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A-Statistical Convergence And A-Statistical Monotonicity*

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Abstract

The aim of the present paper is to give some properties of A-statistical convergence of sequences. We give definition of A-statistical monotonicity, upper and lower peak points of sequences. The relation between these concepts and A-statistical monotonicity is investigated. Also, some results given in [11] are generalized.

1 Introduction and Some Definitions

Statistical convergence of real or complex valued sequences was firstly introduced by Fast [5] and Steinhaus [16] in the journal Colloquim Math. independently in 1951. Since then some properties of statistical convergence have been studied in [6, 7, 9, 15]. The idea of this subject depends on asymptotic density of the subset K of natural numbers $\mathbb{N}$ (see [3, 4]).

Let K be a subset of natural numbers $\mathbb{N}$ and

$$K(n) := \{k \in K : k \leqslant n\}.$$

Then, the asymptotic density of the set $K \subset \mathbb{N}$ is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n}|K(n)|,$$

if the limit exists. The vertical bars above indicate the cardinality of the set $K(n)$.

A real or complex valued sequence $x = (x_n)$ is said to be statistically convergent to the number l, if for every $\varepsilon > 0$, the set

$$K(n, \varepsilon) = \{k : k \leqslant n \text{ and } |x_k - l| \geqslant \varepsilon\}$$

has asymptotic density zero, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |K(n, \varepsilon)| = 0$$

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and it is denoted by $x_n \to l(S)$ or $st - \lim_{n\to\infty} x_n = l$.

Statistically convergence is deeply connected to the strongly Cesàro summability and uniform summability (see [10]).

Let $A = (a_{nk})$ be a matrix. If the matrix $A = (a_{nk})$ transforms convergent sequences to convergent sequences with the same limit, then it is called regular. The following theorem gives the conditions for a matrix to be regular:

THEOREM 1.1. ([18], p.165) $A = (a_{nk})$ is a regular matrix if and only if the following conditions hold

(i) $\sup_n \sum |a_{nk}| < \infty$,

(ii) $a_{nk} \to 0$ $(n \to \infty,\ k \text{ fixed})$, and

(iii) $\sum a_{nk} \to 1$ $(n \to \infty)$.

A-density of a subset K of the natural numbers $\mathbb{N}$ is defined as

$$\delta_A(K) := \lim_{n\to\infty} \sum_{k\in K} a_{nk},$$

if the limit exists.

The sequence $x = (x_n)$ is A-statistically convergent to $l \in \mathbb{R}$, if for every $\varepsilon > 0$, the set $K(n,\varepsilon) := \{k \ : \ k \leqslant n,\ |x_k - l| \geqslant \varepsilon\}$ has A-density zero [8]. It is denoted by $x_n \to l(A - st)$.

2 Some Results About A-Statistical Convergence

The space of all complex valued sequences $x = (x_n)$ will be denoted by $\mathbb{C}^{\mathbb{N}}$. In many circumstances we refer to $\mathbb{C}^{\mathbb{N}}$ as the space of arithmetical functions $f : \mathbb{N} \to \mathbb{C}$, specially, when f reflects the multiplicative structure of $\mathbb{N}$. This is the case for additive and multiplicative functions.

Throughout this article, the matrix $A = (a_{nk})$ is non-negative and regular.

Define the function $d_A : \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}} \to [0,\infty)$ as follows,

$$d_A(x,y) := \limsup_{n\to\infty} \sum_{k\leqslant n} a_{nk}\varphi(|x_k - y_k|)$$

for $x = (x_n)$, $y = (y_n) \in \mathbb{C}^{\mathbb{N}}$ and φ is the function $\varphi : [0,\infty) \to [0,\infty)$ where

$$\varphi(t) := \begin{cases} t & \text{if } t \leqslant 1, \\ 1 & \text{if } t > 1. \end{cases}$$

It is clear that, the function d_A is a semi-metric and it is called A-semi-metric on $\mathbb{C}^{\mathbb{N}}$.

THEOREM 2.1. The sequence $x = (x_n)$ is A-statistically convergent to $l \in \mathbb{R}$ if and only if $d_A(x,y) = 0$ where $y_n = l$ for all $n \in \mathbb{N}$.

PROOF. Let us assume $d_A(x,y)=0$ where $y_n=l$ for all $n\in\mathbb{N}$. Then, if $\varepsilon>0$,

$$\limsup_{n\to\infty}\sum_{\substack{k\leqslant n\\|x_k-l|\geqslant\varepsilon}} a_{nk} \le \begin{cases} \frac{1}{\varepsilon}\limsup\limits_{n\to\infty}\sum_{k\leqslant n} a_{nk}\varphi(|x_k-l|), & \varepsilon\le|x_k-l|\le 1,\\ \limsup\limits_{n\to\infty}\sum_{k\leqslant n} a_{nk}\varphi(|x_k-l|), & |x_k-l|>1,\end{cases}$$

$$\leqslant \max\left\{1,\frac{1}{\varepsilon}\right\}\limsup_{n\to\infty}\sum_{k\leqslant n} a_{nk}\varphi(|x_k-l|)=\max\left\{1,\frac{1}{\varepsilon}\right\}d_A(x,l).$$

This calculation implies that $x_n\to l(A-st)$.

Now, assume that $x=(x_n)$ is A-statistically convergent to $l\in\mathbb{R}$. Then for every $\varepsilon>0$,

$$\sum_{k\leqslant n} a_{nk}\varphi(|x_k-y_k|) = \sum_{\substack{k\leqslant n\\|x_k-l|<\varepsilon}} a_{nk}\varphi(|x_k-l|)+\sum_{\substack{k\leqslant n\\|x_k-l|\geqslant\varepsilon}} a_{nk}\varphi(|x_k-l|)$$

$$\leqslant \varepsilon\sum_{k\leqslant n} a_{nk}+\sum_{\substack{k\leqslant n\\|x_k-l|\geqslant\varepsilon}} a_{nk}.$$

Then, since $A=(a_{nk})$ is a regular matrix, we have

$$d_A(x,y)=\limsup_{n\to\infty}\sum_{k\leqslant n} a_{nk}\varphi(|x_k-y_k|)\leqslant\varepsilon\limsup_{n\to\infty}\sum_{k\leqslant n} a_{nk}+\limsup_{n\to\infty}\sum_{\substack{k\leqslant n\\|x_k-l|\geqslant\varepsilon}} a_{nk}\leqslant\varepsilon$$

and this implies that

$$d_A(x,y)\leqslant\varepsilon$$

for any $\varepsilon>0$ where $y=(y_n)$ and $y_n=l$ $(n\in\mathbb{N})$. This ends the proof.

COROLLARY 2.1. If the sequence $x=(x_k)$ is convergent to l (in the usual case) then $d_A(x,y)=0$, where $y_n=l$ for all $n\in\mathbb{N}$.

PROOF. Let us assume $x=(x_k)$ is convergent to l, i.e., for each $\varepsilon>0$, there exists at least an $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $|x_n-l|<\frac{\varepsilon}{2}$ holds for all $n>n_0$. Therefore, since A is a regular matrix

$$\begin{aligned} d_A(x,y) &= \limsup_{n\to\infty}\sum_{k\leqslant n} a_{nk}\varphi(|x_k-l|)\\ &= \limsup_{n\to\infty}\left[\sum_{k\leqslant n_0} a_{nk}\varphi(|x_k-l|)+\sum_{n_0+1\leqslant k\leqslant n} a_{nk}\varphi(|x_k-l|)\right]\\ &\leqslant \limsup_{n\to\infty}\sum_{k\leqslant n_0} a_{nk}+\limsup_{n\to\infty}\sum_{k=n_0+1}^{n} a_{nk}|x_k-l|\\ &\leqslant n_0\limsup_{n\to\infty} a_{nk}+\varepsilon\limsup_{n\to\infty}\sum_{k=n_0+1}^{n} a_{nk}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

REMARK 2.1. The converse of Corollary 2.1 is not true in general. To see this, let us consider the sequence $x = (x_n)$ where

$$x_n = \begin{cases} \sqrt{n} & \text{for } n = m^2 \text{ and } m = 1, 2, ..., \\ 0 & \text{otherwise,} \end{cases}$$

and the regular matrix $A = C_1$, the Cesàro matrix. It is clear that

$$d_A(x, 0) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k\leqslant n} \varphi(|x_k - 0|) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{k} = 0.$$

But the sequence above is not convergent to 0 in the usual case.

Let f be an arithmetical function. With $M_A(f)$, we denote A-value of f,

$$M_A(f) := \lim_{n\to\infty} \sum_{k\leqslant n} a_{nk} f(k)$$

if the limit exists.

THEOREM 2.2. Assume that $f : \mathbb{N} \to \mathbb{C}$ is bounded and A-statistically convergent to L and $H \subset \mathbb{N}$ is an arbitrary subset of $\mathbb{N}$ which has finite A-density $\delta_A(H)$. Then, $M_A(1_H f) = L\delta_A(H)$.

PROOF. From the following inequality the proof can be obtained easily:

$$\left|\sum_{k\in H} a_{nk} f(k) - \sum_{k\in H} a_{nk} L\right| \leqslant \sum_{\substack{k\in H \\ |f(k)-L|<\varepsilon}} a_{nk} |f(k) - L| + \sum_{\substack{k\in H \\ |f(k)-L|\geq\varepsilon}} a_{nk} |f(k) - L|$$

$$\leqslant \quad \varepsilon \sum_{k\in H} a_{nk} + \sum_{\substack{k\in H \\ |f(k)-L|\geq\varepsilon}} a_{nk} |f(k) - L| < \varepsilon\delta_A(H) + \varepsilon < \varepsilon(\delta_A(H) + 1) < \varepsilon.$$

3 A-Statistical Monotone Sequence

Statistical monotonicity for real valued sequences has been defined and studied in [11]

In this section, A-statistical monotonicity will be defined and its relation between A-statistical convergence will be investigated.

DEFINITION 3.1. A sequence $x = (x_n)$ is called A-statistical monotone increasing (decreasing), if there exists a subset H of the natural numbers $\mathbb{N}$ with $\delta_A(H) = 1$ such that the sequence $x = (x_n)$ is monotone increasing (decreasing) on H. A sequence $x = (x_n)$ is called A-statistical monotone sequence if it is A-statistical monotone increasing or A-statistical monotone decreasing.

Now, we list some results about A-statistical monotonicity:

PROPOSITION 3.1. If $x = (x_n)$ is monotone sequence then $x = (x_n)$ is A-statistical monotone. The converse is not true.

PROOF. Assume $x = (x_n)$ is a nondecreasing sequence. That is, for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$. So, we can consider the set $H = \mathbb{N}$. Since, $\delta_A(H) = 1$, we see that $x = (x_n)$ is A-statistical monotone increasing. The proof can be obtained by the same way when the sequence is monotone decreasing.

Let us consider the sequence $x = (x_n)$ as

$$x_n = \begin{cases} 1 & \text{for } n = m^2 \text{ and } m = 1, 2, ..., \\ n & \text{otherwise,} \end{cases}$$

and the matrix $A = C_1$. It is clear that $x = (x_n)$ is A-statistical monotone increasing but it is not monotone increasing.

THEOREM 3.1. If the sequence $x = (x_n)$ is A-statistical monotone increasing or A-statistical monotone decreasing, then the set

$$\{k \in \mathbb{N} : x_{k+1} < x_k\} \text{ or } \{k \in \mathbb{N} : x_{k+1} > x_k\}$$

has A-density zero respectively.

PROOF. Let us assume that $x = (x_n)$ is A-statistical monotone increasing. That is, there exist a subset H of $\mathbb{N}$ with $\delta_A(H) = 1$ such that (x_n) is monotone increasing on H, i.e.,

$$x_n \leqslant x_{n+1} \text{ for all } n \in H.$$

Therefore, the inclusion

$$\{k \in \mathbb{N} : x_{k+1} < x_k\} \subset \mathbb{N} - H$$

and the inequality

$$\delta_A(\{k \in \mathbb{N} : x_{k+1} < x_k\}) \leqslant \delta_A(\mathbb{N} - H) = 0$$

hold. From this argument the assertion is satisfied.

REMARK 3.1. The converse of Theorem 3.1 is not true in general. This can be seen by looking at the example given in (page 264, [11]).

THEOREM 3.2. Let $x = (x_n)$ be a sequence. If $x = (x_n)$ bounded and A-statistical monotone, then $x = (x_n)$ is A-statistical convergent.

PROOF. We shall give the proof only for A-statistical monotone increasing sequence. From the definition of A-statistical monotonicity of $x = (x_n)$, there exists a subset H of $\mathbb{N}$ such that $\delta_A(H) = 1$ and $x = (x_n)_{n \in H}$ is monotone increasing. Let us denote the element of H by k_n.

Without lost of generality, we may assume that k_n is a monotone increasing sequence of natural numbers. Then (x_{k_n}) is the monotone increasing subsequence of (x_n).

Since the sequence $x = (x_n)$ is bounded, we see that the subsequence (x_{k_n}) is also bounded. Therefore, the subsequence (x_{k_n}) is convergent to $\sup x_{k_n}$. It means that, for every $\varepsilon > 0$ there exists a positive natural number $k_{n_0} = k_{n_0}(\varepsilon) \in \mathbb{N}$ such that

$$|x_{k_n} - \sup x_{k_n}| < \varepsilon$$

holds for all $k_n > k_{n_0}$.

Since all convergent sequence is A-statistical convergent, we see that

$$x_{k_n} \to \sup x_{k_n} \quad (A - st)$$

and

$$\begin{aligned} K(n) \quad : & = \{k \leqslant n : |x_k - \sup x_k| \geqslant \varepsilon\} \\ = & \quad \{k \leqslant n : k \neq k_n \text{ and } |x_k - \sup x_k| \geqslant \varepsilon\} \\ & \quad \cup \{k \leqslant n : k = k_n \text{ and } |x_k - \sup x_k| \geqslant \varepsilon\} \\ = & \quad K^1(n) \cup K^2(n). \end{aligned}$$

Since $K^1(n) \subset \mathbb{N} - H$ and $x_{k_n} \to l(A-st)$, we see that $\delta(K^1(n)) = 0$ and $\delta(K^2(n)) = 0$ respectively. Therefore, $x_n \to l(A - st)$.

REMARK 3.2. Boundedness of A-statistical monotone sequence is sufficient but not necessary for A-statistical convergence in general. To see this let us consider the matrix $A = C_1$ and sequence $x = (x_n)$ where

$$x_n = \begin{cases} m & \text{for } n = m^2 \text{ and } n \in \mathbb{N}, \\ \frac{1}{n} & \text{for } n \neq m^2. \end{cases}$$

It is easy to see that the sequence $x = (x_n)$ is not bounded but it is statistical monotone decreasing and statistically convergent to zero.

REMARK 3.3. For a bounded sequence, A-statistical monotonicity is sufficient but not necessary for A-statistical convergence. Let us consider the matrix $A = C_1$ and the sequence $x = (x_n)$ defined by

$$x_n := \begin{cases} \frac{1}{n} & \text{for } n \text{ is odd}, \\ -\frac{1}{n} & \text{for } n \text{ is even}. \end{cases}$$

It is clear that $x = (x_n)$ is bounded and statistical convergent to zero but it is not statistical monotone.

DEFINITION 3.2. The real number sequence $x = (x_n)$ is said to be A-statistical bounded if there is a number $M > 0$ such that

$$\delta_A(\{n \in \mathbb{N} : |x_n| > M\}) = 0.$$

REMARK 3.4. For A-statistical convergence, boundedness and A-statistical monotonicity is sufficient but not necessary.

Let us consider the Cesàro matrix and the sequence $x = (x_n)$ defined by

$$x_n = \begin{cases} n & \text{for } n \text{ is an odd square,} \\ 2 & \text{for } n \text{ is an even square,} \\ \frac{1}{n} & \text{for } n \text{ is an odd nonsquare,} \\ -\frac{1}{n} & \text{for } n \text{ is an even nonsquare.} \end{cases} \tag{1}$$

Obviously, $x = (x_n)$ is statistical convergent to zero but the sequence is neither A-statistical bounded nor statistical monotone.

By using Definition 3.2, we may give the weak converse of Theorem 3.2 without proof as follows:

THEOREM 3.3. A-statistical monotone sequence $x = (x_n)$ is A-statistical convergent if and only if $x = (x_n)$ is A-statistical bounded.

4 Peak Points and A-Statistical Monotonicity

In [11] the concept of peak points of real valued sequences has been defined and its relation between statistical monotonicity and statistical convergence has been investigated.

Let us recall the definitions of upper and lower peak points:

DEFINITION 4.1.([11]). The element x_k is called an upper (or lower) peak point of the sequence $x = (x_n)$ if $x_k \geqslant x_l$ (respectively $x_l \geqslant x_k$) holds for all $l \geqslant k$.

The element x_k is called a peak point of the sequence if it is an upper peak point or lower peak point.

THEOREM 4.1. If the index set of peak points of the sequence $x = (x_n)$ has A-density 1, then the sequence $x = (x_n)$ is A-statistical monotone.

PROOF. Let us denote the index set of upper peak points of the sequence $x = (x_n)$,

$$H := \{k : x_k \text{ is an upper peak point of } (x_n)\}.$$

There exist a monotone increasing sequence (k_n) of positive natural number such that the set H can be represented as

$$H := \{k_1 < k_2 < k_3 < ...\}$$

with $\delta_A(H) = 1$.

Since, x_{k_n} is an upper peak point for all $n \in \mathbb{N}$, the following

$$x_{k_1} > x_{k_2} > x_{k_3} > ... > x_{k_n} > ...$$

inequalities hold. So, x_n is A-statistical monotone decreasing. By using the same arguments for lower peak points the proof is obtained easily.

Let n_k be a strictly increasing sequence of positive natural numbers and $x = (x_n)$ be a real valued sequence. Define $\tilde{x} = (x_{n_k})$ and $K_{\tilde{x}} = \{n_k : k \in \mathbb{N}\}$.

DEFINITION 4.2. The subsequence $\tilde{x} = (x_{n_k})$ of $x = (x_n)$ is called (I) A-dense subsequence if $\delta_A(K_{\tilde{x}}) = 1$,
(II) A-empty subsequence if $\delta_A(K_{\tilde{x}}) = 0$.

DEFINITION 4.3. The sequences $x = (x_n)$ and $y = (y_n)$ are called A-statistical equivalent if there exists a subset M of $\mathbb{N}$ with $\delta_A(M) = 1$ such that $x_n = y_n$ for all $n \in M$. A-statistical equivalence is denoted by $x \overset{(A)}{\asymp} y$.

In the following we list some properties of A-statistical monotonicity and peak points:
(I) Every A-dense subsequence of an A-statistical monotone sequence is A-statistical monotone.
(II) Let $x = (x_n)$ and $y = (y_n)$ be A-statistical equivalent sequences, i.e. $x \overset{(A)}{\asymp} y$. Then, $x = (x_n)$ is A-statistical monotone if and only if $y = (y_n)$ is A-statistical monotone.
(III) If the sequence $x = (x_n)$ is A-statistical monotone, then it has at least an A-dense and an A-empty subsequence.
The converse is not true. Let us consider the Cesàro matrix and the sequence $x = (x_n)$ given in (1). The subsequence

$$\begin{aligned}(y_n) &= (x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, ..)\\ &= \left(\frac{-1}{2}, \frac{1}{3}, 2, \frac{1}{5}, \frac{-1}{6}, \frac{1}{7}, \frac{-1}{8}, \frac{-1}{10}, \frac{1}{11}, \frac{-1}{12}, \frac{1}{13}, \frac{-1}{14}, \frac{1}{15}, 2, ...\right)\end{aligned}$$

and

$$(z_n) = (x_1, x_9, x_{25}, x_{49}, x_{81}, ...) = (1, 9, 25, 49, 81, ...)$$

are C_1-dense and C_1-empty subsequence of $x = (x_n)$. But it is not statistical monotone.
(IV) If the index set of peak points of the sequence has A-density 1, then it has monotone A-dense and A-empty subsequences.

THEOREM 4.2. Under the condition of Theorem 4.1, if the sequence x_n is A-statistical bounded then it is A-statistical convergent.

PROOF. From the assumption of Theorem 4.1, we can assume $x = (x_n)$ is A-statistical monotone increasing and A-statistical bounded. Therefore, the proof is obtained by using Theorem 3.3.

5 Inclusion Results for C_λ and D_λ

Let $\lambda = \lambda(n)$ be a strictly increasing sequence of positive natural numbers such that $\lambda(0) = 0$. C_λ and D_λ asymptotic density of a subset of K of natural numbers $\mathbb{N}$ is defined by

$$\lim_{n\to\infty} \frac{1}{\lambda(n)} |\{k \in K : k \leqslant \lambda(n)\}|$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda(n) - \lambda(n-1)} |\{k \in K : \lambda(n-1) < k \leqslant \lambda(n)\}|$$

respectively.

The more detailed knowledge about the densities C_λ and D_λ can be found in [2], [14, 17] respectively.

THEOREM 5.1. If the sequence $x = (x_n)$ is D_λ-statistical monotone, then it is C_λ-statistical monotone.

PROOF. We shall apply the technique which was used by Agnew in his paper [1]. Assume, $x = (x_n)$ is a D_λ-statistical monotone. That is, there is a subset H of $\mathbb{N}$ such that $\delta_{D_\lambda}(H) = 1$, and (x_n) is monotone on H. Let us denote the set $\{k \; : \; k \leqslant \lambda(n), \; k \in H\}$ by $H(n)$. The set $H(n)$ can be represented as

$$\begin{aligned} H(n) &= \{k \in H : \lambda(0) + 1 \leqslant k \leqslant \lambda(1)\} \cup \{k \in H : \lambda(1) + 1 \leqslant k \leqslant \lambda(2)\} \cup \\ &\quad ... \cup \{k \in H : \lambda(n-1) + 1 \leqslant k \leqslant \lambda(n)\} \\ &= \bigcup_{j=1}^{n} \{k \in H : \lambda(j-1) + 1 \leq k \leq \lambda(j)\}, \end{aligned}$$

for an arbitrary $n \in \mathbb{N}$. From this representation we have

$$\begin{aligned} |H(n)| &= |\{k \in H : 1 \leqslant k \leqslant \lambda(1)\}| + |\{k \in H : \lambda(1) + 1 \leqslant k \leqslant \lambda(2)\}| + \\ &\quad ... + |\{k \in H : \lambda(n-1) + 1 \leqslant k \leqslant \lambda(n)\}| \\ &= \sum_{j=1}^{n} |\{k \in H : \lambda(j-1) + 1 \leqslant k \leqslant \lambda(j)\}| \end{aligned}$$

and

$$\frac{1}{\lambda(n)} |H(n)| = \sum_{j=1}^{n} \frac{\lambda(j) - \lambda(j-1)}{\lambda(n)} \frac{1}{\lambda(j) - \lambda(j-1)} |\{k \in H : \lambda(j-1) + 1 \leqslant k \leqslant \lambda(j)\}| .$$

Let us consider the matrix $T = (t_{n,k})$ defined by

$$t_{n,k} := \begin{cases} \frac{\lambda(k) - \lambda(k-1)}{\lambda(n)} & \text{for } k = 1, 2, ..., n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the matrix T is regular, and

$$\delta_{C_\lambda}(H) = T\delta_{D_\lambda}(H).$$

Since, $\lim_{n\to\infty}(\delta_{D_\lambda}(H))_n = 1$ and T is regular,

$$\delta_{C_\lambda}(H) = 1.$$

It means that the sequence $x = (x_n)$ is D_λ-statistically monotone.

THEOREM 5.2. Let $E = \{\lambda(n)\}$ be an infinite subset of $\mathbb{N}$ with $\lambda(0) = 0$. Then, a C_λ-statistical monotone sequence is also D_λ-statistical monotone sequence if and only if

$$\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1.$$

PROOF. Let x be a C_λ-statistical monotone sequence. That is, there exists a subset H of $\mathbb{N}$ such that $\delta_{C_\lambda}(H) = 1$ and (x_n) is monotone on H. From the definition of $\delta_{D_\lambda}(H)$ we have

$$\begin{aligned} & \lim_{n\to\infty} \frac{|\{k \in \mathbb{N} : \lambda(n-1)+1 \leqslant k \leqslant \lambda(n)\}|}{\lambda(n)-\lambda(n-1)} \\ = \ & \frac{\lambda(n)}{\lambda(n)-\lambda(n-1)} \frac{|\{k \in H : 1 \leqslant k \leqslant \lambda(n)\}|}{\lambda(n)} \\ & - \frac{\lambda(n-1)}{\lambda(n)-\lambda(n-1)} \frac{|\{k \in H : 1 \leqslant k \leqslant \lambda(n-1)\}|}{\lambda(n-1)}. \end{aligned}$$

If we let the matrix $T = (t_{n,k})$ defined by

$$t_{n,k} = \begin{cases} \frac{\lambda(n)}{\lambda(n)-\lambda(n-1)} & \text{for } k = n, \\ -\frac{\lambda(n-1)}{\lambda(n)-\lambda(n-1)} & \text{for } k = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain $(\delta_{D_\lambda}(H))_n = (T(\delta_{C_\lambda}(H)))_n$.

Therefore, $\delta_{D_\lambda}(H)$ is obtained by $\delta_{C_\lambda}(H)$ if and only if T is regular. Thus, T will be regular if and only if the sequence

$$\left\{ \frac{\lambda(n)}{\lambda(n)-\lambda(n-1)} + \frac{\lambda(n-1)}{\lambda(n)-\lambda(n-1)} \right\}_{n\in\mathbb{N}} \tag{2}$$

is bounded. After simple calculation (2) is bounded if and only if $\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1$.

The following corollaries are simple consequences of Theorem 6.2 in (page 208, [11]).

COROLLARY 5.1. $E = \{\lambda(n)\}$ and $F = \{\mu(n)\}$ be an infinite subset of $\mathbb{N}$. If $F - E$ is finite, then C_λ-statistical (D_λ-st.) monotonicity implies C_μ-statistical (D_μ-st.) monotonicity.

COROLLARY 5.2. Assume $F \Delta E$ is finite. Then C_λ (D_λ) statistical monotonicity implies if C_μ (respectively D_μ) statistical monotonicity and vice versa.

COROLLARY 5.3. Let $E = \{\lambda(n)\}$ be an infinite subset of $\mathbb{N}$ and

$$\limsup_{n\to\infty} \frac{\lambda(n+1)}{\lambda(n)} = 1.$$

Then, the sequence $x = (x_n)$ is C_λ-statistical monotone if and only if it is statistical monotone.

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