation (1) is either oscillatory or tends to zero as $t \to \infty$.

PROOF. Assume that equation (1) has a bounded nonoscillatory solution y. Without loss of generality, assume that y is eventually positive(the proof is similar when y is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Furthermore, suppose that y does not tend to zero as $t \to \infty$. By (1) and (2), we have

$$\Delta^n z(t) = -q(t) h(y(\sigma(t))) \leq 0 \text{ for } t \geq t_1. \tag{7}$$

It follows that $\Delta^\alpha z(t)$ for $\alpha = 0, 1, 2, \ldots, n-1$ is strictly monotone and eventually of constant sign. Since y is bounded and does not tend to zero as $t \to \infty$, and by virtue of (C_1), $\lim_{t\to\infty} p(t) f(y(\tau(t))) = 0$. Then we can find a $t_2 \geq t_1$ such that $z(t) = y(t) + p(t) f(y(\tau(t))) > 0$ eventually and $z(t)$ is also bounded for sufficiently large $t \geq t_2$. Because n is even, $(n+m)$ odd for $\Delta^n z(t) \leq 0$, $z(t) > 0$ is bounded and $m = 1$ (otherwise, $z(t)$ is not bounded by Lemma 1) there exists $t_3 \geq t_2$ such that

$$(-1)^{i+1} \Delta^i z(t) > 0 \text{ for } t \geq t_3 \text{ and } i = 0, 1, 2, \ldots, n-1. \tag{8}$$

In particular, since $\Delta z(t) > 0$ for $t \geq t_3$, z is increasing. Since y is bounded, $\lim_{t\to\infty} p(t) f(y(\tau(t))) = 0$ by (C_1). Then there exists $t_4 \geq t_3$ by (2) such that

$$y(t) = z(t) - p(t) f(y(\tau(t))) \geq \frac{1}{2} z(t) > 0 \text{ for } t \geq t_4.$$

We may find a $t_5 \geq t_4$ such that

$$y(\sigma(t)) \geq \frac{1}{2} z(\sigma(t)) > 0 \text{ for } t \geq t_5. \tag{9}$$

From (7) and (9), we can obtain the result of

$$\Delta^n z(t) + q(t) h\left(\frac{1}{2} z(\sigma(t))\right) \leq 0 \text{ for } t \geq t_5. \tag{10}$$

Since $z(t)$ is defined for $t \geq t_2$, we apply directly Lemma 2 (second part, since z is positive and increasing) to obtain that $z(t) > 0$ with $\Delta^n z(t) \leq 0$ for $t \geq t_2$ and not identically zero. It follows from Lemma 2 that

$$y(\sigma(t)) \geq \frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1} \Delta^{n-1} z(\sigma(t)) \text{ for } t \geq 2^{n-1} t_1. \tag{11}$$

Using (C_3) and (9), we find that for $t \geq t_6 \geq t_5$,

$$\begin{aligned}
h(y(\sigma(t))) &\geq H(y(\sigma(t))) \\
&\geq H\left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1} \Delta^{n-1} z(\sigma(t))\right) \\
&\geq KH\left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right) H(\Delta^{n-1} z(\sigma(t))) \\
&\geq K\gamma H\left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right) \Delta^{n-1} z(\sigma(t)).
\end{aligned}$$

It follows from (7) and the above inequality, that $\{\Delta^{n-1} z(t)\}$ is an eventually positive solution of

$$\Delta w(t) + q(t) K\gamma H\left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right) w(\sigma(t)) \leq 0.$$

By a well-know result (see Theorem 3.1 in [5]), the difference equation

$$\Delta w(t) + q(t)K\gamma H\left(\frac{1}{2}\frac{1}{(n-1)!}\left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right)w(\sigma(t)) = 0 \text{ for } t \geq t_7 \geq t_6$$

has an eventually positive solution. This contradicts the fact that (1) is oscillatory, and the proof is completed.

Thus, from Theorem 2 and Theorem 2.3 in [6] (see also Example 3.2 in [6]), we can obtain the following corollary.

COROLLARY 1. If

$$\liminf_{t\to\infty}\sum_{s=\sigma(t)}^{t-1} q(s)H\left(\frac{1}{2}\frac{1}{(n-1)!}\left(\frac{\sigma(s)}{2^{n-1}}\right)^{n-1}\right) > \frac{1}{\mathrm{e}K\gamma}, \tag{12}$$

then every bounded solution of equation (1.1) either is oscillatory or tends to zero as $t \to \infty$.

When $p(t) \equiv 0$ and $n = 2$, Corollary 3 yields that if

$$\liminf_{t\to\infty}\sum_{s=\sigma(t)}^{t-1} q(s)H\left(\frac{1}{4}\sigma(s)\right) > \frac{1}{\mathrm{e}K\gamma},$$

then

$$\Delta^2 y(t) + q(t)\,h(y(\sigma(t))) = 0 \text{ for } t \geq t_0 \tag{13}$$

is oscillatory. These results have been established in [6, 12, 13] and the references cited therein.

EXAMPLE 1. We consider difference equation of the form

$$\Delta^3\left[y(t) + e^{-5t^2}\sin t\left[y^2(t-5) + 2y(t-5)\right]\right] + t^2y^2(t-3) = 0 \text{ for } t \geq 2, \tag{14}$$

where $n = 3$, $q(t) = t^2$, $\sigma(t) = t-3$, $\tau(t) = t-5$, $p(t) = e^{-5t^2}\sin t$, $f(y) = y^2 - 2y$, and $h(y) = y^2$. Hence, we have

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\frac{1}{\mathrm{e}^{5t^2}}\sin t = 0 \text{ and } \sum_{s=t_0}^{\infty} s^{n-1}q(s) = \sum_{s=t_0}^{\infty} s^4 = \infty.$$

Since Conditions $(C1)$ and $(C2)$ of the Theorem 1 are satisfied, every bounded solution of (14) oscillates or tends to zero at infinity.

EXAMPLE 2. We consider difference equation of the form

$$\Delta^4\left[y(t) + \left(-\frac{1}{2}\right)^t y(t-2)\right] + \frac{1}{t^2}y^3(t-3) = 0, \tag{15}$$

where $n = 4$, $\tau(t) = t - 2$, $p(t) = (-1/2)^t$, $q(t) = 1/t^2$, $\sigma(t) = t - 3$, and $h(y) = y^3$. By taking $H(u) = u$,

$$\liminf_{t\to\infty} \sum_{s=t-3}^{t-1} \frac{1}{s^2}\frac{1}{2}\frac{1}{3!}\left(\frac{s-3}{2^3}\right)^3 > \frac{1}{\mathrm{e}}.$$

We check that all the conditions of Theorem 2 are satisfied, every bounded solution of (15) oscillates or tends to zero at infinity.

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