Applied Mathematics E-Notes, 13(2013), 261-269 © ISSN 1607-2510
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Stability Of Unilateral Problem With Nonconvex Constraints*

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Received 22 October 2013

Abstract

In this paper, we focuses on stability, asymptotical stability and finite-time stability for a class of differential inclusions governed by a nonconvex superpotential. This problem is known by "evolution hemivariational inequalities". After proposing an existence result of solutions, we give the stability results in terms of smooth Lyapunov functions subjected to some conditions described in terms of the orbital derivatives.

1 Introduction

The origin of stability theory comes from mechanics and specially from the study of motion of particles. Today, stability theory plays a dominant role in the study of mechanical, electrical, economical models, control theory and many branches of sciences. The most important stability concept is *stability in the sense of Lyapunov.* Lyapunov proposes two methods in order to solve the stability problem. One of these methods is based on the study of the behavior of special functions called *Lyapunov's function.* This method avoids the calculation of an explicit solution of the problem. But, it requires to find good Lyapunov candidate functions compatible with the problem, and the disadvantage is that there is no straightforward construction of Lyapunov functions.

In this paper, we will mainly focus on the stability of differential inclusions governed by a Clarke subdifferential. Such problem is known by *evolution hemivariational inequalities.* The nonsmoothness of these dynamical systems comes from the fact that there motion is subject to velocity jumps or/and discontinuous forces. The notion of hemivariational inequalities was introduced by P. D. Panagionotopoulos (see. [12, 13]) with the help of the generalized Clarke gradient. In this case, we say that the nonconvex constraints derive from a nonconvex superpotentials. In [1, 2, 9], the authors give some stability results for nonsmooth dynamical system involving convex superpotentials. So, the aim here is the study of the stability of differential inclusions involving nonconvex superpotentials. The stability results are given in terms of smooth Lyapunov functions. To get our goal, we use the concept of *orbital derivatives* introduced by Filippov in [8]. The orbital derivatives are also used in the study of finite time stability. Where, it seems difficult to give a necessary and sufficient condition to prove it.

*Mathematics Subject Classifications:47J40, 49J53.

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The contents of the paper are as follows. In Section 2, we establish definitions, notations and review some basic results from nonsmooth analysis. In addition, we briefly formulate our problem in terms of differential inclusions and give an existence result. In Section 3, we introduce the notion of orbital derivatives in order to develop a Lyapunov like theorem for stability and asymptotic stability for differential inclusions with Clarke subdifferential. Section 4 is dedicated for the study of finite time stability. Instead, we propose a sufficient and a necessary conditions given in terms of Lyapunov functions and their orbital derivatives.

2 Definitions and Notations

The goal of this section is to formulate differential inclusions with Clarke subdifferential (or first-order evolution hemivariational inequalities) and give an existence result of solutions to such problems. We start by giving some notations and definitions which are used in the sequel.

Specifically, we denote by $\|\cdot\|$ the norm of $\mathbb{R}^n$ associated to the usual inner product $\langle\cdot,\cdot\rangle$. For $\rho > 0$ and $x \in \mathbb{R}^n$, we denote by $\mathbb{B}_\rho$ and $\mathbb{B}_\rho(x)$, the closed balls of radius ρ, centered at the origin and at the point x respectively.

Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set valued map. In this paper, we assume that F satisfies the following hypotheses:

i) For all $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.

ii) The growth condition: For some positive constant κ_F and for all $x \in \mathbb{R}^n$,
$v \in F(x) \Longrightarrow \|v\| \le \kappa_F(1 + \|x\|)$.

iii) F is upper semi-continuous at every point $x \in \mathbb{R}^n$; that means for every $\varepsilon > 0$, there is a $\rho > 0$ such that for $y \in \mathbb{B}_\rho(x)$, $F(y) \subseteq F(x) + \mathbb{B}_\varepsilon$.

Let $j : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function; we denote by $j^\circ(x; d)$ the generalized directional derivative of j at a point $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$,

$$j^\circ(x; d) := \limsup_{w \to x, \tau \to 0^+} \frac{j(w + \tau d) - j(w)}{\tau},$$

provided the limit exists in $\mathbb{R}^n$. We say that a vector ξ is a Clarke subgradient of j at $x \in \mathbb{R}^n$ if, for all $d \in \mathbb{R}^n$, $j^\circ(x; d) \ge \langle \xi, d\rangle$.

The set of such ξ is called Clarke subdifferential and denoted by $\partial_c j(x)$. Note that, the Clarke subdifferential is a nonempty compact convex set. Moreover, the set valued map $\partial_c j(x)$ is upper semi-continuous (see.[6]).

Consider the standard differential inclusions given by

$$(D.I.)\begin{cases} \dot{x}(t) \in F(x(t)),\ t \in [0, T], \\ x(0) = x_0 \end{cases}$$

An existence result for $(D.I.)$ can be found in [8] and where we recall it in the following theorem.

THEOREM 1. Let $F(x)$ be a set valued map which fulfills the hypotheses (i)–(iii). Then, for each $x_0 \in \mathbb{R}^n$ there exists at least a function $x(t) : [0, T] \to \mathbb{R}^n$ satisfying $(D.I.)$.

Our main interest is the study of the stability of the first-order differential inclusions of the form:

$$(P) \begin{cases} -\dot{x}(t) \in \partial_c j(x(t)) + F(x(t)), \text{ a.e. } t \geq 0, \\ x(0) = x_0. \end{cases}$$

As a Consequence of Theorem 1, the following gives an existence result of the solution of the problem (P).

THEOREM 2. Let j be a locally Lipschitz function and let F be a set valued map satisfying the hypotheses (i)–(iii). Suppose also j and F satisfy the problem (P). Furthermore, suppose that the set valued map $\partial_c j(x)$ satisfies also a growth condition (see condition iii) and consider the constant is κ_j. Then, for each $u_0 \in \mathbb{R}^n$, there exists at least a solution for the problem (P).

PROOF. For $T > 0$ fixed, we set $\mathcal{F} := -F - \partial_c j$. Obviously, the set valued map $\mathcal{F}$ verifies the hypotheses (i)–(iii). Then, there exists a positive constant $\kappa := \kappa_F + \kappa_j$ such that:

$$v \in \mathcal{F}(x) \Longrightarrow \|v\| \leq \kappa(1 + \|x\|).$$

Thus, by Theorem 1, for each $x_0 \in \mathbb{R}^n$ and for $t \in [0, T]$, the problem $\dot{x}(t) \in \mathcal{F}(x(t))$, has at least a solution. Thus, for T chosen arbitrarily, we deduce the existence of a solution of the problem (P).

By definition of the Clarke subdifferential, the problem (P) can be equivalently written as the following *evolution hemivariational inequality*:

$$\begin{cases} \text{Find } x(t) \in C^0([0, +\infty); \mathbb{R}^n) \text{ and } \dot{x}(t) \in L^1([0, +\infty); \mathbb{R}^n) \text{ such that} \\ \langle \dot{x}(t) + F(x(t)), v \rangle + j^\circ(x(t); v) \geq 0, \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq 0 \\ x(0) = x_0. \end{cases}$$

In literature, there are many results concerning the study of the theory of the hemivariational inequalities and their applications. For that, we invite the readers to consult the results presented in [12, 13].

REMARK 1. From the formulation of problem (P), we remark that the multifunctions F and $\partial_c j$ verify the same assumptions. Here one question might arise: Why do we have to consider both F and $\partial_c j$ when we can study the system (P) only with $\partial_c j$?

The response to this question is simple. In fact, consider the second-order differential inclusion involving a nonconvex superpotential defined as follows:

$$m\ddot{x}(t) + b\dot{x}(t) + cx(t) \in -\partial_c j(\dot{x}(t)), \quad m > 0. \tag{1}$$

Using the transformation $[x_1 \;\; x_2]^T = [x \;\; \dot{x}]^T$, we can convert the system (1) into a first-order differential inclusion similar to (P) where we remark that the multifunction F

appears in the inclusion. From here, we can justify the presence of F in the formulation of (P).

In this paper, we suppose that the hypotheses of Theorem 2 hold. In addition, for $x_0 \in \mathbb{R}^n$, we denote by $\mathcal{S}_{x_0}$ the set of solutions of the problem (P). Finally, we denote by $\mathcal{K}$ and $\mathcal{K}_\infty$, the sets respectively defined by

$$\mathcal{K} \ : \ = \big\{g : [0,\rho] \to \mathbb{R}^+ : g \text{ is strictly increasing and continuous on } [0,\rho] \text{ with } g(0)=0\big\},$$

and

$$\mathcal{K}_\infty \ : \ = \Big\{g : \mathbb{R}^+ \to \mathbb{R}^+ : g \text{ is strictly increasing and continuous on } \mathbb{R}^+ \text{ with } g(0)=0 \text{ and } \lim_{x\to+\infty} g(x) = +\infty\Big\}.$$

3 Stability Results

In this section, we develop stability and asymptotic stability results for problem (P). To attain this aim, we start by recalling the definitions of the stability and the asymptotic stability of the problem (P) (in the sense of Lyapunov). Then, we introduce the notion of orbital derivative of a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$.

DEFINITION 1. We say that the system (P) is

1. *Stable*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x_0 \in \mathbb{B}_\delta$ and all the solutions $x(t) \in \mathcal{S}_{x_0}$, the following holds: $x(t) \in \mathbb{B}_\varepsilon, \forall t \geq 0$.
2. *Attractive*, if there exists $\delta > 0$ such that for each $x_0 \in \mathbb{B}_\delta$ and all the solutions $x(t) \in \mathcal{S}_{x_0}$, we have $\lim_{t\to+\infty} \|x(t)\| = 0$.
3. *Asymptotically stable*, if it is stable and attractive.

Let us now recall the notion of orbital derivative which will be used in our study of the stability. This notion was introduced by Filippov in [8].

DEFINITION 2. For a function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ and for a function $x \in \mathcal{S}_{x_0}$, the set

$$\dot{V}(x) := \{\zeta \in \mathbb{R} : \exists v \in \partial_c j(x), \zeta = \langle -\nabla V(x), F(x) + v\rangle\}.$$

is called the set of *orbital derivatives* associated to V and x.

As the set $\partial_c j(x)$ is convex and compact, we can deduce that $\dot{V}(x)$ is a convex, closed and bounded set. Thus, $\dot{V}(x)$ can be considered as a closed interval of $\mathbb{R}$ and represented as $\dot{V}(x) := \left[\dot{V}_{\inf}(x), \dot{V}^{\sup}(x)\right]$.

The values $\dot{V}_{\inf}(x)$ and $\dot{V}^{\sup}(x)$ are called respectively the *upper* and the *lower orbital derivatives* of V and they are defined by

$$\dot{V}_{\inf}(x) := \inf_{v\in\partial_c j(x)} \langle -\nabla V(x), F(x)+v\rangle;$$

$$\dot{V}^{\sup}(x) := \sup_{v\in\partial_c j(x)} \langle -\nabla V(x), F(x)+v\rangle.$$

REMARK 2. Let $x(t)$ be an element of $\mathcal{S}_{x_0}$ and let $V \in C^1(\mathbb{R}^n;\mathbb{R})$. Consider the function $t \mapsto L(t) := V(x(t))$. By definition of the function L, it is easy to remark that $\nabla L(t) = \frac{d}{dt}V(x(t)) \in \dot{V}(x(t))$ a.e. $t \geq 0$.

Now, we can give a version of Lyapunov's local stability theorem for problem (P).

THEOREM 3. Assume that the hypotheses of Theorem 2. hold and suppose that there exist $\rho > 0$ and a positive definite function $V \in C^1(\mathbb{B}_\rho,\mathbb{R})$ such that

$$\dot{V}^{\sup}(x) \leq 0,\ \forall x \in \mathbb{B}_\rho. \tag{2}$$

Then, the origin is stable for (P). In that case, every positive definite function V verifying (2) is called a *Lyapunov's function* for problem (P).

PROOF. As the function V is $C^1(\mathbb{B}_\rho,\mathbb{R})$ and is definite positive, then there exists a function $\psi : [0,\rho] \to \mathbb{R}^+$ with $\psi \in \mathcal{K}$ such that

$$V(x) \geq \psi(\|x\|), \forall x \in \mathbb{B}_\rho. \tag{3}$$

Let $\varepsilon \in]0,\rho[$. Using the continuity of V and the fact that $V(0) = 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any $x_0 \in \mathbb{B}_\delta$, we have $|V(x_0)| < \psi(\varepsilon)$.

Furthermore, let $\eta = \eta(\varepsilon)$ with $0 < \eta < \min(\varepsilon,\delta)$. We would like to show that, for any $x_0 \in \mathbb{B}_\eta$ and for all $x \in \mathcal{S}_{x_0}$, we have $\|x(t)\| < \varepsilon, \forall t \geq 0$.

Let us prove it by contradiction. Suppose that there exists $t_1 > 0$ such that $\|x(t_1)\| \geq \varepsilon$. We have $\|x_0\| < \varepsilon$.

By the continuity of the function $t \mapsto x(t)$ and according to the mean value theorem, there exists $\tau > 0$ such that, $\|x(\tau)\| = \varepsilon$.

Consider now the function $L(t)$ introduced in Remark 2.. $L(t)$ is strictly decreasing on $[0,\tau]$. In fact, the function L is absolutely continuous on $[0,\tau]$, then it is differentiable almost everywhere on $[0,\tau]$ and we have $\nabla L(t) = \frac{d}{dt}V(x(t)) \in \dot{V}(x(t))$ a.e. $t \in [0,\tau]$. Further, using hypothesis (2), we obtain that, $\dot{V}(x(t)) \subset]-\infty,0]$ a.e. $t \in [0,\tau]$. Then, $\nabla L(t) \leq 0$ a.e. $t \in [0,\tau[$.

Second, by combining the fact that $\|x(\tau)\| = \varepsilon$ and condition (3), we get $L(\tau) = V(x(\tau)) \geq \psi(\varepsilon)$, and so, $L(t) \leq L(0) = V(x_0) < \psi(\varepsilon)$, $\forall t \in [0,\tau[$. By tending t to τ, we get $L(\tau) \leq L(0) < \psi(\varepsilon)$, which contradicts (3). Thus, the origin is stable for (P).

EXAMPLE 1. Consider the problem (P) with $F = 0$ and $j : \mathbb{R}^2 \to \mathbb{R}$ defined by $j(x_1,x_2) = |x_1| + |x_2|$. The Clarke subdifferential of j is given by

$$\partial_c j(x_1,x_2) = \begin{cases} (sign(x_1), sign(x_2)) & \text{if } (x_1,x_2) \neq (0,0), \\ \overline{\text{co}}\ \{(-1,-1);\ (-1,1),(1,-1);\ (1,1)\} & \text{if } (x_1,x_2) = (0,0), \end{cases}$$

where

$$sign\,(x) = \begin{cases} 1 & \text{if } x > 0, \\ [-1,1] & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Consider the function $V(x_1, x_2) = \frac{\|(x_1,x_2)\|^2}{2}$. We have

$$\dot{V}(x_1, x_2) = \begin{cases} \{-|x_1| - |x_2|\} & \text{if } (x_1, x_2) \neq (0,0), \\ \{0\} & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

So that, $\dot{V}(x_1, x_2) \subset]-\infty, 0]$, for all $(x_1, x_2) \in \mathbb{R}^2$. Thus, by THEOREM 3, the trivial solution is stable.

Let us introduce an asymptotic stability result for problem (P) by the following theorem:

THEOREM 4. Assume that the hypotheses of Theorem 2. hold and suppose that there exist $\rho > 0$, $\lambda > 0$ and a definite positive function $V \in C^1(\mathbb{B}_\rho, \mathbb{R})$ such that, for all $x \in \mathbb{B}_\rho$

$$\dot{V}^{\sup}(x) \leq -\lambda V(u). \tag{4}$$

Then, the trivial solution of problem (P) is *asymptotically stable*.

PROOF. First, it is easy to see that, the stability follows from Theorem 3. We still have to prove that, the solution of problem (P) is attracted by the origin. Let $L(t)$ be the function defined in Remark 2. Condition (4) means that $\dot{V}(x) \subset]-\infty, -\lambda V(x)]$, and we have

$$\nabla L(t) \leq -\lambda L(t),\ a.e.\, t \geq 0. \tag{5}$$

By integrating (5), we get $L(t) \leq L(0)e^{-\lambda t}$, $t \geq 0$.

As the function $V \in C^1(\mathbb{B}_\rho, \mathbb{R})$ is definite positive then there exists a function $\psi : [0, \rho] \to \mathbb{R}^+$ such that $\psi \in \mathcal{K}$ and verifies (3) in proof of Theorem 3. Then, we obtain that

$$0 < \psi(\|x(t)\|) \leq L(0)e^{-\lambda t},\ t \geq 0. \tag{6}$$

Finally, by definition of the function ψ and by tending t to $+\infty$ in (6), we get $\lim_{t\to+\infty} \|x(t)\| = 0$, and the attractivity condition holds.

4 Finite-Time Stability

By the asymptotic stability of Theorem 4, we showed that the solution of problem (P) is attracted by the equilibrium point (the origin). But, this concept lacks of informations concerning the time of convergence. For this, let us introduce the notion of *finite-time stability*, as follows:

DEFINITION 3. For all $x_0 \in \mathbb{R}^n$, we denote by $\mathcal{S} := \bigcup_{x_0 \in \mathbb{R}^n} \mathcal{S}_{x_0}$. We say that the origin is *finite-time stable* for the problem (P) if, it is *stable*, and for all $x_0 \in \mathbb{R}^n$, there exists a function $T_f^* : \mathcal{S} \to \mathbb{R}^+$ such that $x(t) = 0$ for all $t \geq T_f^*$. The function T_f^* is called the *settling-time function* of the system (P).

Note that, if T_f^* exists and is continuous, then for all $x_0 \in \mathbb{R}^n$, we introduce the *settling-time with respect to initial conditions of problem* (P), defined as

$$T_f(x_0) := \sup_{x(t)\in\mathcal{S}_{x_0}} T_f^*(x(t)) < +\infty.$$

4.1 Sufficient Condition

We have the following theorem.

THEOREM 5. Assume that the hypotheses of Theorem 2 hold and suppose that there exist $\rho > 0$ and a definite positive function $V \in C^1(\mathbb{B}_\rho, \mathbb{R})$. If there exists a function $g \in \mathcal{K}_\infty$ such that, for all $\varepsilon > 0$, the integral $\int_0^\varepsilon \frac{dz}{g(z)}$ converges and

$$\dot{V}^{\sup}(x) \leq -g(V(x)), \;\; \forall x \in \mathbb{B}_\rho. \tag{7}$$

Then, the origin is *finite-time stable* for (P).

PROOF. From condition (7), we deduce that the origin is asymptotically stable for (P) according to Theorem 4. Then, for a $x(t) \in \mathcal{S}_{x_0}$, we have $x(t)$ is attracted by the origin with the settling-time $T_f^*(x(t)) \in [0, +\infty]$.

Let us show that $T_f^*(x(t)) < +\infty$. First, consider now the function $L(t)$ defined in Remark 2.. From the proof of Theorem 3, the function $L(t)$ is strictly decreasing for $t > 0$. Second, consider the substitution $[0, T_f^*(x(t))] \to [0, L(0)]$ given by $z = L(t)$. Then,

$$\int_{L(0)}^{0} \frac{dz}{-g(z)} = \int_0^{T_f^*(x(t))} \frac{\nabla L(t)}{-g(L(t))}\; dt. \tag{8}$$

From condition (7), we get $\dot{V}(x) \subset]-\infty, -g(V(x))]$ and as $\nabla L(t) \in \dot{V}(x(t))$ almost everywhere $t \geq 0$ according to Remark 2. We obtain than

$$\nabla L(t) \leq -g(L(t)), \;\; a.e. \;\; t \geq 0. \tag{9}$$

By combining equations (8) and (9), we get

$$T_f^*(x(t)) = \int_0^{T_f^*(x(t))} dt \leq \int_0^{T_f^*(x(t))} \frac{\nabla L(x(t))}{-g(L(x(t)))} dt = \int_0^{L(x_0)} \frac{dz}{g(z)} < +\infty.$$

Finally, as $\int_0^{L(x_0)} \frac{dz}{g(z)}$ is $x(t)-$ independent, we can deduce that the settling time with respect to the initial conditions $T_f(x_0)$ is finite. Thus, the problem (P) is finite time stable.

EXAMPLE 2. Consider the system introduced in Example 1.. To study the finite-time stability, we apply Theorem 5. Let us set $g(x) = \sqrt{x}$, for $x > 0$. It is clear that the function $g \in \mathcal{K}_\infty$. Otherwise, for $(x_1, x_2) \neq (0, 0)$, we have $\dot{V}^{\sup}(x) \leq -g(V(x))$, for all $x \in \mathbb{B}_1$. Thus, the origin is finite-time stable for the system.

4.2 Necessary Condition

We have the following theorem.

THEOREM 6. Assume that the hypotheses of Theorem 2. hold and suppose that there exist $\rho > 0$ and a definite positive function $V \in C^1(\mathbb{B}_\rho, \mathbb{R})$. If the trivial solution of problem (P) is *finite-time stable* and if there exists a function $g \in \mathcal{K}_\infty$ such that

$$\dot{V}_{\inf}(x) \geq -g(V(x)). \tag{10}$$

Then, for all $\varepsilon > 0$, the improper integral $\int_0^\varepsilon \frac{dz}{g(z)}$ is convergent.

PROOF. Consider the function $L(t)$ defined in Remark 2. and consider the substitution introduced in the proof of Theorem 5., we have:

$$\int_{L(0)}^{0} \frac{dz}{-g(z)} = \int_0^{T_f^*(x(t))} \frac{\nabla L(t)}{-g(L(t))}\, dt. \tag{11}$$

By (10), we have $\dot{V}(x) \subset [-g(V(x)), 0]$ and by the fact that $\nabla L(t) \in \dot{V}(x(t))$ almost everywhere $t \geq 0$, we obtain that

$$\nabla L(t) \geq -g(L(t)), \;\; a.e. \;\; t \geq 0. \tag{12}$$

By combining equations (11), (12) and as the origin is finite-time stable, we deduce that

$$\int_0^{L(x_0)} \frac{dz}{g(z)} = \int_0^{T_f^*(x(t))} \frac{\nabla L(x(t))}{-g(L(x(t)))} dt \leq T_f^*(u(t)) < +\infty$$

REMARK 4. Note that in Theorem 5, we can take $g(x) = x^\alpha$, with $\alpha \in]0, 1[$. Thus, for some $c > 0$, condition (7) can be reformulated as $\dot{V}^{\sup}(x) \leq -c(V(x))^\alpha$.

This condition is used in many references as [4, 5]. In practice, the function $g(x)$ is usually chosen equals to $h(x)$ just to ensure the fact that the improper integral $\int_0^\varepsilon \frac{dz}{g(z)}$ converges.

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