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Existence Results For Dirichlet Problems With Degenerated p-Laplacian And p-Biharmonic Operators*

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Abstract

In this article, we prove the existence and uniqueness of solutions for the Dirichlet problem

$$(P)\begin{cases} \Delta(\omega(x)|\Delta u|^{p-2}\Delta u) - \operatorname{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega \\ u(x) = 0, & \text{in } \partial\Omega \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N\geq 2)$, $f\in L^{p'}(\Omega,\omega)$ and $G/\omega\in[L^{p'}(\Omega,\omega)]^N$.

1 Introduction

The main purpose of this paper (see Theorem 3.2) is to establish the existence and uniqueness of solutions for the Dirichlet problem

$$(P)\begin{cases} \Delta(\omega(x)|\Delta u|^{p-2}\Delta u) - \operatorname{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega \\ u(x) = 0, & \text{in } \partial\Omega \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded open set, $f\in L^{p'}(\Omega,\omega)$, $G/\omega\in[L^{p'}(\Omega,\omega)]^N$, ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$ such that $0<\omega(x)<\infty$ a.e. $x\in\mathbb{R}^N$), Δ is the Laplacian operator and $1<p<\infty$, $p\neq 2$.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1, 4, 5, 7, 8, 12]). The type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B. Muckenhoupt in the early 1970's (see [8]). These classes have found many useful applications in harmonic analysis (see [9, 11]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^N$ often belong to A_p (see [3, 12]). There are, in fact, many interesting examples of weights (see [7] for p-admissible weights).

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In the non-degenerate case (i.e. with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [6]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [2]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator have been studied by many authors (see [10] and the references therein), and the degenerated p-Laplacian has been studied in [3].

The paper is organized as follow. In Section 2 we present the definitions and basic results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2 Definitions and Basic Results

By a weight we shall mean a locally integrable function ω on $\mathbb{R}^N$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will be denoted by μ. Thus,

$$\mu(E) = \int_E \omega(x)dx \text{ for measurable sets } E \subset \mathbb{R}^N.$$

DEFINITION 2.1. Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant $C = C(p, \omega)$ such that, for every ball $B \subset \mathbb{R}^N$

$$\begin{aligned} \left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} &\le C \text{ if } p > 1, \\ \left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\operatorname*{ess\,sup}_{x\in B} \frac{1}{\omega(x)}\right) &\le C \text{ if } p = 1, \end{aligned}$$

where $|\cdot|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$.

If $1 < q \le p$, then $A_q \subset A_p$ (see [5, 7, 12] for more information about A_p-weights). As an example of an A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if $-N < \alpha < N(p-1)$ (see [11], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$, then $\omega(x) = e^{\alpha\,\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [9]).

REMARK 2.2. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \le C\frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see *15.5 strong doubling property* in [7]). Therefore, if $\mu(E) = 0$, then $|E| = 0$. Thus, if $\{u_n\}$ is a sequence of functions defined in B and $u_n \to u$ μ-a.e. then $u_n \to u$ a.e..

DEFINITION 2.3. Let ω be a weight. We shall denote by $L^p(\Omega,\omega)$ $(1 \le p < \infty)$ the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

We denote $[L^p(\Omega,\omega)]^N = L^p(\Omega,\omega) \times ... \times L^p(\Omega,\omega)$.

REMARK 2.4. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\mathrm{loc}}(\Omega)$ (see [12], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

DEFINITION 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty$, k be a non-negative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)dx + \sum_{1\le|\alpha|\le k} \int_\Omega |D^\alpha u(x)|^p \omega(x)dx\right)^{1/p}. \quad (1)$$

We also define the space $W^{k,p}_0(\Omega,\omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,p}_0(\Omega,\omega)} = \left(\sum_{1\le|\alpha|\le k} \int_\Omega |D^\alpha u(x)|^p \omega(x)dx\right)^{1/p}.$$

The dual space of $W^{1,p}_0(\Omega,\omega)$ is the space $[W^{1,p}_0(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega)$,

$$W^{-1,p'}(\Omega,\omega) = \{T = f - \mathrm{div}(G) : G = (g_1, ..., g_N),\ \frac{f}{\omega},\ \frac{g_j}{\omega} \in L^{p'}(\Omega,\omega)\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \le \omega(x) \le C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $W^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested in all above such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basic result.

THEOREM 2.6 (The weighted Sobolev inequality). Let $\Omega \subset R^N$ be a bounded open set and let ω be an A_p-weight, $1 < p < \infty$. Then there exists positive constants C_Ω and δ such that, for all $f \in C_0^\infty(\Omega)$ and $1 \leq \eta \leq N/(N-1) + \delta$,

$$\|f\|_{L^{\eta p}(\Omega,\omega)} \leq C_\Omega \||\nabla f|\|_{L^p(\Omega,\omega)}. \tag{2}$$

PROOF. See [4], Theorem 1.3.

3 Weak Solutions

We denote by $X = W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,\omega)$ with the norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p\,\omega\,dx + \int_\Omega |\Delta u|^p\,\omega\,dx\right)^{1/p}.$$

In this section we prove the existence and uniqueness of weak solutions $u \in X$ to the Dirichlet problem

$$(P)\begin{cases} \Delta(\omega(x)|\Delta u|^{p-2}\Delta u) - \operatorname{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \operatorname{div}(G(x)), & \text{in } \Omega \\ u(x) = 0, \quad \text{in } \partial\Omega \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\omega[L^{p'}(\Omega,\omega)]^N$.

DEFINITION 3.1. We say that $u \in X$ is a weak solution for problem (P) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,\omega(x)\,dx + \int_\Omega \omega(x)\,|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,dx = \int_\Omega f\,\varphi\,dx + \int_\Omega \langle G,\nabla\varphi\rangle\,dx, \tag{3}$$

for all $\varphi \in X$, with $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\omega \in [L^{p'}(\Omega,\omega)]^N$.

THEOREM 3.2. Let $\omega \in A_p$, $1 < p < \infty$, $f/\omega \in L^{p'}(\Omega,\omega)$ and $G/\omega \in [L^{p'}(\Omega,\omega)]^N$. Then the problem (P) has a unique solution $u \in X$.

PROOF. (I) *Existence*. By Theorem 2.6, we have that

$$\begin{aligned} \left|\int_\Omega f\varphi dx\right| &\leq \left(\int_\Omega \left|\frac{f}{\omega}\right|^{p'}\omega\,dx\right)^{1/p'}\left(\int_\Omega |\varphi|^p\omega dx\right)^{1/p} \\ &\leq C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\nabla\varphi\|_{L^p(\Omega,\omega)} \\ &\leq C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_X, \end{aligned} \tag{4}$$

and

$$
\begin{aligned}
\left|\int_\Omega \langle G, \nabla\varphi\rangle dx\right| &\leq \int_\Omega |\langle G, \nabla\varphi\rangle| dx \qquad (5)\\
&\leq \int_\Omega |G||\nabla\varphi| dx \\
&= \int_\Omega \frac{|G|}{\omega}|\nabla\varphi|\omega dx \\
&\leq \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\nabla\varphi\|_{L^p(\Omega,\omega)} \\
&\leq \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\varphi\|_X. \qquad (6)
\end{aligned}
$$

Define the functional $J_p : X \to \mathbb{R}$ by

$$J_p(\varphi) = \frac{1}{p}\int_\Omega |\Delta\varphi|^p\omega dx + \frac{1}{p}\int_\Omega |\nabla\varphi|^p\omega dx - \int_\Omega f\varphi dx - \int_\Omega \langle G, \nabla\varphi\rangle dx.$$

Using (4), (5) and Young's inequality, we have that

$$
\begin{aligned}
J_p(\varphi) &\geq \frac{1}{p}\int_\Omega |\Delta\varphi|^p\omega dx + \frac{1}{p}\int_\Omega |\nabla\varphi|^p\omega dx \\
&\quad - \left(C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\|\varphi\|_X \\
&\geq \frac{1}{p}\int_\Omega |\Delta\varphi|^p\omega dx + \frac{1}{p}\int_\Omega |\nabla\varphi|^p\omega dx - \frac{1}{p}\|\varphi\|_X^p - \frac{1}{p'}\left[C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right. \\
&\quad \left. + \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right]^{p'} \\
&= -\frac{1}{p'}\left[C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right]^{p'},
\end{aligned}
$$

that is, J_p is bounded from below.

Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$J_p(u_n) \to \inf_{\varphi\in X} J_p(\varphi).$$

Then for n large enough, we obtain that

$$0 \geq J_p(u_n) = \frac{1}{p}\int_\Omega |\Delta u_n|^p\,\omega\,dx + \frac{1}{p}\int_\Omega |\nabla u_n|^p\,\omega\,dx - \int_\Omega f\,u_n\,dx - \int_\Omega \langle G, \nabla u_n\rangle\,dx,$$

and we get (by Theorem 2.6)

$$\begin{aligned} \|u_n\|_X^p &\leq p\left(\int_\Omega f u_n dx + \int_\Omega \langle G, \nabla u_n\rangle dx\right) \\ &\leq p\left(\left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|u_n\|_{L^p(\Omega,\omega)} + \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)} \|\nabla u_n\|_{L^p(\Omega,\omega)}\right) \\ &\leq p\left(C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right) \|\nabla u_n\|_{L^p(\Omega,\omega)} \\ &\leq p\left(C_\Omega \|\frac{f}{\omega}\|_{L^{p'}(\Omega,\omega)} + \|\frac{G}{\omega}\|_{L^{p'}(\Omega,\omega)}\right) \|u_n\|_X. \end{aligned}$$

Hence

$$\|u_n\|_X \leq \left[p\left(C_\Omega \left\|\frac{f}{\omega}\right\|_{L^{p'}(\Omega,\omega)} + \left\|\frac{G}{\omega}\right\|_{L^{p'}(\Omega,\omega)}\right)\right]^{1/(p-1)}.$$

Therefore $\{u_n\}$ is bounded in X. Since X is reflexive, there exists a $u \in X$ such that $u_n \rightharpoonup u$ in X. Since

$$X \ni \varphi \mapsto \int_\Omega f\varphi dx + \int_\Omega \langle G, \nabla\varphi\rangle dx$$

and $\varphi \mapsto \|\nabla\varphi\|_{L^p(\Omega,\omega)} + \|\Delta\varphi\|_{L^p(\Omega,\omega)}$ are continuous then J_p is continuous. Moreover since $1 < p < \infty$ we have that J_p is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J_p(u) \leq \liminf_n J_p(u_n) = \inf_{\varphi \in X} J_p(\varphi),$$

and thus u is a minimizer of J_p on X. For any $\varphi \in X$ the function

$$\begin{aligned} \lambda \quad \mapsto \quad & \frac{1}{p}\int_\Omega |\Delta(u+\lambda\varphi)|^p \omega dx + \frac{1}{p}\int_\Omega |\nabla(u+\lambda\varphi)|^p \omega dx - \int_\Omega (u+\lambda\varphi) f dx \\ & - \int_\Omega \langle G, \nabla(u+\lambda\varphi)\rangle dx \end{aligned}$$

has a minimum at $\lambda = 0$. Hence

$$\frac{d}{d\lambda} J_p(u+\lambda\,\varphi)\bigg|_{\lambda=0} = 0, \ \forall \varphi \in X.$$

We have that

$$\frac{d}{d\lambda}\,(|\nabla(u+\lambda\varphi|^p\omega) = p\{|\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u, \nabla\varphi\rangle + \lambda\,|\nabla\varphi|^2)\}\omega,$$

and

$$\frac{d}{d\,\lambda}\,(|\Delta(u+\lambda\,\varphi)|^p\omega) = p|\Delta u + \lambda\Delta\varphi|^{p-2}(\Delta u + \lambda\Delta\varphi)\Delta\varphi\omega,$$

and we obtain that

$$\begin{aligned}
0 &= \frac{d}{d\lambda}J_p(u+\lambda\,\varphi)\Big|_{\lambda=0}\\
&= \Bigg[\frac{1}{p}\left(p\int_\Omega |\Delta u+\lambda\,\Delta\varphi|^{p-2}(\Delta u+\lambda\,\Delta\varphi)\,\Delta\varphi\,\omega\,dx\right)\\
&\quad+\frac{1}{p}\left(p\int_\Omega |\nabla(u+\lambda\,\varphi)|^{p-2}(\langle\nabla u,\nabla\varphi\rangle+\lambda|\nabla\varphi|^2)\omega dx\right)\\
&\quad-\int_\Omega \varphi f dx-\int_\Omega\langle G,\nabla\varphi\rangle dx\Bigg]\Bigg|_{\lambda=0}\\
&= \int_\Omega |\Delta u|^{p-2}\Delta u\Delta\varphi\omega dx+\int_\Omega|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,\omega\,dx\\
&\quad-\int_\Omega f\varphi dx-\int_\Omega\langle G,\nabla\varphi\rangle\,dx.
\end{aligned}$$

Therefore

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta\varphi\omega dx+\int_\Omega|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\omega dx=\int_\Omega f\varphi dx+\int_\Omega\langle G,\nabla\varphi\rangle\,dx,$$

for all $\varphi\in X$, that is, $u\in X$ is a solution of problem (P).

(II) *Uniqueness.* If $u_1,u_2\in X=W^{2,p}(\Omega,\omega)\cap W_0^{1,p}(\Omega,\omega)$ are two weak solutions of problem (P), we have (for $i=1,2$)

$$\int_\Omega |\Delta u_i|^{p-2}\Delta u_i\Delta\varphi\omega dx+\int_\Omega|\nabla u_i|^{p-2}\langle\nabla u_i,\nabla\varphi\rangle\omega dx=\int_\Omega f\varphi dx+\int_\Omega\langle G,\nabla\varphi\rangle dx,$$

for all $\varphi\in X$. Hence

$$\begin{aligned}
&\int_\Omega |\left(\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\right)\Delta\varphi\omega dx\\
&+\int_\Omega\left(|\nabla u_1|^{p-2}\langle\nabla u_1,\nabla\varphi\rangle-|\nabla u_2|^{p-2}\langle\nabla u_2,\nabla\varphi\rangle\right)\omega dx\\
=&\ 0.
\end{aligned}$$

Taking $\varphi=u_1-u_2$, and using that for every $x,y\in\mathbb{R}^N$ there exist two positive constants α_p and β_p such that

$$\alpha_p(|x|+|y|)^{p-2}\,|x-y|\le\langle|x|^{p-2}x-|y|^{p-2}y,x-y\rangle\le\beta_p\,(|x|+|y|)^{p-2}|x-y|,$$

(see Proposition 17.3 in [2]) we obtain

$$\begin{aligned} 0 &= \int_\Omega \left(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\right)(\Delta u_1 - \Delta u_2)\,\omega dx \\ &+ \int_\Omega \left(|\nabla u_1|^{p-2}\langle\nabla u_1, \nabla u_1 - \nabla u_2\rangle - |\nabla u_2|^{p-2}\langle\nabla u_2, \nabla u_1 - \nabla u_2\rangle\right)\omega dx \\ &= \int_\Omega \left(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\right)(\Delta u_1 - \Delta u_2)\,\omega dx \\ &\quad + \int_\Omega \langle|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\omega dx \\ &\geq \alpha_p \int_\Omega (|\Delta u_1| + |\Delta u_2|)^{p-2}(|\Delta u_1 - \Delta u_2|)\omega dx \\ &\quad + \alpha_p \int_\Omega (|\nabla u_1| + |\nabla u_2|)^{p-2}\,|\nabla u_1 - \nabla u_2|\omega dx. \end{aligned} \tag{7}$$

Therefore $\Delta u_1 = \Delta u_2$ and $\nabla u_1 = \nabla u_2$ μ-a.e. and since $u_1, u_2 \in X$, then $u_1 = u_2$ a.e.. (by Remark 2.2).

EXAMPLE. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $w(x,y) = (x^2+y^2)^{-1/2}$ ($\omega \in A_3$, $p = 3$),

$$f(x,y) = \frac{\cos(xy)}{(x^2+y^2)^{1/6}} \text{ and } G(x,y) = \left(\frac{\sin(x+y)}{(x^2+y^2)^{1/6}}, \frac{\sin(xy)}{(x^2+y^2)^{1/6}}\right).$$

By Theorem 3.2, the problem

$$\begin{cases} \Delta((x^2+y^2)^{-1/2}|\Delta u|\Delta u) - \operatorname{div}[(x^2+y^2)^{-1/2}|\nabla u|\nabla u] = f(x) - \operatorname{div}(G(x)), \text{ in } \Omega \\ u(x) = 0, \text{ in } \partial\Omega \end{cases}$$

has a unique solution $u \in X = W^{2,3}(\Omega,\omega) \cap W_0^{1,3}(\Omega,\omega)$.

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