Oscillatory Behavior Of A Higher-Order Nonlinear Neutral Type Functional Difference Equation With Oscillating Coefficients

Emrah Karaman†, Mustafa Kemal Yıldız‡

Received 4 October 2014

Abstract

In this work, we shall consider oscillation of bounded solutions of higher-order nonlinear neutral delay difference equations of the following type

$$\Delta^n [y(t) + p(t) f(y(\tau(t)))] + q(t) h(y(\sigma(t))) = 0, \ t \in \mathbb{N},$$

where $n \in \{2, 3, \ldots\}$ is fixed and can take both odd and even values, $\{p(t)\}_{t=1}^{\infty}$ is a sequence of reals such that $\lim_{t \to \infty} p(t) = 0$, $\{q(t)\}_{t=1}^{\infty}$ is a nonnegative sequence of reals, and $\{\tau(t)\}_{t=1}^{\infty}$ and $\{\sigma(t)\}_{t=1}^{\infty}$ are sequences of integers tending to infinity asymptotically and bounded above by $\{t\}_{t=1}^{\infty}$, and $f, h \in C(\mathbb{R}, \mathbb{R})$.

1 Introduction

We consider the higher-order nonlinear difference equation of the form

$$\Delta^n [y(t) + p(t) f(y(\tau(t)))] + q(t) h(y(\sigma(t))) = 0 \ \text{for} \ \ t \in \mathbb{N}, \ (1)$$

where $n \in \{2, 3, \ldots\}$ is fixed, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $p : \mathbb{N} \to \mathbb{R} = (-\infty, \infty)$, $\{p(t)\}_{t=1}^{\infty}$ is a sequence of real such that $\lim_{t \to \infty} p(t) = 0$, and it is an oscillating function; $q : \mathbb{N} \to [0, \infty)$, $\tau(t) : \mathbb{N} \to \mathbb{Z}$ ($\mathbb{Z}$ denotes the set of integers) with $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$, $\sigma(t) : \mathbb{N} \to \mathbb{Z}$ ($\mathbb{Z}$ denotes the set of integers) with $\sigma(t) \leq t$, for all $t \in \mathbb{N}$ and $\sigma(t) \to \infty$ as $t \to \infty$, $f(u), h(u) \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing functions (1), $uf(u) > 0$ and $uh(u) > 0$, for all $u \neq 0$, we mean any function $y(t) : \mathbb{Z} \to \mathbb{R}$, which is defined for all $t \geq \min_{i \geq 0} \{\tau(i), \sigma(i)\}$, and satisfies equation (1) for sufficiently large $t$. As it is customary, a solution $\{y(t)\}$ is said to be oscillatory if the terms $y(t)$ of the sequence are not eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory. In this paper, we restrict our attention to real valued solutions $y$.

Recently, much research has been done on the oscillatory and asymptotic behavior of solutions of higher-order delay and neutral type difference equations. The results

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*Mathematics Subject Classifications: 35C20, 35D10.
†Department of Mathematics, Karabük University, Karabük, 78050 Turkey
‡Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, 03200 Turkey
obtained here are an extension of work in [7]. Most of the known results are for special cases of equation (1) and related equations; see, for example, [1, 2, 3, 16].

The purpose of this paper is to study oscillatory behavior of bounded solutions of equation (1). For the general theory of difference equations, one can refer to [1, 2, 3, 10, 11, 12, 15]. Many references to applications of the difference equations can be found in [10, 11, 12].

For the sake of convenience, we let

\[ N(a) = \{a, a+1, \ldots\}, \ N(a, b) = \{a, a+1, \ldots, b\}, \]

and the function \( z(t) \) is defined by

\[ z(t) = y(t) + p(t)f(y(\tau(t))). \quad (2) \]

### 2 Some Auxiliary Lemmas

In this section, we present the known results.

**LEMMA 1 ([2]).** Let \( y(t) \) be defined for \( t \geq t_0 \in \mathbb{N} \), and \( y(t) > 0 \) with \( \Delta^n y(t) \) of constant sign for \( t \geq t_0 \), \( n \in \mathbb{N}(1) \), and not identically zero. Then there exists an integer \( m \in [0, n] \) satisfying either \((n + m)\) is even for \( \Delta^n y(t) \geq 0 \) or \((n + m)\) is odd for \( \Delta^n y(t) \leq 0 \) such that

(i) if \( m \leq n - 1 \) implies \((-1)^{m+i} \Delta^i y(t) > 0 \) for all \( t \geq t_0 \) and \( m \leq i \leq n - 1 \),

(ii) if \( m \geq 1 \) implies \( \Delta^i y(t) > 0 \) for all large \( t \geq t_0 \) and \( 1 \leq i \leq m - 1 \).

**LEMMA 2 ([2]).** Let \( y(t) \) be defined for \( t \geq t_0 \), and \( y(t) > 0 \) with \( \Delta^n y(t) \leq 0 \) for \( t \geq t_0 \) and not identically zero. Then there exists a large \( t_1 \geq t_0 \), such that

\[ y(t) \geq \frac{1}{(n-1)!} (t - t_1)^{n-1} \Delta^{n-1} y(2^{n-1} t_1), \quad t \geq t_1, \]

where \( m \) is defined as in Lemma 2. Furthermore, if \( y(t) \) is increasing, then

\[ y(t) \geq \frac{1}{(n-1)!} \left( \frac{t}{2^{n-1}} \right)^{n-1} \Delta^{n-1} y(t), \quad t \geq 2^{n-1} t_1. \]

### 3 Main Results

In this section, we present main results and give some examples.

**THEOREM 1.** Assume than \( n \) is odd and the following assertions \((C_1)-(C_2)\) hold:

\[(C_1) \ \lim_{t \to \infty} p(t) = 0, \]
\[(C_2) \ \sum_{s = t_0}^{\infty} s^{n-1} q(s) = \infty. \]
Then every bounded solution of equation (1) either is oscillatory or tends to zero as \( t \to \infty \).

PROOF. Assume that equation (1) has a bounded nonoscillatory solution \( y \). Without loss of generality, assume that \( y \) is eventually positive (the proof is similar when \( y \) is eventually negative). That is, \( y(t) > 0 \), \( y(\tau(t)) > 0 \), and \( y(\sigma(t)) > 0 \) for \( t \geq t_1 \geq t_0 \). Furthermore, we assume that \( y(t) \) does not tend to zero as \( t \to \infty \). By (1) and (2), we have that

\[
\Delta^n z(t) = -q(t) h(y(\sigma(t))) \leq 0 \quad \text{for} \quad t \geq t_1.
\]  

That is, \( \Delta^n z(t) \leq 0 \). It follows that \( \Delta^\alpha z(t) \) for \( \alpha = 0, 1, 2, \ldots, n-1 \) is strictly monotone and eventually of constant sign. Since \( \lim_{t \to \infty} p(t) = 0 \), there exists \( t_2 \geq t_1 \) such that \( z(t) > 0 \) for \( t \geq t_2 \). Since \( y \) is bounded, and by virtue of \((C_1)\) and (2), there exists \( t_3 \geq t_2 \) such that \( z(t) \) is also bounded for \( t \geq t_3 \). Because \( n \) is odd, \( z(t) \) is bounded and \( m = 0 \) (otherwise, \( z(t) \) is not bounded by Lemma 1), there exists \( t_4 \geq t_3 \) such that for \( t \geq t_4 \), we have \((-1)^i \Delta^i z(t) > 0 \) for \( i = 0, 1, 2, \ldots, n-1 \). In particular, since \( \Delta z(t) < 0 \), \( z(t) \) is decreasing. Since \( z \) is bounded, we obtain that \( \lim_{t \to \infty} z(t) = L \) where \(-\infty < L < \infty \). Assume that \( 0 \leq L < \infty \). Let \( L > 0 \). Then there exist a constant \( c > 0 \) and \( t_5 \geq t_4 \) such that \( z(t) > c > 0 \) for \( t \geq t_5 \). Since \( y \) is bounded, \( \lim_{t \to \infty} p(t)f(y(\tau(t))) = 0 \) by \((C_1)\). Therefore, there exist a constant \( c_1 > 0 \) and \( t_6 \) with \( t_6 \geq t_5 \) such that

\[
y(t) = z(t) - p(t)f(y(\tau(t))) > c_1 > 0 \quad \text{for} \quad t \geq t_6.
\]

So we may find \( t_7 \) with \( t_7 \geq t_6 \) such that \( y(\sigma(t)) > c_1 > 0 \) for \( t \geq t_7 \). From (3), we have

\[
\Delta^n z(t) \leq -q(t) h(c_1) \quad \text{for} \quad t \geq t_7.
\]  

If we multiply (4) by \( t^n-1 \), and summing it from \( t_7 \) to \( t-1 \), we obtain

\[
F(t) - F(t_7) \leq -h(c_1) \sum_{s=t_7}^{t-1} q(s) s^{n-1},
\]

where

\[
F(t) = \sum_{\gamma=2}^{n-1} (-1)^\gamma \Delta^n t^{n-1} \Delta^{n-\gamma-1} z(t + \gamma).
\]

Since \((-1)^i \Delta^i z(t) > 0 \) for \( i = 0, 1, 2, \ldots, n-1 \) and \( t \geq t_4 \), we have \( F(t) > 0 \) for \( t \geq t_7 \). From (5), we have

\[
-F(t_7) \leq -h(c_1) \sum_{s=t_7}^{t-1} q(s) s^{n-1}.
\]

By \((C_2)\), we obtain

\[
-F(t_7) \leq -h(c_1) \sum_{s=t_7}^{t-1} q(s) s^{n-1} = -\infty \quad \text{as} \quad t \to \infty.
\]
This is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{t \to \infty} z(t) = 0$. Since $y$ is bounded, and by virtue of (C1) and (2), we obtain

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t) - \lim_{t \to \infty} p(t) f(y(\tau(t))) = 0.$$  

Now, let us consider the case of $y(t) < 0$ for $t \geq t_1$. By (1) and (2),

$$\Delta^n z(t) = -q(t) h(y(\sigma(t))) \geq 0 \text{ for } t \geq t_1.$$

That is, $\Delta^n z(t) \geq 0$. It follow that $\Delta^n z(t)$ for $\alpha = 0, 1, 2, \ldots, n-1$ is strictly monotone and eventually constant sign. Since $\lim_{t \to \infty} p(t) = 0$, there exists $t_2 \geq t_1$, such that $z(t) < 0$ for $t \geq t_2$. Since $y(t)$ is bounded, by virtue of (C1) and (2), there exists $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Assume that $x(t) = -z(t)$. Then $\Delta^n x(t) = -\Delta^n z(t)$. Therefore, $x(t) > 0$ and $\Delta^n x(t) \leq 0$ for $t \geq t_3$. From this, we observe that $x(t)$ is bounded. Because $n$ is odd, $x(t)$ is bounded and $m = 0$ (otherwise, $x(t)$ is not bounded by Lemma 1) there exists $t_4 \geq t_3$ such that $(-1)^i \Delta^i x(t) > 0$ for $i = 0, 1, 2, \ldots, n-1$ and $t \geq t_4$. That is, $(-1)^i \Delta^i z(t) < 0$ for $i = 0, 1, 2, \ldots, n-1$ and $t \geq t_4$. In particular, we have $\Delta z(t) > 0$ for $t \geq t_4$. Therefore, $z(t)$ is increasing. So, we can assume that $\lim_{t \to \infty} z(t) = L$ where $-\infty < L \leq 0$. As in the proof of $y(t) > 0$, we may prove that $L = 0$. As for the rest, it is similar to the case $y(t) > 0$. That is, $\lim_{t \to \infty} y(t) = 0$. This contradicts our assumption. Hence, the proof is completed.

**THEOREM 2.** Assume that $n$ is even and the following condition $(C_3)$ holds:

$(C_3)$ there exists a function $H : \mathbb{R} \to \mathbb{R}$ such that $H$ is continuous and nondecreasing, and satisfies the inequality

$$-H(-uv) \geq H(uv) \geq KH(u)H(v) \text{ for } u, v > 0,$$

where $K$ is a positive constant, and

$$|h(u)| \geq |H(u)|, \quad \frac{H(u)}{u} \geq \gamma > 0 \quad \text{and} \quad H(u) > 0 \text{ for } u \neq 0.$$

and every bounded solution of the first-order delay difference equation

$$\Delta w(t) + q(t)K^n H \left( \frac{1}{2} \left( \frac{1}{(n-1)!} \left( \frac{\sigma(t)}{2^{n-1}} \right)^{n-1} \right) w(\sigma(t)) = 0 \right)$$

is oscillatory.

Then every bounded solution of equation (1) is either oscillatory or tends to zero as $t \to \infty$.

**PROOF.** Assume that equation (1) has a bounded nonoscillatory solution $y$. Without loss of generality, assume that $y$ is eventually positive (the proof is similar when $y$ is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Furthermore, suppose that $y$ does not tend to zero as $t \to \infty$. By (1) and (2), we have

$$\Delta^n z(t) = -q(t) h(y(\sigma(t))) \leq 0 \text{ for } t \geq t_1.$$  

(7)
It follows that $\Delta^\alpha z(t)$ for $\alpha = 0, 1, 2, \ldots, n-1$ is strictly monotone and eventually of constant sign. Since $y$ is bounded and does not tend to zero as $t \to \infty$, and by virtue of $(C_1)$, $\lim_{t \to \infty} p(t) f(y(\tau(t))) = 0$. Then we can find a $t_2 \geq t_1$ such that $z(t) = y(t) + p(t) f(y(\tau(t))) > 0$ eventually and $z(t)$ is also bounded for sufficiently large $t \geq t_2$. Because $n$ is even, $(n + m)$ odd for $\Delta^\alpha z(t) \leq 0$, $z(t) > 0$ is bounded and $m = 1$ (otherwise, $z(t)$ is not bounded by Lemma 1) there exists $t_3 \geq t_2$ such that

$$(-1)^{i+1} \Delta^i z(t) > 0 \text{ for } t \geq t_3 \text{ and } i = 0, 1, 2, \ldots, n-1. \quad (8)$$

In particular, since $\Delta z(t) > 0$ for $t \geq t_3$, $z$ is increasing. Since $y$ is bounded, $\lim_{t \to \infty} p(t) f(y(\tau(t))) = 0$ by $(C_1)$. Then there exists $t_4 \geq t_3$ by $(2)$ such that

$$y(t) = z(t) - p(t) f(y(\tau(t))) \geq \frac{1}{2} z(t) > 0 \text{ for } t \geq t_4.$$ 

We may find a $t_5 \geq t_4$ such that

$$y(\sigma(t)) \geq \frac{1}{2} z(\sigma(t)) > 0 \text{ for } t \geq t_5. \quad (9)$$

From $(7)$ and $(9)$, we can obtain the result of

$$\Delta^\alpha z(t) + q(t) h\left(\frac{1}{2} z(\sigma(t))\right) \leq 0 \text{ for } t \geq t_5. \quad (10)$$

Since $z(t)$ is defined for $t \geq t_2$, we apply directly Lemma 2 (second part, since $z$ is positive and increasing) to obtain that $z(t) > 0$ with $\Delta^\alpha z(t) \leq 0$ for $t \geq t_2$ and not identically zero. It follows from Lemma 2 that

$$y(\sigma(t)) \geq \frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1} \Delta^{n-1} z(\sigma(t)) \text{ for } t \geq 2^{n-1} t_1. \quad (11)$$

Using $(C_3)$ and $(9)$, we find that for $t \geq t_6 \geq t_5$,

$$h(y(\sigma(t))) \geq H(y(\sigma(t))) \geq H \left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1} \Delta^{n-1} z(\sigma(t))\right) \geq K H \left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right) H(\Delta^{n-1} z(\sigma(t))) \geq K \gamma H \left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right) \Delta^{n-1} z(\sigma(t)).$$

It follows from $(7)$ and the above inequality, that $\{\Delta^{n-1} z(t)\}$ is an eventually positive solution of

$$\Delta w(t) + q(t) K \gamma H \left(\frac{1}{2} \frac{1}{(n-1)!} \left(\frac{\sigma(t)}{2^{n-1}}\right)^{n-1}\right) w(\sigma(t)) \leq 0.$$
By a well-known result (see Theorem 3.1 in [5]), the difference equation
\[ \Delta w(t) + q(t)K\gamma H \left( \frac{1}{2} \left( \frac{1}{(n-1)!} \left( \frac{\sigma(t)}{2^{n-1}} \right)^{n-1} \right) \right) w(\sigma(t)) = 0 \text{ for } t \geq t_\gamma \geq t_6 \]
has an eventually positive solution. This contradicts the fact that (1) is oscillatory, and the proof is completed.

Thus, from Theorem 2 and Theorem 2.3 in [6] (see also Example 3.2 in [6]), we can obtain the following corollary.

COROLLARY 1. If
\[ \liminf_{t \to \infty} \sum_{s=\sigma(t)}^{t-1} q(s)H \left( \frac{1}{2} \left( \frac{1}{(n-1)!} \left( \frac{\sigma(s)}{2^{n-1}} \right)^{n-1} \right) \right) > \frac{1}{eK\gamma}, \tag{12} \]
then every bounded solution of equation (1.1) either is oscillatory or tends to zero as \( t \to \infty \).

When \( p(t) \equiv 0 \) and \( n = 2 \), Corollary 3 yields that if
\[ \liminf_{t \to \infty} \sum_{s=\sigma(t)}^{t-1} q(s)H \left( \frac{1}{4} \sigma(s) \right) > \frac{1}{eK\gamma}, \]
then
\[ \Delta^2 y(t) + q(t) h(y(\sigma(t))) = 0 \text{ for } t \geq t_0 \tag{13} \]
is oscillatory. These results have been established in [6, 12, 13] and the references cited therein.

EXAMPLE 1. We consider difference equation of the form
\[ \Delta^3 \left[ y(t) + e^{-5t^2} \sin t \left( y^2(t-5) + 2y(t-5) \right) \right] + t^2y^2(t-3) = 0 \text{ for } t \geq 2, \tag{14} \]
where \( n = 3, q(t) = t^2, \sigma(t) = t-3, \tau(t) = t-5, p(t) = e^{-5t^2} \sin t, f(y) = y^2 - 2y, \) and \( h(y) = y^2 \). Hence, we have
\[ \lim_{t \to \infty} p(t) = \lim_{t \to \infty} \frac{1}{e^{5t^2}} \sin t = 0 \text{ and } \sum_{s=t_0}^{\infty} s^{n-1}q(s) = \sum_{s=t_0}^{\infty} s^4 = \infty. \]
Since Conditions \((C1)\) and \((C2)\) of the Theorem 1 are satisfied, every bounded solution of (14) oscillates or tends to zero at infinity.

EXAMPLE 2. We consider difference equation of the form
\[ \Delta^4 \left[ y(t) + \left( -\frac{1}{2} \right)^t y(t-2) \right] + \frac{1}{4^2}y^3(t-3) = 0, \tag{15} \]
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where \( n = 4, \tau(t) = t - 2, p(t) = (-1/2)^4, q(t) = 1/t^2, \sigma(t) = t - 3, \) and \( h(y) = y^3. \)

By taking \( H(u) = u, \)

\[
\lim_{t \to \infty} \inf \sum_{s=t-3}^{t-1} \frac{1}{s^2} \left( \frac{1}{2} \right)^3 \frac{1}{s^3} \left( \frac{s^3 - 3}{2^3} \right) > \frac{1}{e}.
\]

We check that all the conditions of Theorem 2 are satisfied, every bounded solution of (15) oscillates or tends to zero at infinity.

References


