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Analytical Solution Of The Black-Scholes Equation By Using Variational Iteration Method*

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Abstract

In this paper, we shall use the variational iteration method to solve the Black-Scholes equation and will obtain a closed form of the solution. In this method, the problems are initially approximated with possible unknowns, then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory.

1 Introduction

Options are widely used on markets and exchanges. The pricing of options is a central problem in financial investment. The famous Black-Scholes model [9, 12] is a convenient way to calculate the price of an option. The equation assumes the existence of perfect capital markets and the security prices are log normally distributed or, equivalently, the log-returns are normally distributed. To these, one adds the assumptions that trading in all securities is continuous and that the distribution of the rates of return is stationary. The linear parabolic partial differential equation of Black-Scholes equation for valuing an option with value u is

$$u_t + \frac{\sigma^2}{2}x^2 u_{xx} + (r-\tau)xu_x - ru = 0, \tag{1}$$

where r is the risk-free rate, σ is the volatility, and τ is the dividend yield. The numerical solution of this equation has been of paramount interest due to the governing partial differential equation, which is very difficult to generate stable and accurate solutions. In this paper, we research the equation (1) with a view to obtaining the analytical solution to the terminal condition

$$u(x,T) = g(x), \tag{2}$$

by using the variational iteration method (VIM) [1, 2, 4, 7, 8, 10, 11]. We suppose throughout that g has derivatives of all orders.

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2 Variational Iteration Method

In this method, the solution of a differential equation with a linearization assumption is used as an initial approximation, then a more precise approximation at some special point can be obtained. To illustrate this method, consider the differential equation in the formal form

$$Lu(x,t)+Nu(x,t)=g(x,t), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x,t)$ is an inhomogeneous term.

According to the VIM, we can construct a correctional functional as

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left(Lu_n(x,s)+N\tilde{u}_n(x,s)-g(x,s)\right)ds, \tag{4}$$

where λ is a general Lagrangian multiplier [5, 6], and can be identified optimally via the variational theory, the subscript n denotes the n-th order approximation, $\tilde{u}_n$ is considered as a restricted variation [5, 6], i.e., $\delta\tilde{u}_n=0$. Eq. (4) is called a correction functional. The successive approximations u_{n+1}, $n\geq 0$, of the solution u will be readily obtained by suitable choice of trial function u_0. Consequently, the solution is given as

$$u(x,t)=\lim_{n\to\infty}u_n(x,t). \tag{5}$$

3 Applications

In this section, we illustrate the proposed method to Black-Scholes equation.

3.1 Case I

According to the VIM, we construct a correction functional for Eq. (1) in the form

$$u_{n+1}(x,t)=u_n(x,t)+\int_t^T \lambda\left((u_n)_s+\frac{\sigma^2}{2}x^2(\tilde{u}_n)_{xx}+(r-\tau)x(\tilde{u}_n)_x-r(\tilde{u}_n)\right)ds, \tag{6}$$

where λ is a general Lagrange multiplier, and $\tilde{u}$ denotes the restricted variation, i.e., $\delta\tilde{u}=0$.

The correction functional (6)

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t)+\delta\int_t^T \lambda\left((u_n)_s+\frac{\sigma^2}{2}x^2(\tilde{u}_n)_{xx}+(r-\tau)x(\tilde{u}_n)_x-r(\tilde{u}_n)\right)ds\\
&= \delta u_n(x,t)+\delta\int_t^T \lambda(u_n)_s ds\\
&= \delta u_n(x,t)-\lambda\delta u_n(x,s)|_{s=t}-\int_t^T \lambda'\delta u_n(x,s)ds=0,
\end{aligned}$$

yields the following stationary condition

$$-\lambda'(s)=0,$$
$$1-\lambda(s)|_{s=t}=0.$$

Therefore, the Lagrange multiplier is

$$\lambda(s) = 1.$$

Substituting this value of the Lagrange multiplier into the functional (6) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_t^T \left((u_n)_s + \frac{\sigma^2}{2} x^2 (u_n)_{xx} + (r-\tau) x (u_n)_x - r(u_n) \right) ds. \quad (7)$$

We start with an initial approximation: $u_0(x,t) = g(x)$, and using the iteration formula (7), we obtain the following successive approximations

$$u_1(x,t) = g(x) + (T-t) \left(-rg(x) + x(r-\tau)xg'(x) + \frac{1}{2} x^2 \sigma^2 g''(x) \right),$$

$$\begin{aligned} u_2(x,t) &= g(x) + (T-t) \left(-rg(x) + x(r-\tau)xg'(x) + \frac{1}{2} x^2 \sigma^2 g''(x) \right) \\ &\quad + \frac{1}{8}(T-t)^2 \left[4r^2 g(x) - (4r^2 - 4\tau^2) x g'(x) \right. \\ &\quad + (4r^2 - 8r\tau + 4\tau^2 + 4r\sigma^2 - 8\tau\sigma^2 + 2\sigma^4) x^2 g''(x) \\ &\quad \left. + (4r\sigma^2 - 4\tau\sigma^2 + 4\sigma^4) x^3 g^{(3)}(x) + x^4 \sigma^4 g^{(4)}(x) \right], \end{aligned}$$

$$\vdots$$

i.e.,

$$u_n(x,t) = \sum_{k=0}^{n} \left[\sum_{m=0}^{2k} \left\{ \sum_{v=0}^{m} \frac{(-1)^{(m-v)}}{v!(m-v)!} \left(\left(\frac{\sigma^2 v}{2} + r \right)(v-1) - \tau v \right)^k \right\} x^m g^{(m)}(x) \right] \frac{(T-t)^k}{k!},$$

where $n \geq 0$. Then, the exact solution is given as series

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= \sum_{k=0}^{\infty} \left[\sum_{m=0}^{2k} \left\{ \sum_{v=0}^{m} \frac{(-1)^{(m-v)}}{v!(m-v)!} \left(\left(\frac{\sigma^2 v}{2} + r \right)(v-1) - \tau v \right)^k \right\} x^m g^{(m)}(x) \right] \frac{(T-t)^k}{k!}. \end{aligned} \quad (8)$$

LEMMA 1. Assume

$$\varphi_k(x) = \sum_{m=0}^{2k} \left\{ \sum_{v=0}^{m} \frac{(-1)^{(m-v)}}{v!(m-v)!} \left(\left(\frac{\sigma^2 v}{2} + r \right)(v-1) - \tau v \right)^k \right\} x^m g^{(m)}(x).$$

The defined function $\varphi_k(x)$ satisfies the recursion

$$\frac{\sigma^2}{2} x^2 \varphi_k''(x) + (r-\tau) x \varphi_k'(x) - r\varphi_k(x) = \varphi_{k+1}(x) \text{ for all } k \in \mathbb{N}_0. \quad (9)$$

PROOF. See [Lemma 3.1, 3].

THEOREM 1. The function $u(x,t)$ defined by

$$u(x,t)=\sum_{k=0}^{\infty}\varphi_k(x)\frac{(T-t)^k}{k!}, \tag{10}$$

satisfies equation (1).

PROOF. Substituting $u(x,t)$ into the equation (1) gives

$$\begin{aligned}
& u_t+\frac{\sigma^2}{2}x^2u_{xx}+(r-\tau)xu_x-ru \\
= & \sum_{k=0}^{\infty}\varphi_k(x)\frac{-k(T-t)^{(k-1)}}{k!}+\sum_{k=0}^{\infty}\frac{\sigma^2}{2}x^2\varphi_k''(x)\frac{(T-t)^k}{k!} \\
& +\sum_{k=0}^{\infty}(r-\tau)x\varphi_k'(x)\frac{(T-t)^k}{k!}-\sum_{k=0}^{\infty}r\varphi_k(x)\frac{(T-t)^k}{k!} \\
= & -\sum_{k=1}^{\infty}\varphi_k(x)\frac{(T-t)^{(k-1)}}{(k-1)!}+\sum_{k=0}^{\infty}(\frac{\sigma^2}{2}x^2\varphi_k''(x)+(r-\tau)x\varphi_k'(x)-r\varphi_k(x))\frac{(T-t)^k}{k!} \\
= & -\sum_{k=1}^{\infty}\varphi_k(x)\frac{(T-t)^{(k-1)}}{(k-1)!}+\sum_{k=0}^{\infty}\varphi_{k+1}(x)\frac{(T-t)^k}{k!} \\
= & -\sum_{k=1}^{\infty}\varphi_k(x)\frac{(T-t)^{(k-1)}}{(k-1)!}+\sum_{k=1}^{\infty}\varphi_k(x)\frac{(T-t)^{(k-1)}}{(k-1)!} \\
= & \ 0.
\end{aligned}$$

3.2 Case II

According to the VIM, we construct a correction functional for Eq. (1) in the following form

$$u_{n+1}(x,t)=u_n(x,t)+\int_t^T\lambda\left((u_n)_s-ru_n+\frac{\sigma^2}{2}x^2(\tilde{u}_n)_{xx}+(r-\tau)x(\tilde{u}_n)_x\right)ds. \tag{11}$$

The correction functional for (11) is

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t)+\delta\int_t^T\lambda\left((u_n)_s-ru_n+\frac{\sigma^2}{2}x^2(\tilde{u}_n)_{xx}+(r-\tau)x(\tilde{u}_n)_x\right)ds \\
&= \delta u_n(x,t)+\delta\int_t^T\lambda\left((u_n)_s-ru_n\right)ds \\
&= \delta u_n(x,t)-\lambda\delta u_n(x,s)|_{s=t}-\int_t^T\left(\lambda'(s)-r\lambda(s)\right)\delta u_n(x,s)ds=0,
\end{aligned}$$

which yields the following stationary condition

$$\begin{aligned} -\lambda'(s) + r\lambda(s) &= 0, \\ 1 - \lambda(s)|_{s=t} &= 0. \end{aligned}$$

Hence, the Lagrange multiplier is

$$\lambda(s) = e^{r(s-t)}.$$

Substituting this value of the Lagrange multiplier into the functional (11) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_t^T e^{r(s-t)} \left((u_n)_s + \frac{\sigma^2}{2} x^2 (u_n)_{xx} + (r-\tau) x (u_n)_x - r(u_n) \right) ds. \tag{12}$$

We start with an initial approximation: $u_0(x,t) = g(x)$, and using the iteration formula (12), we obtain the following successive approximations

$$u_1(x,t) = g(x) + \frac{1}{r}(e^{r(T-t)} - 1) \left(-rg(x) + x(r-\tau)xg'(x) + \frac{1}{2}x^2\sigma^2 g''(x) \right),$$

$$\begin{aligned} u_2(x,t) \quad = \quad & g(x) - \frac{2}{r}\left(1 - e^{r(T-t)} + \frac{1}{2}r(T-t)e^{r(T-t)}\right) \\ & \times \left(-rg(x) + x(r-\tau)xg'^2 \frac{\sigma^2}{2} g''(x)\right) \\ & + \frac{1}{4r^2}\left(1 - e^{r(T-t)} + r(T-t)e^{r(T-t)}\right) \\ & \times [4r^2 g(x) - (4r^2 - 4\tau^2)xg'(x) \\ & + (4r^2 - 8r\tau + 4\tau^2 + 4r\sigma^2 - 8\tau\sigma^2 + 2\sigma^4)x^2 g''(x) \\ & + (4r\sigma^2 - 4\tau\sigma^2 + 4\sigma^4)x^3 g^{(3)}(x) + x^4\sigma^4 g^{(4)}(x)], \end{aligned}$$

$$\vdots$$

where $u_n(x,t)$ is an approximation of the solution. In fact, the cases I and II show flexibility of the VIM method where it can be used in solving the differential equations.

4 Conclusions

In this work, the VIM is applied to obtain the solution of Black-Scholes equation in two cases. A theoretical analysis and closed form of the solution was presented for the Black-Scholes equation. The results clearly indicate the reliability and accuracy of the proposed technique.

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