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A Theorem On Characteristic Equations And Its Application To Oscillation Of Functional Differential Equations*

Shao Yuan Huang† and Sui Sun Cheng‡

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Abstract

We consider the oscillation of a class of second order delay functional differential equations. This relatively difficult problem is completely solved by applying the Cheng-Lin envelope method to find the exact conditions for the absence of real roots of the associated characteristic function. Several specific examples are also included to illustrate these conditions.

1 Introduction

In this paper, we intend to consider delay differential equations with constant coefficients of the form

$$N''(t) + aN'(t - \tau_1) + bN(t - \tau_2) = 0 \tag{1}$$

where $a, b \in \mathbf{R}$, $\tau_1, \tau_2 \geq 0$ and $a, b > 0$, and to find the exact region containing these parameters such that all solutions of equation (1) are oscillatory.

One motivation for studying (1) is that linearization of nonlinear equations (e.g. [1] and [4])

$$N''(t) + f(N'(t - \tau_1), N(t - \tau_2)) = 0 \tag{2}$$

where f is continuous on $\mathbf{R}^2$, leads us to equation (1). Many qualitative properties of the nonlinear equation (2) can then be inferred from the oscillatory properties of (1).

As another motivation, impulsive differential equations are mathematical apparatus for simulation of different dynamical processes and phenomena observed in nature (e.g. [7]). For this reason, many impulsive differential equations are studied and their qualitative properties investigated (e.g. [2, 5, 8, 9, 10]). In particular, consider

$$N''(t) + aN'(t - \tau_1) + bN(t - \tau_2) = 0,\ t \in [0, \infty)\backslash\Gamma, \tag{3}$$

$$N(t_k^+) = a_k N(t_k),\ k \in \mathbf{N}, \tag{4}$$

$$N'(t_k^+) = a_k N'(t_k),\ k \in \mathbf{N}, \tag{5}$$

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†Department of Matheamtics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China
‡Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China

where $a, b \in \mathbf{R}$, $\tau_1, \tau_2 \geq 0$, $a_k > 0$ for $k \in \mathbf{N}$, and Γ is a set of positive numbers $t_0, t_1, t_2, \cdots$ satisfying $t_0 < t_1 < t_2 < \cdots$. The basic concepts relevant to system (3)-(5) can be found in the appendix. Furthermore, we have the following

LEMMA 1.1. Let $a, b \in \mathbf{R}$, $\tau_1, \tau_2 \geq 0$, $a_k > 0$ for $k \in \mathbf{N}$, and $\Gamma = \{t_0, t_1, t_2, ...\}$, and let the function

$$A(s,t) = \begin{cases} \prod\limits_{s \leq t_k < t} a_k & \text{if } [s,t) \cap \Upsilon \neq \emptyset \\ 1 & \text{if } [s,t) \cap \Upsilon = \emptyset \end{cases} \quad \text{for } t \geq s \geq 0.$$

Assume that there exist positive numbers α_1 and α_2 such that $A(t - \tau_i, t) = \alpha_i$ for $i = 1, 2$. Then the system (3)-(5) has a nonoscillatory solution if, and only if, the equation

$$\lambda^2 + \frac{a}{\alpha_1}\lambda e^{-\lambda\tau_1} + \frac{b}{\alpha_2}e^{-\lambda\tau_2} = 0 \tag{6}$$

has a real root.

The proof of Lemma 1.1 is presented in the Appendix. Here we note that equation (6) is exactly of the form (7).

It is well-known that all solutions of equation (1) are oscillatory if, and only if, the characteristic equation

$$\lambda^2 + a\lambda e^{-\lambda\tau_1} + be^{-\lambda\tau_2} = 0 \tag{7}$$

has no real roots. Clearly, (6) can be 'absorbed' into (7). Therefore, studying the problem of absence of real roots of (7) becomes the main issue.

We rewrite equation (7) as

$$\lambda^2 + x\lambda e^{-\lambda\tau_1} + ye^{-\lambda\tau_2} = 0 \tag{8}$$

where $\lambda, x, y \in \mathbf{R}$ and $\tau_1, \tau_2 \geq 0$ with $\tau_1\tau_2 \neq 0$, and intend to find the exact region containing the parameters such that the equation (8) has no real roots. We note that when $\tau_1 = \tau_2 = 0$, it is a quadratic polynomial so we may ignore this easy case. The other case, however, is a relatively difficult one. Fortunately, we have an envelope method which can be used to handle the existence of real roots of functions (e.g. [6]) This method is formalized recently and presented in the book [3]. We will apply this method together with several new ideas and techniques to tackle our problem.

2 Preliminary

To facilitate discussion, we first recall a few basic concepts and tool explained in [3]. Let Θ_0 be the null function, that is $\Theta_0(x) = 0$ for all $x \in \mathbf{R}$. Given an interval I in $\mathbf{R}$, the chi-function $\chi_I : I \to R$ is defined by $\chi_I(x)$ is equal to 1 if $x \in I$ and 0 elsewhere. The restriction of a real function f defined over an interval J (which is not disjoint from I) will be written as $f\chi_I$, so that $f\chi_I$ is now defined on $I \cap J$ and

$$(f\chi_I)(x) = f(x), \; x \in I \cap J.$$

A point in the plane is said to be a *dual point* of order m of the plane curve S, where m is a nonnegative integer, if there exist exactly m mutually distinct tangents of S that also pass through it. The set of all dual points of order m of S in the plane is called the dual set of order m of S. We remark that $m = 0$ is allowed. In this case, there are no tangents of S that pass through the point in consideration.

Let $\{C_\lambda : \lambda \in I\}$, where I is a real interval, be a family of plane curves. With each C_λ, suppose we can associate just one point P_λ in each C_λ such that the totality of these points form a curve S. Then S is called an **envelope** of the family $\{C_\lambda | \lambda \in I\}$ if the curves C_λ and S share a common tangent line at the common point P_λ. Suppose we have a family of curves in the x, y-plane implicitly defined by

$$F(x, y, \lambda) = 0, \ \lambda \in I,$$

where I is an interval of $\mathbf{R}$. Then it is well known that the envelope S is described by a pair of parametric functions $(\psi(\lambda), \phi(\lambda))$ that satisfy

$$\begin{cases} F(\psi(\lambda), \phi(\lambda), \lambda) = 0, \\ F'_\lambda(\psi(\lambda), \phi(\lambda), \lambda) = 0, \end{cases}$$

for $\lambda \in I$, provided some "good conditions" are satisfied. In particular, let $\alpha, \beta, \gamma : I \to \mathbf{R}$. Then for each fixed $\lambda \in I$, the equation

$$L_\lambda : \alpha(\lambda)x + \beta(\lambda)y = \gamma(\lambda), \ (\alpha(\lambda), \beta(\lambda)) \neq 0, \tag{9}$$

defines a straight line L_λ in the x, y-plane, and we have a collection $\{L_\lambda : \lambda \in I\}$ of straight lines. For such a collection, we have the following result.

THEOREM 2.1 (see [3, Theorems 2.3 and 2.5]). Let α, β, γ be real differentiable functions defined on the interval I such that $\alpha(\lambda)\beta'(\lambda) - \alpha'(\lambda)\beta(\lambda) \neq 0$ and $\beta(\lambda) \neq 0$ for $\lambda \in I$. Let Φ be the family of straight lines of the form (9). Let the curve S be defined by the functions $x = \psi(\lambda), y = \phi(\lambda)$:

$$\psi(\lambda) = \frac{\beta'(\lambda)\gamma(\lambda) - \beta(\lambda)\gamma'(\lambda)}{\alpha(\lambda)\beta'(\lambda) - \alpha'(\lambda)\beta(\lambda)}, \quad \phi(\lambda) = \frac{\alpha(\lambda)\gamma'(\lambda) - \alpha'(\lambda)\gamma(\lambda)}{\alpha(\lambda)\beta'(\lambda) - \alpha'(\lambda)\beta(\lambda)}, \quad \lambda \in I. \tag{10}$$

Suppose ψ and ϕ are smooth functions over I and one of the following cases holds: (i) $\psi'(\lambda) \neq 0$ for $\lambda \in I$; (ii) $\psi'(\lambda) \neq 0$ for $I \backslash \{d\}$ where $d \in I$ and $\lim_{\lambda \to d^-} \phi'(\lambda)/\psi'(\lambda)$ as well as $\lim_{\lambda \to d^+} \phi'(\lambda)/\psi'(\lambda)$ exist and are equal. Then S is the envelope of the family Φ.

THEOREM 2.2 (see [3, Theorem 2.6]). Let Λ be an interval in $\mathbf{R}$, and α, β, γ be real differentiable functions defined on Λ such that $\alpha(\lambda)\beta'(\lambda) - \alpha'(\lambda)\beta(\lambda) \neq 0$ for $\lambda \in \Lambda$. Let Φ be the family of straight lines of the form (9), where $\lambda \in \Lambda$, and let the curve S be the envelope of the family Φ. Then the point (α, β) in the plane is a dual point of order m of S, if, and only if, the function $\alpha(\lambda)\alpha + \beta(\lambda)\beta - \gamma(\lambda)$, as a function of λ, has exactly m mutually distinct roots in Λ.

Let g be a function defined on an interval I with $c = \inf I$ and $d = \sup I$. Note that c or d may be infinite, or may be outside the interval I, and that $g(c^+)$, $g(d^-)$, $g'(c^+)$ or $g'(d^-)$ may not exist. For $\lambda \in (c, d)$, let

$$L_{g|\lambda}(x) = g'(\lambda)(x - \lambda) + g(\lambda),\ x \in R. \tag{11}$$

In case d is finite and $g(d^-)$, $g'(d^-)$ exist, we let

$$L_{g|d}(x) = g'(d^-)(x - d) + g(d^-),\ x \in R, \tag{12}$$

and in case c is finite and $g(c^+), g'(c^+)$ exist, we let

$$L_{g|c}(x) = g'(c^+)(x - c) + g(c^+),\ x \in R. \tag{13}$$

When d is finite, we say $g \sim H_{d^-}$ if $\lim_{\lambda \to d^-} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha < d$; and similarly when c is finite, $g \sim H_{c^+}$ if $\lim_{\lambda \to c^+} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha > c$. In case d is infinite, we say $g \sim H_{+\infty}$ if $\lim_{\lambda \to +\infty} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha \in \mathbf{R}$; and similarly, when c is infinite, we say $g \sim H_{-\infty}$ if $\lim_{\lambda \to -\infty} L_{g|\lambda}(\alpha) = -\infty$ for any $\alpha \in \mathbf{R}$.

There is a convenient criterion for the determination of functions with the above stated properties.

LEMMA 2.3.([3, Lemmas 3.1 and 3.5]). Let $g : (c, d) \to \mathbf{R}$ be a smooth and strictly convex function. (i) Assume $d < +\infty$. If $g'(d^-) = +\infty$, then $g \sim H_{d^-}$. (ii) Assume $d = +\infty$. If $g'(+\infty) = +\infty$, or, $g'(+\infty) = 0$ and $g(+\infty) = -\infty$, then $g \sim H_{+\infty}$.

The description of the distribution of dual points of a plane curve can be cumbersome. For this reason, it is convenient to introduce several notations. We say that a point (a, b) in the plane is strictly above (above, strictly below, below) the graph of a function g if a belongs to the domain of g and $g(a) < b$ (respectively $g(a) \le b$, $g(a) > b$ and $g(a) \ge b$). The notation is $(a, b) \in \vee(g)$ (respectively $(a, b) \in \overline{\vee}(g)$, $(a, b) \in \wedge(g)$ and $(a, b) \in \underline{\wedge}(g)$). Suppose we now have two real functions g_1 and g_2 defined one real subsets I_1 and I_2 respectively. We say that $(a, b) \in \vee(g_1) \oplus \vee(g_2)$ if $a \in I_1 \cap I_2$ and $b > g_1(a)$ and $b > g_2(a)$, or, $a \in I_1 \backslash I_2$ and $b > g_1(a)$, or, $a \in I_2 \backslash I_1$ and $b > g_2(a)$. The notations $(a, b) \in \overline{\vee}(g_1) \oplus \vee(g_2)$, $(a, b) \in \overline{\vee}(g_1) \oplus \wedge(g_2)$, etc. are similarly defined. If we now have n real functions $g_1, ..., g_n$ defined on intervals $I_1, ..., I_n$ respectively, we write $(a, b) \in \vee(g_1) \oplus \vee(g_2) \oplus \cdots \oplus \vee(g_n)$ if $a \in I_1 \cup I_2 \cup \cdots \cup I_n$, and if

$$a \in I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_m} \Rightarrow b > g_{i_1}(a), b > g_{i_2}(a), ..., b > g_{i_m}(a),\ i_1, ..., i_m \in \{1, ..., n\}.$$

The notations $(a, b) \in \overline{\vee}(g_1) \oplus \overline{\vee}(g_2) \oplus \cdots \oplus \overline{\vee}(g_n)$, etc. are similarly defined.

We will utilize several theorems in [3] (Theorems 3.6, 3.7, 3.10, 3.11, 3.17, 3.18, 3.19, 3.20, A3, A5, A8 and A16) which are relevant to the distribution maps for dual points. However, two more new results are needed (see Lemmas 2.4 and 2.5 below).

By Theorems 3.6 and 3.10 in [3], we may easily show the following lemma.

LEMMA 2.4. Let $a > 0$, $G_1 \in C^1(0, a)$ and $G_2 \in C^1(-\infty, a]$. Suppose the following hold:

(i) G_1 is strictly concave on $(0,a)$ such that $G_1(a^-)$ and $G_1'(a^-)$ exist, $G_1(0^+) = -\infty$ and $G_1\,\tilde{}\,H_{0^+}$;

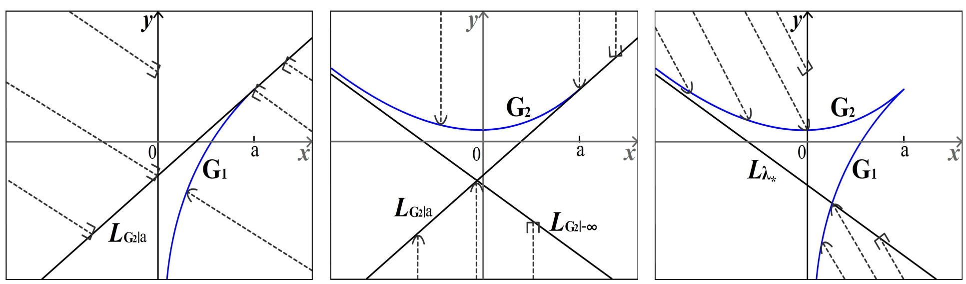
(ii) G_2 is strictly convex on $(-\infty, a]$ such that $L_{G_2|-\infty}$ exists;

(iii) $G_1^{(v)}(a^-) = G_2^{(v)}(a)$ for $v = 0, 1$.

Then the intersection of the dual sets of order 0 of G_1 and G_2 is

$$\vee(G_2\chi_{(-\infty,0]}) \oplus \wedge(G_1) \oplus \underline{\triangle}(L_{G_2|-\infty}).$$

See Figure 1.



(a) (b) (c)

Figure 1: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems 3.6 and 3.10]) to yield (c).

By Theorems 3.8 and A.5 in [3], we may show the following lemma.

LEMMA 2.5. Let $b > 0 > a$, $G_1 \in C^1(a,0)$, $G_2 \in C^1[a,b]$ and $G_3 \in C^1(-\infty, b)$. Suppose the following hold:

(i) G_1 is strictly convex on $[a,0)$ such that $G_1(0^+)$ exists and $G_1\,\tilde{}\,H_{0^-}$;

(ii) G_2 is strictly concave on (a,b) such that $G_2(a^+)$, $G_2'(a^+)$, $G_2(b^-)$ and $G_2'(b^-)$ exist

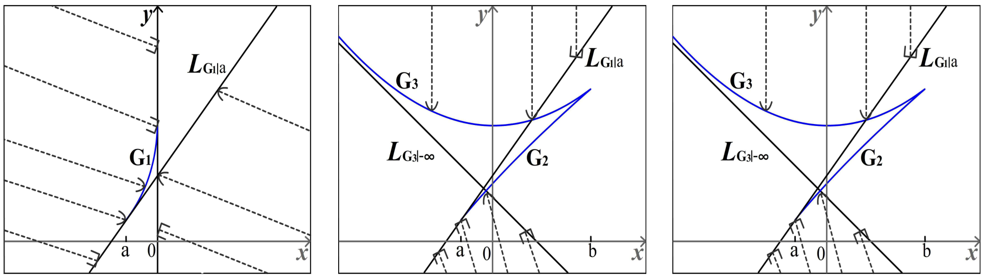
(iii) G_3 is strictly convex on $(-\infty, b]$ such that $L_{G_3|-\infty}$ exists ;

(iv) $G_1^{(v)}(a) = G_2^{(v)}(a^+)$ and $G_2^{(v)}(b^-) = G_3^{(v)}(b)$ for $v = 0, 1$.

Then the intersection of the dual sets of order 0 of G_1 G_2 and G_3 is

$$\vee(G_2\chi_{(-\infty,0]}) \oplus \wedge(G_2\chi_{[0,b]}) \oplus \underline{\triangle}(L_{G_3|-\infty}\chi_{[0,\infty]}).$$

See Figure 2.



(a) (b) (c)

Figure 2: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems 3.8 and A.5]) to yield (c).

3 Cubic Polynomial

Before studying our problem, we need to consider the distribution of real roots of cubic polynomial

$$P(\lambda|\widetilde{x}, \widetilde{y}, d) = \lambda^3 + \widetilde{x}\lambda^2 + \widetilde{y}\lambda + d \tag{14}$$

for $\lambda \in \mathbf{R}$ where $\widetilde{x}, \widetilde{y} \in \mathbf{R}$ and $d > 0$. Let

$$\begin{aligned} \Omega_{ij}(d) \quad = \quad & \{(\widetilde{x}, \widetilde{y}) \in \mathbf{R}^2 : P(\lambda|\widetilde{x}, \widetilde{y}, d) \text{ has } i \text{ distinct positive roots} \\ & \text{and } j \text{ distinct negative roots}\} \end{aligned}$$

for nonnegative integers i and j such that $i + j \leq 3$. We will apply the Cheng-Lin envelope method to find the exact sets $\Omega_{01}(d)$, $\Omega_{11}(d)$, $\Omega_{02}(d)$, $\Omega_{21}(d)$ and $\Omega_{03}(d)$ for any $d > 0$. We note that $P(0|\widetilde{x}, \widetilde{y}, d) \neq 0$ for all $\widetilde{x}, \widetilde{y} \in \mathbf{R}$. For each $\lambda \in \mathbf{R}\backslash\{0\}$, let L_λ be the straight line in the plane defined by

$$L_\lambda : \widetilde{x}\lambda^2 + \widetilde{y}\lambda = -\lambda^3 - d. \tag{15}$$

Note that L_λ defined by (15) is of the form (9) and $\alpha'(\lambda)\beta(\lambda) - \alpha(\lambda)\beta'(\lambda) = -\lambda^2 \neq 0$ for $\lambda \in \mathbf{R}\backslash\{0\}$. From (10), we let S be the curve defined by the parametric functions

$$\widetilde{x}(\lambda) = -2\lambda + \frac{d}{\lambda^2} \text{ and } \widetilde{y}(\lambda) = \lambda^2 - \frac{2d}{\lambda} \text{ for } \lambda \neq 0. \tag{16}$$

By Theorem 2.1, S is the envelope of the family $\{L_\lambda : \lambda \in \mathbf{R}\backslash\{0\}\}$ where L_λ is defined by (15). We have

$$\widetilde{x}(-d^{1/3}) = 3d^{1/3}, \ \widetilde{y}(-d^{1/3}) = 3d^{2/3},$$

$$\lim_{\lambda\to-\infty} (\widetilde{x}(\lambda), \widetilde{y}(\lambda)) = (\infty, \infty),$$

$$\lim_{\lambda\to 0^+} (\widetilde{x}(\lambda), \widetilde{y}(\lambda)) = (sgn(d)\infty, sgn(-d)\infty),$$

$$\lim_{\lambda\to 0^-} (\widetilde{x}(\lambda), \widetilde{y}(\lambda)) = (sgn(d)\infty, sgn(d)\infty),$$

$$\lim_{\lambda\to\infty} (\widetilde{x}(\lambda), \widetilde{y}(\lambda)) = (-\infty, \infty),$$

$$\widetilde{x}'(\lambda) = -2\frac{\lambda^3 + d}{\lambda^3} \text{ and } \widetilde{y}'(\lambda) = 2\frac{\lambda^3 + d}{\lambda^2} \text{ for } \lambda \neq 0. \tag{17}$$

For $\lambda \neq -d^{1/3}$, we further have

$$\frac{d\widetilde{y}}{d\widetilde{x}}(\lambda) = -\lambda \text{ and } \frac{d^2\widetilde{y}}{d\widetilde{x}^2}(\lambda) = \frac{\lambda^3}{2(\lambda^3 + d)}. \tag{18}$$

In view of (17), $\widetilde{x}(\lambda)$ is strictly increasing on $(-d^{1/3}, 0)$ and strictly decreasing on $(-\infty, -d^{1/3}) \cup (0, \infty)$, and $\widetilde{y}(\lambda)$ is strictly increasing on $(-d^{1/3}, \infty)$ and strictly decreasing on $(-\infty, -d^{1/3})$. We can see that the curve S is composed of two pieces S_1 and S_2 restricted respectively to $(-\infty, 0)$ and $(0, \infty)$. We can further see that the curve S_1 is composed of S_{11} and S_{12} restricted respectively to $(-\infty, -d^{1/3}]$ and $(-d^{1/3}, \infty)$. S_{11} is the graph of a function $y = S_{11}(x)$ which is strictly increasing, strictly convex, and smooth over $[3d^{1/3}, \infty)$ (see Figure 3(a)); S_{12} is the graph of a function $y = S_{12}(x)$ which is strictly increasing, strictly concave, and smooth over $(3d^{1/3}, \infty)$ (see Figure 3(a)); and S_2 is the graph of a function $y = S_2(x)$ which is strictly decreasing, strictly convex, and smooth over $\mathbf{R}$ (see Figure 3(b)). We have

$$S_{11}^{(v)}(3d^{1/3}) = S_{12}^{(v)}(\left(3d^{1/3}\right)^+),\ v = 1, 2.$$

Furthermore, $S_{11} \sim H_{+\infty}$, $S_{12} \sim H_{+\infty}$ and $S_2 \sim H_{-\infty}$ by Lemma 2.3 and (18). We have the following lemma.

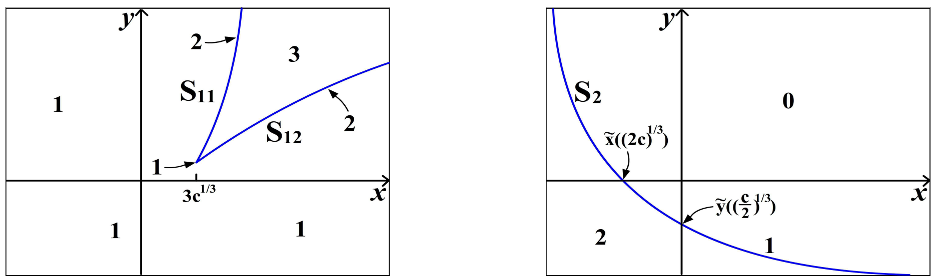
LEMMA 3.1. Assume that $d > 0$. Let the functions $\widetilde{x}(\lambda)$ and $\widetilde{y}(\lambda)$ be defined by (16). The curve S_1 is described by $(x(\lambda), y(\lambda))$ for $\lambda < 0$, and curve S_2 is described by $(x(\lambda), y(\lambda))$ for $\lambda > 0$. Then the curve S_1 lies in the first quadrant, and the curve S_2 does not pass through the first quadrant.

PROOF. In view of (16), we may observe that $\widetilde{x}(\lambda) > 0$ and $\widetilde{y}(\lambda) > 0$ for $\lambda < 0$. It follows that the curve S_1 lies in the first quadrant. We further observe that $\widetilde{x}(\lambda) = 0$ if, and only if, $\lambda = (d/2)^{1/3}$, and that $\widetilde{y}(\lambda) = 0$ if, and only if, $\lambda = (2d)^{1/3}$. Then $\widetilde{x}((2d)^{1/3}) < 0$ and $\widetilde{y}((d/2)^{1/3}) < 0$. Since S_2 is strictly decreasing, we can see that any point in the first quadrant does not lie on the curve S_2. The proof is complete.

By Lemma 3.1, the graph S_2 cannot intersect with the graph S_1. By Theorems 3.11 and 3.17 in [3], we can see that the dual set of order 1 of S_1 is $\mathbf{R}^2 \backslash \left(\underline{\triangle}(S_{11}\chi_{(0,\infty)}) \oplus \overline{\nabla}(S_{12})\right)$, the dual set of order 2 of S_1 is

$$\{(x, y) \in \mathbf{R}^2 : x > 3d^{1/3} \text{ and } y = S_{1i}(x) \text{ for some } i = 1, 2\}$$

and the dual set of order 3 of S_1 is $\wedge(S_{11}\chi_{(0,\infty)}) \oplus \vee(S_{12})$ (see Figure 3(a)). By Theorem 3.20 in [3], the dual set of order 0 of S_2 is $\vee(S_2)$, the dual set of order 1 of S_2 is $\{(x, y) \in \mathbf{R}^2 : y = S_2(x)\}$, and the dual set of order 2 of S_2 is $\wedge(S_2)$ (see Figure 3(b))



(a) (b)

Figure 3

By Theorem 2.2, we note that for any $i, j \geq 0$ with $i + j \leq 3$, $(\widetilde{x}, \widetilde{y}) \in \Omega_{ij}(d)$ if, and only if $(\widetilde{x}, \widetilde{y})$ is the dual point of order j of S_1 and is the dual point of order i of S_2. So we have the following theorem.

THEOREM 3.2. Assume that $d > 0$. Let $\widetilde{x}(\lambda)$ and $\widetilde{y}(\lambda)$ be defined by (16). Then

$$\Omega_{01}(d) = \vee(S_3) \backslash \left(\underline{\triangle}(S_1 \chi_{(3d^{1/3},\infty)}) \oplus \overline{\nabla}(S_2) \right), \tag{19}$$

$$\Omega_{11}(d) = \left\{ (x, y) \in \mathbf{R}^2 : y = S_3(x) \right\}, \tag{20}$$

$$\Omega_{02}(d) = \left\{ (x, y) \in \mathbf{R}^2 : y = S_1(x) \text{ or } y = S_2(x) \right\}, \tag{21}$$

$$\Omega_{21}(d) = \wedge(S_3), \tag{22}$$

and

$$\Omega_{03}(d) = \wedge(S_1 \chi_{(3d^{1/3},\infty)}) \oplus \vee(S_2) \tag{23}$$

where the curve S_1 is described by $(\widetilde{x}(\lambda), \widetilde{y}(\lambda))$ for $\lambda \leq -d^{1/3}$, S_2 is described by $(\widetilde{x}(\lambda), \widetilde{y}(\lambda))$ for $-d^{1/3} < \lambda < 0$ and S_2 is described by $(\widetilde{x}(\lambda), \widetilde{y}(\lambda))$ for $\lambda > 0$. See Figure 4.

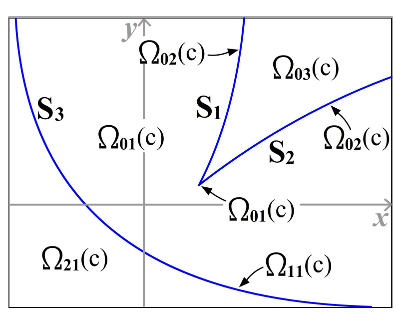


Figure 4

4 Main Results

We recall the equation (8). Let

$$\Omega(\tau_1,\tau_2)=\left\{(x,y)\in\mathbf{R}^2:(8)\text{ has no real roots}\right\}$$

for $\tau_1,\tau_2\geq 0$. We apply the Cheng-Lin envelope method to find the set $\Omega(\tau_1,\tau_2)$ for $\tau_1,\tau_2\geq 0$. For each $\lambda\in\mathbf{R}$, let L_λ be the straight line in the plane defined by

$$L_\lambda: x\lambda e^{-\lambda\tau_1}+ye^{-\lambda\tau_2}=-\lambda^2. \tag{24}$$

Note that L_λ defined by (24) is of the form (9) and

$$\alpha'(\lambda)\beta(\lambda)-\alpha(\lambda)\beta'(\lambda)=e^{-\lambda(\tau_1+\tau_2)}(1+(\tau_2-\tau_1)\lambda) \tag{25}$$

for $\lambda\in\mathbf{R}$. In view of (25), we may consider the following two cases: $\tau_1=\tau_2$ and $\tau_1\neq\tau_2$.

4.1 The case $\tau_1=\tau_2$

From (25), $\alpha'(\lambda)\beta(\lambda)-\alpha(\lambda)\beta'(\lambda)\neq 0$ for $\lambda\in\mathbf{R}$. From (10), we let C be the curve defined by the parametric functions

$$x(\lambda)=\left(-\tau_1\lambda^2+2\lambda\right)e^{\lambda\tau_1}\text{ and }y(\lambda)=\left(\tau_1\lambda^3+\lambda^2\right)e^{\lambda\tau_1}\text{ for }\lambda\in\mathbf{R}. \tag{26}$$

By Theorem 2.1, C is the envelope of the family $\{L_\lambda:\lambda\in\mathbf{R}\}$ where L_λ is defined by (24). We have

$$(x(0),y(0))=(0,0),$$

$$\lim_{\lambda\to-\infty}(x(\lambda),y(\lambda))=(0,0),\quad \lim_{\lambda\to\infty}(x(\lambda),y(\lambda))=(-\infty,\infty),$$

$$x'(\lambda)=-e^{\lambda\tau_1}\left(\tau_1^2\lambda^2+4\tau_1\lambda+2\right)\text{ and }y'(\lambda)=\lambda e^{\lambda\tau_1}\left(\tau_1^2\lambda^2+4\tau_1\lambda+2\right)$$

for $\lambda\in\mathbf{R}$. Furthermore,

$$\frac{y'(\lambda)}{x'(\lambda)}=-\lambda\text{ and }\frac{\frac{d}{d\lambda}\left(\frac{y'(\lambda)}{x'(\lambda)}\right)}{x'(\lambda)}=\frac{e^{-\tau_1\lambda}}{\tau_1^2\lambda^2+4\tau_1\lambda+2} \tag{27}$$

for $\lambda\neq\lambda_1$ and $\lambda\neq\lambda_2$ where

$$\lambda_1=\frac{-2-\sqrt{2}}{\tau_1}\text{ and }\lambda_2=\frac{-2+\sqrt{2}}{\tau_1}.$$

Then x is strictly increasing on (λ_1,λ_2) and strictly decreasing on $(-\infty,\lambda_1)\cup(\lambda_2,\infty)$, and y is strictly increasing on $(\lambda_1,\lambda_2)\cup(0,\infty)$ and strictly decreasing on $(-\infty,\lambda_1)\cup(\lambda_2,0)$. We can see that C is composed of three pieces C_1, C_2 and C_3 restricted respectively to $(-\infty,\lambda_1]$, $(\lambda_1,\lambda_2]$ and (λ_2,∞). C_1 is the graph of a function $y=C_1(x)$ which is strictly increasing, strictly convex, and smooth over $[x(\lambda_1),0)$; C_2 is the graph of a function $y=C_2(x)$ which is strictly increasing, strictly concave and smooth over

$(x(\lambda_1), x(\lambda_2)]$; and C_3 is the graph of a function $y = C_3(x)$ which is strictly convex and smooth over $(-\infty, x(\lambda_2))$. See Figure 5. We have

$$C_1^{(v)}(x(\lambda_1)) = C_2^{(v)}(x(\lambda_1)^+) \text{ and } C_2^{(v)}(x(\lambda_2)) = C_3^{(v)}(x(\lambda_2)^-),\ v = 1, 2.$$

Furthermore, $C_3\tilde{}H_{-\infty}$ by Lemma 2.3 and (27). In view of Theorem (2.2), $\Omega(\tau_1, \tau_1)$ is the intersection of dual sets of order 0 of C_1, C_2, and C_3. By Theorem 3.7 in [3], the dual set of order 0 of C_1 is

$$\vee(L_{\lambda_1}) \oplus \vee(C_1) \cup \{(0, y) : y \geq 0\} \cup \left\{\wedge(L_{\lambda_1}\chi_{[0,\infty)}) \cup \{(0, y) : y < 0\}\right\}.$$

See Figure 5(a). By Theorem A.3 in [3], the intersection of dual sets of order 0 of C_2 and C_3 is $\overline{\nabla}(L_{\lambda_1}) \oplus \vee(C_3)$. See Figure 5(b). So we have $\Omega(\tau_1, \tau_1) = \vee(C_3\chi_{(-\infty,0]})$. See Figure 5(c).

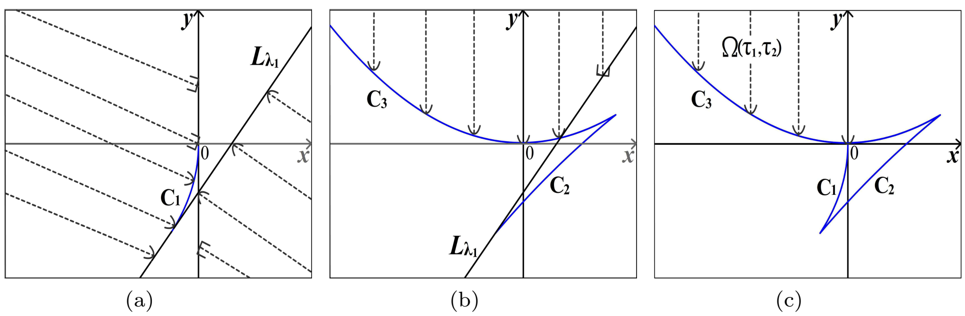


Figure 5: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems 3.7 and A.5) to yield (c).

THEOREM 4.1. Assume that $\tau_1 = \tau_2$. Let $x(\lambda)$ and $y(\lambda)$ be defined by (26). Then the equation (8) has no real roots if, and only if, $(x, y) \in \vee(C)$ where the curve C is described by $(x(\lambda), y(\lambda))$ for $\lambda \geq 0$.

4.2 The case $\tau_1 \neq \tau_2$

Let $\lambda_* = 1/(\tau_1 - \tau_2)$. From (10), we let C be the curve defined by the parametric functions

$$x(\lambda) = -\frac{\tau_2\lambda + 2}{1 + (\tau_2 - \tau_1)\lambda}\lambda e^{\lambda\tau_1} \text{ and } y(\lambda) = \frac{\tau_1\lambda^3 + \lambda^2}{1 + (\tau_2 - \tau_1)\lambda}e^{\lambda\tau_2} \tag{28}$$

for $\lambda \in \mathbf{R}\backslash\{\lambda_*\}$. We have

$$(x(0), y(0)) = (0, 0),$$

$$\lim_{\lambda\to-\infty}(x(\lambda), y(\lambda)) = \begin{cases} (0,0) & \text{if } \tau_1\tau_2 \neq 0 \\ (\infty, 0) & \text{if } \tau_2 > \tau_1 = 0 \\ (0, -\infty) & \text{if } \tau_1 > \tau_2 = 0 \end{cases},$$

$$\lim_{\lambda\to\infty}(x(\lambda),y(\lambda))=\begin{cases}(-\infty,\infty) & \text{if } \tau_2\geq\tau_1>0\\(-\infty,\infty) & \text{if } \tau_2>\tau_1=0\\(\infty,-\infty) & \text{if } \tau_1>\tau_2>0\\(\infty,-\infty) & \text{if } \tau_1>\tau_2=0\end{cases},$$

$$\lim_{\lambda\to\lambda_*^-}(x(\lambda),y(\lambda))=\begin{cases}(\infty,\infty) & \text{if } \tau_1<\tau_2<2\tau_1\\(-\infty,-\infty) & \text{if } 2\tau_1<\tau_2\\(-\infty,\infty) & \text{if } \tau_2<\tau_1\end{cases},$$

and

$$\lim_{\lambda\to\lambda_*^+}(x(\lambda),y(\lambda))=\begin{cases}(-\infty,-\infty) & \text{if } \tau_1<\tau_2<2\tau_1\\(\infty,\infty) & \text{if } 2\tau_1<\tau_2\\(\infty,-\infty) & \text{if } \tau_2<\tau_1\end{cases}.$$

We note that

$$x'(\lambda)=\frac{-e^{\lambda\tau_1}g(\lambda)}{(1+(\tau_2-\tau_1)\lambda)^2} \text{ and } y'(\lambda)=\frac{\lambda e^{\lambda\tau_2}g(\lambda)}{(1+(\tau_2-\tau_1)\lambda)^2} \text{ for } \lambda\in\mathbf{R}\backslash\{\lambda_*\} \tag{29}$$

where

$$g(\lambda)=\tau_1\tau_2(\tau_2-\tau_1)\lambda^3+(\tau_2^2-2\tau_1^2+2\tau_1\tau_2)\lambda^2+2(\tau_1+\tau_2)\lambda+2. \tag{30}$$

We observe that

$$g(\lambda_*)=\frac{2\tau_1-\tau_2}{\tau_1-\tau_2}. \tag{31}$$

Let $\Sigma(\tau_1,\tau_2)=\{\lambda\in\mathbf{R}: g(\lambda)=0\}$. Then

$$\frac{y'(\lambda)}{x'(\lambda)}=-\lambda e^{(\tau_2-\tau_1)\lambda} \text{ and } \frac{\frac{d}{d\lambda}\left(\frac{y'(\lambda)}{x'(\lambda)}\right)}{x'(\lambda)}=\frac{e^{(\tau_2-2\tau_1)\lambda}(1+(\tau_2-\tau_1)\lambda)^3}{g(\lambda)} \tag{32}$$

for $\lambda\in\mathbf{R}\backslash\Sigma(\tau_1,\tau_2)$ and $\lambda\neq\lambda_*$. We need to further analyze the function g in order to understand the standard properties of the curve C. Therefore, we consider five cases. Case 1: $0=\tau_2<\tau_1$; Case 2: $0=\tau_1<\tau_2$; Case 3: $2\tau_1=\tau_2$; Case 4: $0<\tau_1<\tau_2$ and $2\tau_1\neq\tau_2$; and Case 5: $0<\tau_2<\tau_1$.

Case 1. In this case, we note that

$$g(\lambda)=-2\left(\tau_1^2\lambda^2-\tau_1\lambda-1\right) \text{ for } \lambda\in\mathbf{R}.$$

Then $g(\lambda)$ has two real roots λ_1 and λ_2 where

$$\lambda_1=\frac{1-\sqrt{5}}{2\tau_1} \text{ and } \lambda_2=\frac{1+\sqrt{5}}{2\tau_1}.$$

Clearly, $\lambda_1<0<\lambda_*<\lambda_2$. In view of (29), we can see that $x(\lambda)$ is strictly increasing on $(-\infty,\lambda_1)\cup(\lambda_2,\infty)$ and strictly decreasing on $(\lambda_1,\lambda_*)\cup(\lambda_*,\lambda_2)$, and $y(\lambda)$ is strictly increasing on $(-\infty,\lambda_1)\cup(0,\lambda_*)\cup(\lambda_*,\lambda_2)$ and strictly decreasing on $(\lambda_1,0)\cup(\lambda_2,\infty)$. We can further see that C is composed of four pieces C_1, C_2, C_3 and C_4 restricted respectively to $(-\infty,\lambda_1)$, $[\lambda_1,\lambda_*)$, $(\lambda_*,\lambda_2]$ and $(\lambda_2,+\infty)$. Then C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly concave, and smooth over $(0, x(\lambda_1))$; C_2 is the graph of a function $y = C_2(x)$ which is strictly convex and smooth over $(-\infty, x(\lambda_1)]$; C_3 is the graph of a function $y = C_3(x)$ which is strictly decreasing, strictly concave, and smooth over $[x(\lambda_2), x(\lambda_*))$; and C_4 is the graph of a function $y = C_4(x)$ which is strictly decreasing, strictly convex, and smooth over $(x(\lambda_2), \infty)$. See Figure 6. We have

$$C_1^{(v)}(x(\lambda_1)^-) = C_2^{(v)}(x(\lambda_1)) \text{ and } C_3^{(v)}(x(\lambda_2)) = C_4^{(v)}(x(\lambda_2)^+),\ v = 1, 2.$$

By Lemma 2.3 and (32), $C_1 \tilde{} H_{0+}$ and $C_4 \tilde{} H_{+\infty}$. By Lemma 2.4, we can see that the intersection of dual sets of order 0 of C_1 and C_2 is

$$\vee(C_2\chi_{(-\infty,0]}) \cup \{\wedge(C_1) \oplus \underline{\triangle}(L_{\lambda_*})\}.$$

See Figure 6(a). By Theorem A.8 in [3], we can further see that the intersection of dual sets of order 0 of C_3 and C_4 is

$$\vee(C_4) \oplus \overline{\nabla}(L_{\lambda_*}).$$

See Figure 6(b). By Theorem (2.2), $\Omega(\tau_1, 0) = \vee(C_2\chi_{(-\infty,0]})$. See Figure 6(c).

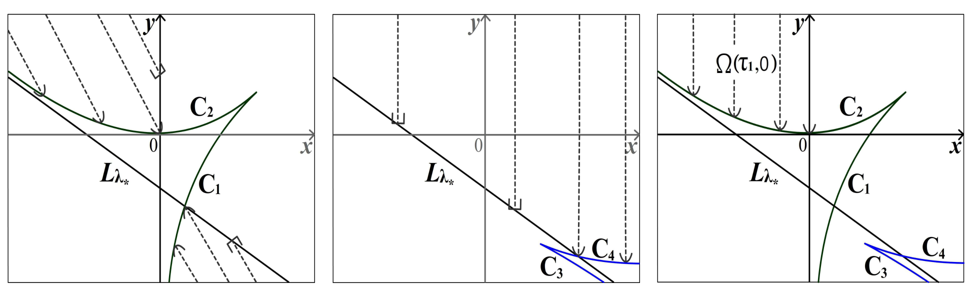


Figure 6: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorem A.8] and Lemma 2.4) to yield (c).

THEOREM 4.2. Assume that $0 = \tau_2 < \tau_1$. Let $x(\lambda)$ and $y(\lambda)$ be defined by (28). Then $\Omega(\tau_1, 0) = \vee(C)$ where the curve C is described by $(x(\lambda), y(\lambda))$ for $0 \le \lambda < 1/\tau_1$.

Case 2. In this case, we note that $\lambda_* < 0$ and

$$g(\lambda) = \tau_2^2\lambda^2 + 2\tau_2\lambda + 2 > 0 \text{ for } \lambda \in \mathbf{R}.$$

By (29), we can see that $x(\lambda)$ is strictly decreasing on $\mathbf{R}\backslash\{\lambda_*\}$, and $y(\lambda)$ is strictly increasing on $(0, \infty)$ and strictly decreasing on $(-\infty, \lambda_*) \cup (\lambda_*, 0)$. We can further see that C is composed of two pieces C_1 and C_2 restricted respectively to $(-\infty, \lambda_*)$ and (λ_*, ∞). Then C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly concave, and smooth over $\mathbf{R}$; and C_2 is the graph of a function $y = C_2(x)$

which is strictly convex and smooth over **R**. By Lemma 2.3 and (32), $C_2\tilde{\ }H_{-\infty}$. We can apply Theorems 3.18 and 3.19 in [3] respectively to see the dual set of order 0 of C_1 and the dual set of order 0 of C_2. See Figures 7(a) and (b). By Theorem 2.2, $\Omega(0,\tau_2) = \vee(C_2)$. See Figure 7(c).

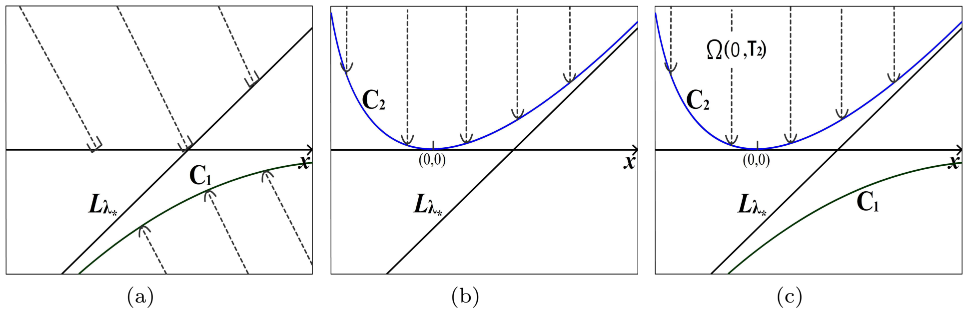


Figure 7: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems 3.18 and 3.19] to yield (c).

THEOREM 4.3. Assume that $0 = \tau_1 < \tau_2$. Let $x(\lambda)$ and $y(\lambda)$ be defined by (28). Then $\Omega(0,\tau_2) = \vee(C)$ where the curve C is described by $(x(\lambda), y(\lambda))$ for $\lambda > 0$.

Case 3. In this case, we note that $x(\lambda) = -2\lambda e^{\lambda\tau_1}$ and $y(\lambda) = \lambda^2 e^{2\lambda\tau_1}$ for $\lambda \in \mathbf{R}\backslash\{\lambda_*\}$. Furthermore, we have

$$\lim_{\lambda\to\lambda_*^-} (x(\lambda), y(\lambda)) = \left(\frac{2}{\tau_1 e}, \frac{1}{\tau_1^2 e^2}\right).$$

Then the curve C is the graph of a function $y = C(x) = x^2/4$ for $x < 2/\tau_1 e$. Clearly, $C\tilde{\ }H_{-\infty}$. By Theorem 3.11 in [3], $\Omega(\tau_1,\tau_2) = \vee(C) \oplus \overline{\vee}(L_{\lambda_*})$. See Figure 8.

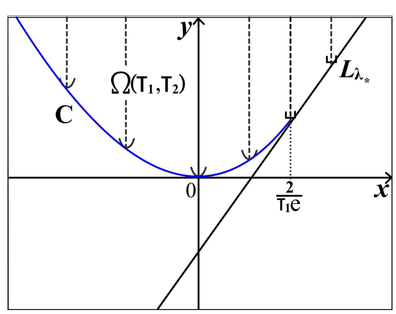


Figure 8

THEOREM 4.4. Assume that $2\tau_1 = \tau_2$. Then

$$\Omega(\tau_1,\tau_2) = \left\{(x,y)\in \mathbf{R}^2 : y > \frac{x^2}{2} \text{ and } x < \frac{2}{\tau_1 e}\right\} \cup \left\{(x,y)\in \mathbf{R}^2 : y \geq \left(\frac{e}{\tau_1}x - \frac{1}{\tau_1^2}\right) e^{-\frac{\tau_2}{\tau_1}} \text{ and } x \geq \frac{2}{\tau_1 e}\right\}.$$

Case 4. Assume that $0 < \tau_1 < \tau_2$ and and $2\tau_1 \neq \tau_2$. Then g is a cubic polynomial with positive leading term. Let

$$A = \frac{\tau_2^2 - 2\tau_1^2 + 2\tau_1\tau_2}{\tau_1\tau_2(\tau_2-\tau_1)},\ B = \frac{2(\tau_1+\tau_2)}{\tau_1\tau_2(\tau_2-\tau_1)} \text{ and } D = \frac{2}{\tau_1\tau_2(\tau_2-\tau_1)}. \tag{33}$$

Clearly, $B > 0$ and $D > 0$. We note that

$$A = \frac{(1+\sqrt{3})\tau_1 + \tau_2}{\tau_1\tau_2(\tau_2-\tau_1)}\left((1-\sqrt{3})\tau_1 + \tau_2\right) > \frac{(1+\sqrt{3})\tau_1 + \tau_2}{\tau_1\tau_2(\tau_2-\tau_1)}(2-\sqrt{3})\tau_1 > 0.$$

By Lemma 3.1, (A, B) belongs to one of $\Omega_{01}(D)$, $\Omega_{02}(D)$ and $\Omega_{03}(D)$. Therefore, we need to consider three cases. Case 4-1: $(A, B) \in \Omega_{01}(D)$; Case 4-2: $(A, B) \in \Omega_{02}(D)$; and Case 4-3: $(A, B) \in \Omega_{03}(D)$.

Case 4-1. In this case, the cubic polynomial

$$\lambda^3 + A\lambda^2 + B\lambda + D = 0$$

has a unique root λ_1 with $\lambda_1 < 0$. It follows that $g(\lambda) < 0$ for $\lambda < \lambda_1$ and $g(\lambda) > 0$ for $\lambda > \lambda_1$. If $\tau_2 > 2\tau_1$, by (31), we may see that $\lambda_1 < \lambda_* < 0$ because of $g(\lambda_*) > 0$. Then x is strictly increasing on $(-\infty, \lambda_1)$ and strictly decreasing on $(\lambda_1, \lambda_*) \cup (\lambda_*, \infty)$, and y is strictly increasing on $(-\infty, \lambda_1) \cup (0, \infty)$ and strictly decreasing on $(\lambda_1, \lambda_*) \cup (\lambda_*, 0)$. We may see that C is composed of three pieces C_1, C_2 and C_3 restricted respectively to $(-\infty, \lambda_{\cdot})$, $[\lambda_1, \lambda_*)$ and (λ_*, ∞). Then C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly convex, and smooth over $(0, x(\lambda_1))$; C_2 is the graph of a function $y = C_2(x)$ which is strictly increasing, strictly concave, and smooth over $(-\infty, x(\lambda_1)]$; and C_3 is the graph of a function $y = C_3(x)$ which is strictly convex and smooth over $(-\infty, \infty)$. We have

$$C_1^{(v)}(x(\widetilde{\lambda})) = C_2^{(v)}(x(\lambda_1)^-),\ v = 1, 2.$$

By Lemma 2.3 and (32), $C_3\tilde{\ }H_{-\infty}$. See Figure 9. By Theorem A.5 in [3], the intersection of dual sets of order 0 of C_1 and C_2 is

$$\{\vee(C_1) \oplus \overline{\nabla}(L_{\lambda_*}) \oplus \overline{\nabla}(\Theta_0)\} \cup \{\wedge(C_2) \oplus \underline{\triangle}(\Theta_0)\}.$$

See Figure 9(a). By Theorem 3.19 in [3], the dual set of order 0 of C_3 is $\vee(C_3)$. See Figure 9(b). Therefore, $\Omega(\tau_1, \tau_2) = \vee(C_1) \oplus \vee(C_3)$. See Figure 9(c).

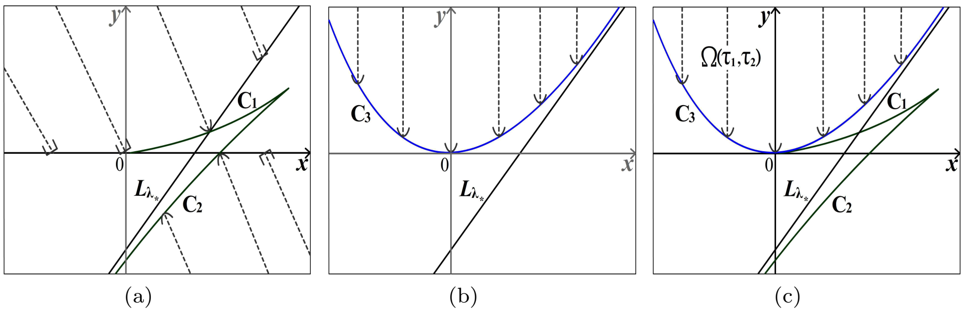


Figure 9: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems A.5 and 3.19]) to yield (c).

If $2\tau_1 > \tau_2$, by (31), we may see that $\lambda_* < \lambda_1 < 0$ because of $g(\lambda_*) < 0$. Then x is strictly increasing on $(-\infty, \lambda_*) \cup (\lambda_*, \widetilde{\lambda})$ and strictly decreasing on $(\widetilde{\lambda}, \infty)$, and y is strictly increasing on $(-\infty, \lambda_*) \cup (\lambda_*, \widetilde{\lambda}) \cup (0, \infty)$ and strictly decreasing on $(\widetilde{\lambda}, 0)$. We may see that C is composed of three pieces C_1, C_2 and C_3 restricted respectively to $(-\infty, \lambda_*)$, $(\lambda_*, \widetilde{\lambda})$ and $[\widetilde{\lambda}, \infty)$. Then C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly convex, and smooth over $(0, \infty)$; C_2 is the graph of a function $y = C_2(x)$ which is strictly increasing, strictly concave, and smooth over $(-\infty, x(\widetilde{\lambda}))$; and C_3 is the graph of a function $y = C_3(x)$ which is strictly convex and smooth over $(-\infty, \widetilde{\lambda}]$. We have

$$C_2^{(v)}(x(\widetilde{\lambda})^-) = C_3^{(v)}(x(\widetilde{\lambda})),\ v = 1, 2.$$

By Lemma 2.3 and (32), $C_3\tilde{\ }H_{-\infty}$. By Theorems 3.10 and A.8 in [3], $\Omega(\tau_1, \tau_2) = \vee(C_1) \oplus \vee(C_3)$. See Figure 10.

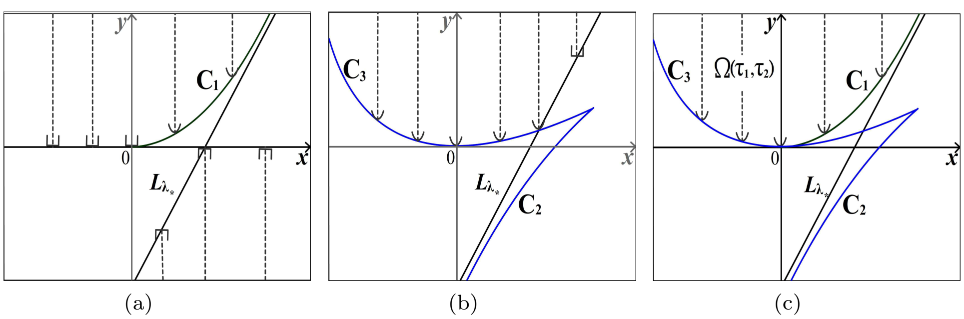


Figure 10: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems 3.10 and A.8] to yield (c).

THEOREM 4.5. Assume that $0 < \tau_1 < \tau_2$ and $(A, B) \in \Omega_{01}(D)$ where A, B and D are defined by (33). Let λ_1 be the root of g defined by (30). Then $\Omega(\tau_1, \tau_2) = \vee(C_1) \oplus \vee(C_2)$ where the curve C_1 is described by $(x(\lambda), y(\lambda))$ for $\lambda < \min\{1/(\tau_1 - \tau_2), \lambda_1\}$, and curve C_2 is described by $(x(\lambda), y(\lambda))$ for $\lambda > \max\{1/(\tau_1 - \tau_2), \lambda_1\}$.

Case 4-2. In this case, we may know that the cubic polynomial

$$\lambda^3 + A\lambda^2 + B\lambda + D = 0$$

has exactly two distinct negative roots λ_1 and λ_2 with $\lambda_1 < \lambda_2$. Then either $g(\lambda) < 0$ for $\lambda < \lambda_1$ and $g(\lambda) > 0$ for $\lambda > \lambda_1$ and $\lambda \neq \lambda_2$, or $g(\lambda) > 0$ for $\lambda > \lambda_2$ and $g(\lambda) < 0$ for $\lambda < \lambda_2$ and $\lambda \neq \lambda_1$. See Figure 11.

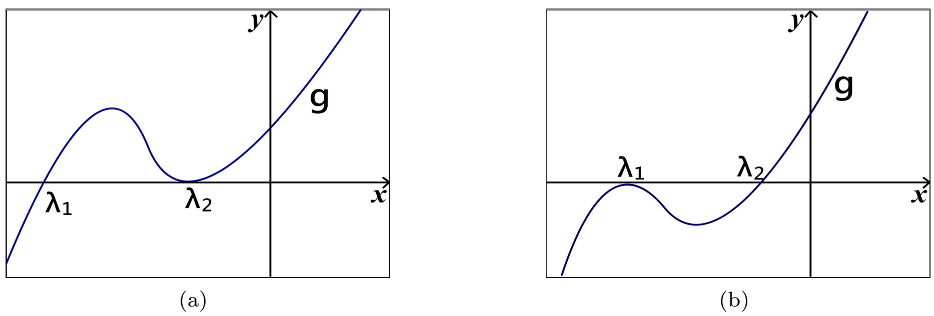


Figure 11

Assume that $g(\lambda) < 0$ for $\lambda < \lambda_1$ and $g(\lambda) > 0$ for $\lambda > \lambda_1$ and $\lambda \neq \lambda_2$. We may note that $g'(\lambda_1) \neq 0$ and $g'(\lambda_2) = 0$. If $2\tau_1 < \tau_2$, we may see that $\lambda_1 < \lambda_* < 0$ because of $g(\lambda_*) > 0$. Then x is strictly increasing on $(-\infty, \lambda_1)$ and strictly decreasing on $(\lambda_1, \lambda_*) \cup (\lambda_*, \infty)$, and y is strictly increasing on $(-\infty, \lambda_1) \cup (0, \infty)$ and strictly decreasing on $(\lambda_1, \lambda_*) \cup (\lambda_*, 0)$. We can see that C is composed of three pieces C_1, C_2,and C_3 restricted respectively to $(-\infty, \lambda_1)$, $[\lambda_1, \lambda_*)$ and (λ_*, ∞). Then C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly convex and smooth over $(0, x(\lambda_1))$; C_2 is the graph of a function $y = C_2(x)$ which is strictly increasing, strictly concave and smooth over $(-\infty, x(\lambda_1)]$; and C_3 is the graph of a function $y = C_3(x)$ which is strictly convex and smooth over $(-\infty, \infty)$. We have

$$C_1^{(v)}(x(\lambda_1)^-) = C_2^{(v)}(x(\lambda_1)) \text{ for } v = 1, 2.$$

By Lemma 2.3 and (32), $C_3\tilde{}H_{-\infty}$. The graph of C is similar to the graph described in Figure 9. So $\Omega(\tau_1, \tau_2) = \vee(C_1) \oplus \vee(C_3)$.

If $2\tau_1 > \tau_2$, we may see that $\lambda_* < \lambda_1 < 0$ because of $g(\lambda_*) < 0$. Then x is strictly increasing on $(-\infty, \lambda_*) \cup (\lambda_*, \lambda_1)$ and strictly decreasing on (λ_1, ∞), and y is strictly increasing on $(-\infty, \lambda_*) \cup (\lambda_*, \lambda_1) \cup (0, \infty)$ and strictly decreasing on $(\lambda_1, 0)$. We may see that C is composed of three pieces C_1, C_2,and C_3 restricted respectively to $(-\infty, \lambda_*)$, $(\lambda_*, \lambda_1]$, and (λ_1, ∞). Then C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly convex, and smooth over $(0, \infty)$; C_2 is the graph of a function $y = C_2(x)$ which is strictly increasing, strictly concave, and smooth over $(-\infty, x(\lambda_1)]$; and C_3 is the graph of a function $y = C_3(x)$ which is strictly convex and smooth over $(-\infty, \infty)$. We have

$$C_2^{(v)}(x(\lambda_1)^-) = C_3^{(v)}(x(\lambda_1)) \text{ for } v = 1, 2.$$

By Lemma 2.3 and (32), $C_3\tilde{\ }H_{-\infty}$. The graph of C is similar to the graph described in Figure 10. So $\Omega(\tau_1,\tau_2)=\vee(C_1)\oplus\vee(C_3)$.

Assume that $g(\lambda)>0$ for $\lambda>\lambda_2$ and $g(\lambda)<0$ for $\lambda<\lambda_2$ and $\lambda\neq\lambda_1$. By discussions similar to those above, we may obtain the same conclusion, $\Omega(\tau_1,\tau_2)=\vee(C_1)\oplus\vee(C_3)$.

THEOREM 4.6. Assume that $0<\tau_1<\tau_2$ and $(A,B)\in\Omega_{02}(D)$ where A, B and D are defined by (33). Let λ_1 be the root of g defined by (30) with $g'(\lambda_1)\neq 0$. Then $\Omega(\tau_1,\tau_2)=\vee(C_1)\oplus\vee(C_3)$. where curve C_1 is described by $(x(\lambda),y(\lambda))$ for $\lambda<\min\{1/(\tau_1-\tau_2),\lambda_1\}$ and curve C_2 is described by $(x(\lambda),y(\lambda))$ for $\lambda>\max\{1/(\tau_1-\tau_2),\lambda_1\}$.

Case 4-3. In this case, we may know that the cubic polynomial

$$\lambda^3+A\lambda^2+B\lambda+D=0$$

has three distinct negative roots λ_1, λ_2 and λ_3 with $\lambda_1<\lambda_2<\lambda_3$. Then $g(\lambda)>0$ on $(\lambda_1,\lambda_2)\cup(\lambda_3,\infty)$ and $g(\lambda)<0$ on $(-\infty,\lambda_1)\cup(\lambda_2,\lambda_3)$. We note that

$$g'(\lambda)=3\tau_1\tau_2(\tau_2-\tau_1)\lambda^2+2(\tau_2^2-2\tau_1^2+2\tau_1\tau_2)\lambda+2(\tau_1+\tau_2)$$

for $\lambda\in\mathbf{R}$. Since $g(\lambda_2)=g(\lambda_3)=0$, by the Mean Value Theorem, we may see that $g'(\lambda)$ has a real roots λ_+ such that $\lambda_2<\lambda_+<\lambda_3$ and

$$\lambda_+=\frac{-(\tau_2^2-2\tau_1^2+2\tau_1\tau_2)+\sqrt{(2\tau_1-\tau_2)\left(2\tau_1^3-\tau_2^3\right)}}{3\tau_1\tau_2(\tau_2-\tau_1)}.$$

We observe that

$$(2\tau_1-\tau_2)(\tau_1+\tau_2)^2=2\tau_1^3-\tau_2^3+3\tau_1^2\tau_2. \tag{34}$$

If $2\tau_1<\tau_2$, by (31), we may see that $g(\lambda_*)>0$. Thus either $\lambda_1<\lambda_*<\lambda_2$ or $\lambda_3<\lambda_*<0$. We claim that the former case $\lambda_1<\lambda_*<\lambda_2$ holds. It is sufficient to prove that $\lambda_+>\lambda_*$. In view of (34), we may see that $(2\tau_1-\tau_2)^2(\tau_1+\tau_2)^2<(2\tau_1-\tau_2)\left(2\tau_1^3-\tau_2^3\right)$, which implies that

$$\begin{aligned}&-\left(\tau_2^2-2\tau_1^2-\tau_1\tau_2\right)+\sqrt{(2\tau_1-\tau_2)\left(2\tau_1^3-\tau_2^3\right)}\\ >\;&(2\tau_1-\tau_2)(\tau_1+\tau_2)+\sqrt{(2\tau_1-\tau_2)^2(\tau_1+\tau_2)^2}\\ =\;&0.\end{aligned}$$

Then

$$\lambda_+=\frac{-(\tau_2^2-2\tau_1^2+2\tau_1\tau_2)+\sqrt{(2\tau_1-\tau_2)\left(2\tau_1^3-\tau_2^3\right)}}{3\tau_1\tau_2(\tau_2-\tau_1)}>\frac{1}{\tau_1-\tau_2}=\lambda_*$$

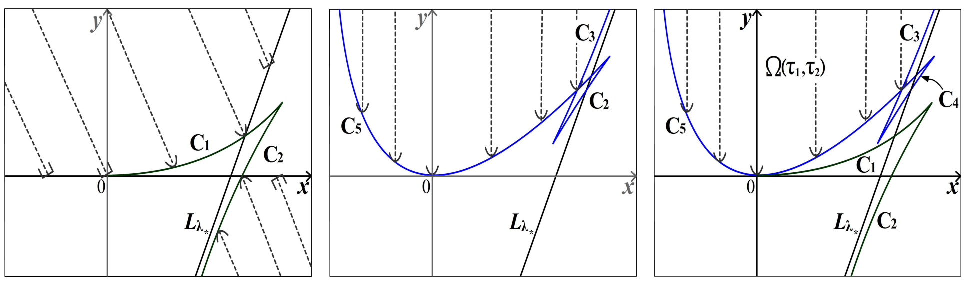
We have verified our assertion. We may now see that C is composed of five pieces C_1, C_2, C_3, C_4 and C_5 restricted respectively to $(-\infty,\lambda_1)$, $[\lambda_1,\lambda_*)$, (λ_*,λ_2), $[\lambda_2,\lambda_3]$ and

(λ_3, ∞). C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly convex, and smooth over $(0, x(\lambda_1))$; C_2 is the graph of a function $y = C_2(x)$ which is strictly increasing, strictly concave, and smooth over $(-\infty, x(\lambda_1)]$; C_3 is the graph of a function $y = C_3(x)$ which is strictly increasing, strictly convex and smooth over $(x(\lambda_2), \infty)$; C_4 is the graph of a function $y = C_4(x)$ which is strictly increasing, strictly concave, and smooth over $[x(\lambda_2), x(\lambda_3)]$; and C_5 is the graph of a function $y = C_5(x)$ which is strictly convex and smooth over $(-\infty, x(\lambda_3)]$. We have

$$C_2^{(v)}(x(\lambda_1)) = C_3^{(v)}(x(\lambda_1)^-),$$

$$C_3^{(v)}(x(\lambda_2)^+) = C_4^{(v)}(x(\lambda_2)) \text{ and } C_4^{(v)}(x(\lambda_3)) = C_5^{(v)}(x(\lambda_3)^-)$$

for $v = 1, 2$. By Lemma 2.3 and (32), $C_5 \tilde{} H_{-\infty}$. By Theorem A.5 in [3], the intersection of dual sets of order 0 of C_1 and C_2 can be seen in Figure 12(a). By Theorem A.16 in [3], the intersection of dual sets of order 0 of C_3, C_4 and C_5 can be seen in Figure 12(b). So $\Omega(\tau_1, \tau_2) = \vee(C_1) \oplus \vee(C_3). \oplus \vee(C_5)$. See Figure 12(c).



(a) (b) (c)

Figure 12: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorems A.5 and A.16]) to yield (c).

If $2\tau_1 > \tau_2$, by (31), we may see that $g(\lambda_*) < 0$. Thus either $\lambda_2 < \lambda_* < \lambda_3$ or $\lambda_* < \lambda_1$. We claim that $\lambda_2 < \lambda_* < \lambda_3$. It is sufficient to prove that $\lambda_+ < \lambda_*$. In view of (34), we can see that $(2\tau_1 - \tau_2)^2 (\tau_1 + \tau_2)^2 > (2\tau_1 - \tau_2)\left(2\tau_1^3 - \tau_2^3\right)$, which implies that

$$-\left(\tau_2^2 - 2\tau_1^2 - \tau_1\tau_2\right) + \sqrt{(2\tau_1 - \tau_2)\left(2\tau_1^3 - \tau_2^3\right)} < 0$$

Then

$$\lambda_+ = \frac{-(\tau_2^2 - 2\tau_1^2 + 2\tau_1\tau_2) + \sqrt{(2\tau_1 - \tau_2)\left(2\tau_1^3 - \tau_2^3\right)}}{3\tau_1\tau_2(\tau_2 - \tau_1)} < \frac{1}{\tau_1 - \tau_2} = \lambda_*.$$

We have verified our assertion. By analyzing the monotonicity of x and y, we may see that C is composed of five pieces C_1, C_2, C_3, C_4 and C_5 restricted respectively to $(-\infty, \lambda_1)$, $[\lambda_1, \lambda_2]$, (λ_2, λ_*), $(\lambda_*, \lambda_3]$ and (λ_3, ∞). Then C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly convex, and smooth over $(0, x(\lambda_1))$; C_2 is the graph of a function $y = C_2(x)$ which is strictly increasing, strictly concave, and smooth over $[x(\lambda_2), x(\lambda_1)]$; C_3 is the graph of a function $y = C_3(x)$ which is strictly increasing, strictly convex and smooth over $(x(\lambda_2),\infty)$; C_4 is the graph of a function $y=C_4(x)$ which is strictly increasing, strictly concave, and smooth over $(-\infty,x(\lambda_3)]$; and C_5 is the graph of a function $y=C_5(x)$ which is strictly convex and smooth over $(-\infty,x(\lambda_3))$. We have

$$C_1^{(v)}(x(\lambda_1)^-)=C_2^{(v)}(x(\lambda_1)),$$

$$C_2^{(v)}(x(\lambda_2))=C_3^{(v)}(x(\lambda_2)^+)\text{ and }C_4^{(v)}(x(\lambda_3))=C_5^{(v)}(x(\lambda_3)^-)$$

for $v=1,2$. By Lemma 2.3 and (32), $C_5\,\tilde{}\,H_{-\infty}$. By Theorem A.8 in [3] and Lemma 2.5, $\Omega(\tau_1,\tau_2)=\vee(C_1)\oplus\vee(C_3)\oplus\vee(C_5)$. See Figure 13(c).

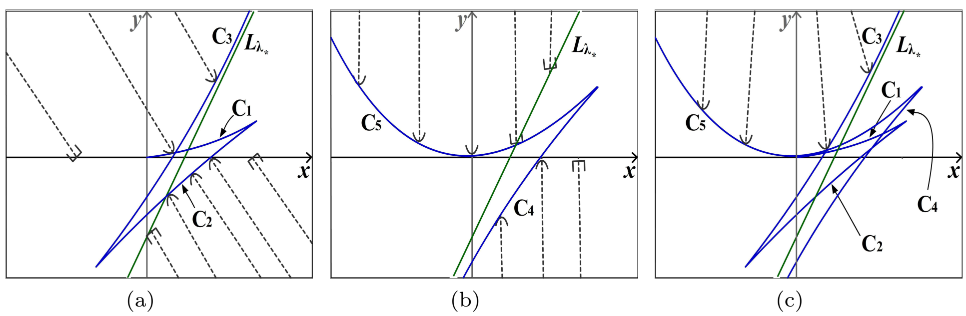


Figure 13: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorem A.8] and Lemma 2.5) to yield (c).

THEOREM 4.7. Assume that $0<\tau_1<\tau_2$ and $(A,B)\in\Omega_{03}(D)$ where A, B and D are defined by (33). Let λ_1, λ_2 and λ_3 be the roots of g defined by (30). Then $\Omega(\tau_1,\tau_2)=\vee(C_1)\oplus\vee(C_3)\oplus\vee(C_5)$ where curve C_1 is described by $(x(\lambda),y(\lambda))$ for $\lambda<\lambda_1$, curve C_2 is described by $(x(\lambda),y(\lambda))$ for $\min\{1/(\tau_1-\tau_2),\lambda_2\}<\lambda<\max\{1/(\tau_1-\tau_2),\lambda_2\}$, and curve C_3 is described by $(x(\lambda),y(\lambda))$ for $\lambda>\lambda_3$.

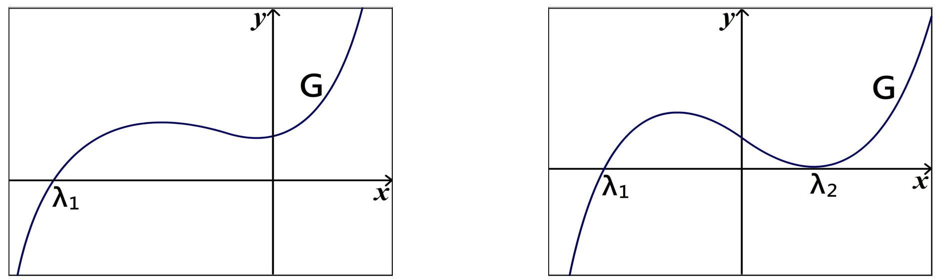
Case 5. Assume that $0<\tau_2<\tau_1$. Let G be a function defined by $G(\lambda)=g(-\lambda)$. We can see that G has exactly m positive roots if, and only if g has exactly m negative roots. We recall the numbers A, B and D defined by 33. Then

$$G(\lambda)=\tau_1\tau_2(\tau_1-\tau_2)\left(\lambda^3-A\lambda^2+B\lambda-D\right).$$

We note that $B<0$ and $-D>0$. Then $(-A,B)\in\Omega_{01}(-D)$ or $(-A,B)\in\Omega_{11}(-D)$ or $(-A,B)\in\Omega_{21}(-D)$. If $(-A,B)\in\Omega_{01}(-D)\cup\Omega_{11}(-D)$, then G has a negative root λ_1, and there exists $\lambda_2>0$ such that $G(\lambda)<0$ for $\lambda<\lambda_1$ and $G(\lambda)>0$ for $\lambda>\lambda_1$ and $\lambda\neq\lambda_2$. See Figure 14.

It implies that $g(\lambda)<0$ for $\lambda>-\lambda_1$ and $g(\lambda)>0$ for $\lambda<-\lambda_1$ and $\lambda\neq-\lambda_2$. Then x is strictly decreasing in $(-\infty,0]$. It is impossible that

$$x(-\infty)=0>x(0)=0.$$



(a) (b)

Figure 14

So we may assume that $(-A, B) \in \Omega_{21}(-D)$. Then g has three real roots λ_1, λ_2 and λ_3 with $\lambda_1 < \lambda_2 < 0 < \lambda_3$, and $g(\lambda) > 0$ on $(-\infty, \lambda_1) \cup (\lambda_2, \lambda_*) \cup (\lambda_*, \lambda_3)$ and $g(\lambda) < 0$ on $(\lambda_1, \lambda_2) \cup (\lambda_3, \infty)$. In view of (31), we can see that $\lambda_1 < \lambda_2 < 0 < \lambda_* < \lambda_3$. By analyzing the monotonicity of x and y, we may see that C is composed of five pieces C_1, C_2, C_3, C_4,and C_5 restricted respectively to $(-\infty, \lambda_1)$, $[\lambda_1, \lambda_2]$, (λ_2, λ_*), $(\lambda_*, \lambda_3]$ and (λ_3, ∞). C_1 is the graph of a function $y = C_1(x)$ which is strictly increasing, strictly convex, and smooth over $(x(\lambda_1), 0)$; C_2 is the graph of a function $y = C_2(x)$ which is strictly increasing, strictly concave, and smooth over $[x(\lambda_1), x(\lambda_2)]$; C_3 is the graph of a function $y = C_3(x)$ which is strictly convex and smooth over $(-\infty, x(\lambda_2))$; C_4 is the graph of a function $y = C_4(x)$ which is strictly decreasing, strictly concave, and smooth over $[x(\lambda_3), \infty)$; and C_5 is the graph of a function $y = C_5(x)$ which is strictly decreasing, strictly convex, and smooth over $(x(\lambda_3), \infty)$. We have

$$C_1^{(v)}(x(\lambda_1)^+) = C_2^{(v)}(x(\lambda_1))$$

$$C_2^{(v)}(x(\lambda_2)) = C_3^{(v)}(x(\lambda_2)^-) \text{ and } C_4^{(v)}(x(\lambda_1)) = C_5^{(v)}(x(\lambda_1)^+)$$

for $v = 1, 2$. By Lemma 2.3 and (32), $C_5\tilde{\ }H_{-\infty}$. By Theorem A.8 in [3] and Lemma 2.5, $\Omega(\tau_1, \tau_2) = \vee(C_3\chi_{(-\infty,0]})$. See Figure 15.

THEOREM 4.8. Assume that $0 < \tau_2 < \tau_1$. Let λ_1 be the positive root of g defined by (30). Then $\Omega(\tau_1, \tau_2) = \vee(D)$. where the curve C is described by $(x(\lambda), y(\lambda))$ for $0 \le \lambda < 1/(\tau_1 - \tau_2)$.

5 Examples

We illustrate our results with two examples.

EXAMPLE 5.1. Consider the equation

$$\lambda^2 + a\lambda e^{-\lambda} + be^{-3\lambda} = 0 \tag{35}$$

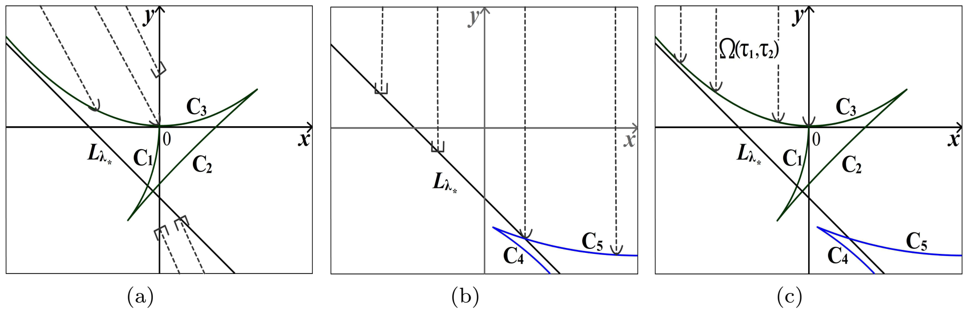


Figure 15: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Theorem A,8] and Lemma 2.5) to yield (c).

where $a, b \in \mathbf{R}$. In view of (33), $A = 13/6$, $B = 4/3$ and $D = 1/3$. Let curve S_1 be described by $(\widetilde{x}(\lambda), \widetilde{y}(\lambda))$ for $\lambda < 0$ and curve S_2 by $(\widetilde{x}(\lambda), \widetilde{y}(\lambda))$ for $\lambda > 0$ where

$$\widetilde{x}(\lambda) = -2\lambda + \frac{1}{3\lambda^2} \text{ and } \widetilde{y}(\lambda) = \lambda^2 - \frac{2}{3\lambda}.$$

We note that

$$\widetilde{x}(-D^{1/3}) = 3^{2/3} \text{ and } \widetilde{y}(-D^{1/3}) = 3^{1/3}.$$

Then $A > \widetilde{x}(-D^{1/3})$ and $B < \widetilde{y}(-D^{1/3})$. See Figure 16. By Theorem 3.2, we may see that $(A, B) \in \Omega_{01}(D)$. We note that

$$g(\lambda) = 6\lambda^3 + 13\lambda^2 + 8\lambda + 2 \text{ for } \lambda \in \mathbf{R}.$$

Then g has the unique negative root $\lambda_1 \approx -1.3719$. Let curve C_1 be described by $(x(\lambda), y(\lambda))$ for $\lambda < \lambda_1$, and curve C_2 by $(x(\lambda), y(\lambda))$ for $\lambda > -0.5$ where

$$x(\lambda) = -\frac{3\lambda + 2}{1 + 2\lambda}\lambda e^{\lambda} \text{ and } y(\lambda) = \frac{\lambda^3 + \lambda^2}{1 + 2\lambda}e^{3\lambda}.$$

By Theorem 4.5, $\Omega(1, 3) = \vee(C_1) \oplus \vee(C_2)$. See Figure 17. So $(a, b) \in \vee(C_1) \oplus \vee(C_2)$ if, and only if, the equation (35) has no real roots. As an application, we see that $(a, b) \in \vee(C_1) \oplus \vee(C_2)$ if, and only if, the all solutions of delay differential equation

$$N''(t) + aN'(t-1) + bN(t-3) = 0$$

are oscillatory. On the other hand, we may also consider the oscillation of the impulsive delay differential equation

$$x''(t) + ax'(t-1) + bx(t-3) = 0, \; t \in [0, \infty)\backslash\Gamma, \tag{36}$$

$$x(t_k^+) = a_k x(t_k), \; k \in \mathbf{N}, \tag{37}$$

$$x'(t_k^+) = a_k x'(t_k), \; k \in \mathbf{N}, \tag{38}$$

where $a_k > 0$ for $k \in \mathbf{N}$. Indeed, we assume that $A(t-1, t) = \alpha_1$ and $A(t-3, t) = \alpha_2$ for $t \geq 0$. By Lemma 1.1, all solutions of system (36)-(38) are oscillatory if, and only if, $(a/\alpha_1, b/\alpha_2) \in \vee(C_1) \oplus \vee(C_2)$.

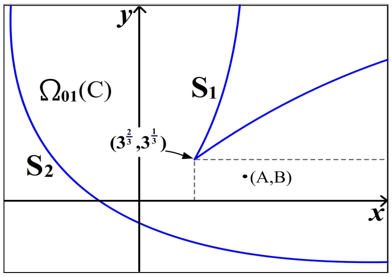


Figure 16

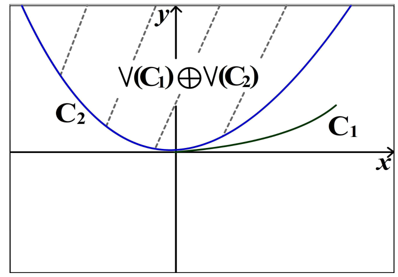


Figure 17

EXAMPLE 5.2. Consider the equation

$$\lambda^2 + a\lambda e^{-2\tau\lambda} + be^{-\tau\lambda} = 0 \tag{39}$$

where $a, b \in \mathbf{R}$. Let the curve C be described by $(x(\lambda), y(\lambda))$ for $0 \leq \lambda < 1/\tau$ where

$$x(\lambda) = -\frac{\tau\lambda + 2}{1 - \tau\lambda}\lambda e^{2\lambda\tau} \text{ and } y(\lambda) = \frac{2\tau\lambda^3 + \lambda^2}{1 - \tau\lambda}e^{\lambda\tau}.$$

By Theorem 4.8, we may see that $(a, b) \in \vee(C)$ if, and only if equation (39) has no real roots. We will give a criterion in order to facilitate determination. Let $0 < k < 1$. Let

$$m(k) = \frac{y(\frac{1}{k\tau})}{x(\frac{1}{k\tau})} = \frac{k+2}{-(2k+1)\,k^2\tau e^{\frac{1}{k}}}.$$

Since C is strictly convex, we may see that the line segment $L_k(x) = m(k)x$ on $(x(\frac{1}{k\tau}), 0)$ lies above the curve C. In other words, the point (a, b) satisfies $x(\frac{1}{k\tau}) < a < 0$ and $L(a) \leq b$ belongs to $\vee(C)$. Therefore, if there exists $0 < k < 1$ such that

$$-\frac{2k+1}{(k-1)\,k\tau}e^{\frac{2}{k}} < a < 0 \text{ and } \frac{-(k+2)}{(2k+1)\,k^2\tau e^{\frac{1}{k}}}a < b,$$

then equation (39) has no real roots. See Figure 18.

6 Appendix

Before proving Lemma 1.1, we need the definition of a solution of impulsive delay differential equation (3)-(5) and some basic concepts. Let Λ_1 and Λ_2 be two subsets of $\mathbf{R}$. We first define two sets

$$PC(\Lambda_1, \Lambda_2) = \{\varphi : \Lambda_1 \to \Lambda_2 | \varphi \text{ is continuous in each interval } \Lambda_1 \cap (t_k, t_{k+1}], k \in \mathbf{N} \cup \{0\} \text{ with discontinuities of the first kind only}\}$$

and

$$PC'(\Lambda_1, \Lambda_2) = \{\varphi \in PC(\Lambda_1, \Lambda_2) | \varphi \text{ is continuously differentiable a.e. in } \Lambda_1\}$$

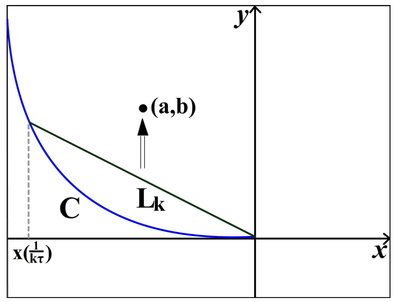


Figure 18

DEFINITION 6.1. Let Λ be an interval in $[0,\infty)$, $T = \inf \Lambda$ and $r_T = \min\{T - \tau_1, T - \tau_2\}$. For any $\phi \in PC([r_T, T], \mathbf{R})$, a function x defined on $[r_T, T] \cup \Lambda$ is said to be a solution of system (3)-(5) on Λ satisfying the initial value condition $x(t) = \phi(t)$ for $t \in [r_T, T]$ if

(i) $x, x' \in PC'(\Lambda, \mathbf{R})$;

(ii) $x(t)$ satisfies (3) a.e. on Λ; and

(iii) $x(t)$ satisfies (4) and (5) on Λ.

DEFINITION 6.2. Let a function $\varphi(t)$ be defined for all sufficiently large t. We say that $\varphi(t)$ is eventually positive (or negative) if there exists a number T such that $\varphi(t) > 0$ (respectively $\varphi(t) < 0$) for every $t \geq T$. We say that $\varphi(t)$ is nonoscillatory if $\varphi(t)$ is eventually positive or eventually negative. Otherwise, $\varphi(t)$ is called oscillatory.

Proof of Lemma 1.1. Let $\tau = \max\{\tau_1, \tau_2\}$. Assume that the system (3)-(5) has a nonoscillatory solution $N(t)$. We may assume that $N(t) > 0$ for $t \geq -\tau$. Let $y(t) = N(t)/A(0,t)$ for $t \geq 0$. We note that

$$y(t_k^+) = \frac{N(t_k^+)}{A(0,t_k^+)} = \frac{a_k N(t_k)}{a_k A(0,t_k)} = y(t_k)$$

and

$$y'(t_k^+) = \frac{N'(t_k^+)}{A(0,t_k^+)} = \frac{a_k N'(t_k)}{a_k A(0,t_k)} = y'(t_k)$$

for $k \in \mathbf{N}$. Then $y(t)$ is a continuously differentiable function on $[0,\infty)$ and satisfies

$$\begin{aligned} & y''(t) + \frac{a}{\alpha_1} y'(t-\tau_1) + \frac{b}{\alpha_2} y(t-\tau_2) \\ = \; & \frac{1}{A(0,t)} \left(N''(t) + a\frac{A(t-\tau_1,t)}{\alpha_1} N'(t-\tau_1) + b\frac{A(t-\tau_2,t)}{\alpha_2} N(t-\tau_2) \right) \\ = \; & \frac{1}{A(0,t)} \left(N''(t) + aN'(t-\tau_1) + bN(t-\tau_2) \right) \\ = \; & 0 \end{aligned}$$

for $t \geq 0$. Since y is not oscillatory, we can see that the equation (6) has a real root. Conversely, assume that λ_1 is the real root of equation (6). Let $N(t) = A(0,t)e^{\lambda_1 t}$ for $t \geq -\tau$. Then

$$N(t_k^+) = A(0, t_k^+)e^{\lambda_1 t_k^+} = a_k A(0, t_k)e^{\lambda_1 t_k} = a_k N(t_k)$$

and

$$N'(t_k^+) = A(0, t_k^+)\lambda_1 e^{\lambda_1 t_k^+} = a_k A(0, t_k)\lambda_1 e^{\lambda_1 t_k} = a_k N'(t_k)$$

for $k \in \mathbf{N}$. Furthermore,

$$\begin{aligned} & N''(t) + aN'(t-\tau_1) + bN(t-\tau_2) \\ = \; & A(0,t)e^{\lambda_1 t}\left(\lambda_1^2 + \frac{a}{A(0,t-\tau_1)}\lambda_1 e^{-\lambda_1\tau_1} + \frac{b}{A(0,t-\tau_2)}e^{-\lambda_1\tau_2}\right) \\ = \; & A(0,t)e^{\lambda_1 t}\left(\lambda_1^2 + \frac{a}{\alpha_1}\lambda_1 e^{-\lambda_1\tau_1} + \frac{b}{\alpha_2}e^{-\lambda_1\tau_2}\right) \\ = \; & 0 \end{aligned}$$

for $t \geq 0$. So $N(t)$ is a positive solution of system (3)-(5). The proof is complete.

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