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Equitable Coloring On Mycielskian Of Wheels And Bigraphs*

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Abstract

In this paper we obtain interesting results regarding the equitable chromatic number $\chi_{=}$ for the Mycielskian of wheel graph $\mu(W_n)$ and the Mycielskian of complete bipartite graph $\mu(K_{m,n})$.

1 Introduction

The notion of equitable coloring was introduced by Meyer [14] in 1973. This model of graph coloring has many applications. Everytime when we have to divide a system with binary conflicting relations into equal or almost equal conflict-free subsystems we can model this situation by means of equitable graph coloring. This subject is widely discussed in literature [1, 2, 3, 6, 7, 12, 13, 16, 17].

One motivation for equitable coloring suggested by Meyer [14] concerns scheduling problems. In this application, the vertices of a graph represent a collection of tasks to be performed, and an edge connects two tasks that should not be performed at the same time. A coloring of this graph represents a partition of tasks into subsets that may be performed simultaneously. Due to load balancing considerations, it is desirable to perform equal or nearly-equal numbers of tasks in each time step, and this balancing is exactly what an equitable coloring achieves. Furmańczyk [7] mentions a specific application of this type of scheduling problem, namely assigning university courses to time slots in a way that spreads the courses evenly among the available time slots and avoids scheduling incompatible pairs of courses at the same time as each other, since then the usage of additional resources (e.g.rooms) is maximal.

A straightforward reduction from graph coloring to equitable coloring by adding sufficiently many isolated vertices to a graph, proves that it is NP-complete to test whether a graph has an equitable coloring with a given number of colors (greater than two). Furmańczyk et al. [8] proved that the problem remains NP-complete for corona graphs. Bodlaender and Fomin [1] showed that equitable coloring problem can

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be solved to optimality in polynomial time for trees (previously known due to Chen and Lih [3]) and outerplanar graphs. A polynomial time algorithm is also known for equitable coloring of split graphs [2], cubic graphs [4] and some coronas [8].

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, we now call the Mycielskian of G, which has the same clique number as G and whose chromatic number equals $\chi(G) + 1$.

Mycielski [15] showed that $\chi(\mu(G)) = \chi(G) + 1$ for any graph G and $\omega(\mu(G)) = \omega(G)$ for any graph G with at least one edge. Besides such interesting properties involving clique numbers and chromatic numbers, Mycielski's graphs also have some other parameters that behave in a predictable way. For example, it was shown by Larsen et al. [11] that $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$ for any graph G, where $\chi_f(G)$ is the fractional chromatic number of G. Mycielski's graphs were also used by Fisher [5] as examples of optimal fractional colorings that have large denominators.

The problem of determining if $\chi_=(G) = \chi(G)$ or $\chi_=(G)$ is 'close to' $\chi(G)-1$ is hard and has been extensively studied for general graphs. This paper presents a polynomial time solution to the Mycielskian of wheel graphs and complete bipartite graphs.

2 Preliminaries

If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable*. The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* of G and is denoted by $\chi_=(G)$.

The *open neighborhood* of a vertex v in a graph G, denoted by $N_G(v)$, is the set of all vertices of G which are adjacent to v. Moreover, $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v in the graph G.

In this paper, by G we mean a connected graph. From a graph G, by Mycielski's construction [9, 10, 15], we get the Mycielskian $\mu(G)$ of G with $V(\mu(G)) = V \cup U \cup \{z\}$, where

$$V = V(G) = \{x_1, \ldots, x_n\}, \quad U = \{y_1, \ldots, y_n\}$$

and

$$E(\mu(G)) = E(G) \cup \{y_i x : x \in N_G(x_i) \cup \{z\}, i = 1, \ldots, n\}.$$

We have the following example of a Mycielskian construction of a complete bigraph (Figure 1).

For any integer $n \geq 4$, the wheel graph W_n is the n-vertex graph obtained by joining a vertex v_1 to each of the $n-1$ vertices $\{w_1, w_2, \ldots, w_{n-1}\}$ of the cycle graph C_{n-1}.

For any set S of vertices of G, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S.

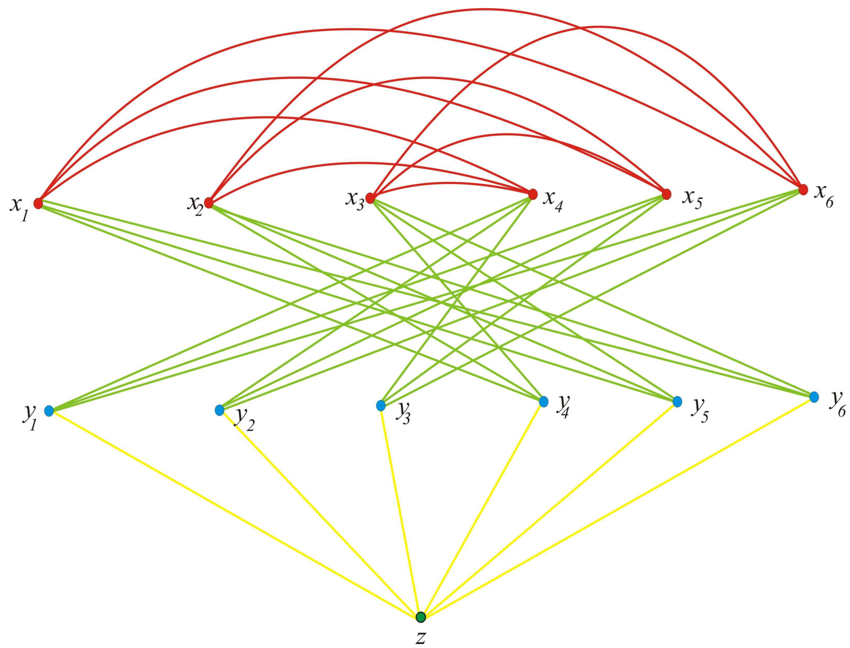

Figure 1: The Mycielskian graph of complete bipartite graph $K_{3,3}$

3 Equitable Coloring of Mycielskian of Wheel Graph

We have the following Theorem 1.

THEOREM 1. Let $\mu(W_n)$ be the Mycielskian of wheel graph. Then

$$\chi_{=}(\mu(W_n)) = \begin{cases} 5, & \text{for } n = 4, \\ \left\lceil \frac{2n+1}{3} \right\rceil, & \text{for } n \geq 5. \end{cases}$$

PROOF. Let

$$V(W_n) = \{x_i : 1 \leq i \leq n\}$$

and

$$E(W_n) = \{x_1 x_i : 2 \leq i \leq n\} \cup \{x_i x_{i+1} : 2 \leq i \leq n-1\} \cup \{x_n x_2\}.$$

By Mycielski's construction,

$$V(\mu(W_n)) = V(W_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{z\}.$$

In $\mu(W_n)$, y_1 is adjacent to each vertex of $\{x_i : 2 \leq i \leq n\}$, each $y_i\,(2 \leq i \leq n)$ is adjacent to the set of vertices of $N(x_i)$. z is adjacent to each vertex of $\{y_i : 1 \leq i \leq n\}$.

1. $n = 4$

Since $\langle\{x_1, x_2, x_3, x_4\}\rangle$ is K_4, each of these vertices receives distinct colors c_1, c_2, c_3, c_4 respectively. The same colors can be used to color the vertices y_1, y_2, y_3 and y_4 in the same order. Since z is adjacent to each of y_1, y_2, y_3 and y_4 there exists a new color to color z. Hence $\chi_=(\mu(W_n)) = 5$ for $n = 4$.

2. $n \geq 5$

By the definition of Mycielskian, x_1 is adjacent to each of $\{x_i : 2 \leq i \leq n\} \cup \{y_i : 2 \leq i \leq n\}$. There exists a color (say c_1), which is assigned to x_1, so that it cannot be assigned to any vertex of $\{x_i : 2 \leq i \leq n\} \cup \{y_i : 2 \leq i \leq n\}$. The color c_1 can be assigned at most two times in $\mu(W_n)$. By the definition of equitable coloring, the number of vertices of $\mu(W_n)$, receiving the same color, is either 2 or 3. Therefore $\chi_=(\mu(W_n)) \geq \left\lceil \frac{2n+1}{3} \right\rceil$.

We use the following partition to color the vertices of $\mu(W_n)$ equitably in the following subcases:

(a) $n = 5$

$$\begin{aligned} V_1 &= \{x_1, y_1\}, \\ V_2 &= \{x_2, x_4, z\}, \\ V_3 &= \{x_3, y_3, x_5\}, \\ V_4 &= \{y_2, y_4, y_5\}. \end{aligned}$$

(b) $n = 6$

$$\begin{aligned} V_1 &= \{x_1, y_1\}, \\ V_2 &= \{x_2, x_5, y_2\}, \\ V_3 &= \{x_3, y_3, y_5\}, \\ V_4 &= \{x_4, y_4, y_6\}, \\ V_5 &= \{x_6, z\}. \end{aligned}$$

(c) $n = 7$

$$\begin{aligned} V_1 &= \{x_1, y_1\}, \\ V_2 &= \{x_2, y_2, y_6\}, \\ V_3 &= \{x_3, y_3, y_7\}, \\ V_4 &= \{x_4, y_4\}, \\ V_5 &= \{x_5, y_5, x_7\}, \\ V_6 &= \{x_6, z\}. \end{aligned}$$

(d) $n \geq 8$

$$V(\mu(W_n)) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z\}.$$

We first partition this vertex set as follows:

$$V(\mu(W_n)) = V' \cup V''$$

where

$$V' = \left\{x_i : 1 \le i \le \left\lceil \frac{2n-2}{3} \right\rceil\right\} \cup \left\{y_i : 1 \le i \le \left\lceil \frac{2n-2}{3} \right\rceil\right\} \cup \left\{x_{\left\lceil \frac{2n+1}{3} \right\rceil}, x_n, z\right\}$$

and

$$V'' = \left\{x_i : \left\lceil \frac{2n+4}{3} \right\rceil \le i \le n-1\right\} \cup \left\{y_i : \left\lceil \frac{2n+1}{3} \right\rceil \le i \le n\right\}.$$

Clearly $V' \cup V'' = V$ and $V' \cap V'' = \emptyset$ with $|V'| = 2\left\lceil \frac{2n-2}{3} \right\rceil + 2$ and $|V''| = 2n - 2\left\lceil \frac{2n+1}{3} \right\rceil + 1$.

Now we partition V' into $\left\lceil \frac{2n+1}{3} \right\rceil$ independent sets each containing two vertices of V' as follows:

$$V_i = \{x_i, y_i\} \left(1 \le i \le \left\lceil \frac{2n-2}{3} \right\rceil\right) \text{ and } V_{\left\lceil \frac{2n+1}{3} \right\rceil} = \left\{x_{\left\lceil \frac{2n+1}{3} \right\rceil}, x_n, z\right\}.$$

Clearly

$$\bigcup_{i=1}^{\left\lceil \frac{2n+1}{3} \right\rceil} V_i = V' \text{ and } V_i \cup V_j = \emptyset$$

for $i \ne j$ and $|V_i| = 2$ for each i. Now we add the vertices of V' into the partition of V'' without violating the independence of each V_i's. Since each $x_i\,(2 \le i \le n)$ is adjacent to y_1 and each $y_i\,(2 \le i \le n)$ is adjacent to x_1, V_1 cannot be altered. Similarly $V_{\left\lceil \frac{2n+1}{3} \right\rceil}$ cannot be altered. Since z is adjacent with each $y_i\,(1 \le i \le n)$. Hence

$$|V'| = V_2 \cup V_3 \cup \cdots \cup V_{\left\lceil \frac{2n-2}{3} \right\rceil}$$

and

$$|V''| = \left(2n - 2\left\lceil \frac{2n+1}{3} \right\rceil + 1\right) \le \left\lceil \frac{2n+1}{3} \right\rceil - 2.$$

Also, $V_2, V_3, \ldots, V_{\left\lceil \frac{2n-2}{3} \right\rceil}$ can be altered by adding at most one vertex from V'' in such a way that the resultant V_i's contain at most three vertices. Hence $||V_i| - |V_j|| \le 1$ for $i \ne j$ and each V_i is an independent set. This partition implies $V(\mu(W_n))$ can be colored equitably with at most $\left\lceil \frac{2n+1}{3} \right\rceil$ colors. Hence $\chi_=(\mu(W_n)) = \left\lceil \frac{2n+1}{3} \right\rceil$ for $n \ge 5$.

This completes the proof of theorem 1.

4 Equitable Coloring of Mycielskian of Bipartite Graph

We have the following Theorem 2

THEOREM 2. Let $\mu(K_{m,n})$ be the Mycielskian of bipartite graph. Then

$$\chi_{=}(\mu(K_{m,n})) = \begin{cases} 3, & \text{if } m = n \leq 4, \\ 4, & \text{if } m = n \geq 5, \\ 3, & \text{if } 2m = n, \\ \left\lceil \frac{m+n}{\min(m,n)} \right\rceil, & \text{otherwise.} \end{cases}$$

PROOF. Let

$$V(\mu(K_{m,n})) = \{x_i : 1 \leq i \leq m\} \cup \{y_i : 1 \leq j \leq n\}$$

and

$$E(K_{m,n}) = \cup_{i=1}^{m} \{e_{ij} = x_i y_j : 1 \leq i \leq n\}.$$

By Mycielski's construction,

$$V(\mu(K_{m,n})) = V(K_{m,n}) \cup \{x_i' : 1 \leq i \leq m\} \cup \{y_j' : 1 \leq j \leq n\} \cup \{z\}.$$

1. $m = n \leq 4$

   Since $\langle \{x_i : 1 \leq i \leq 4\} \rangle$, $\langle \{y_i : 1 \leq i \leq 4\} \rangle$ and $\langle \{y_i' : 1 \leq i \leq 4\} \rangle$ are totally disconnected, each of these subgraphs can be assigned with single color.

   Assign the color c_1 to each of $\{x_i : 1 \leq i \leq 4\}$, c_2 to each of $\{y_i : 1 \leq i \leq 4\}$ and c_3 to each of $\{y_i' : 1 \leq i \leq 4\}$. Assign c_1 to x_1', c_2 to z and c_3 to x_2'. By this process of assigning color, each color is assigned exactly 4 times. To color the remaining vertices x_3' and x_4', we can use the colors c_1 and c_3 without affecting the equitable condition. Hence $\chi_{=}(\mu(K_{m,n})) = 3$ for $m = n \leq 4$.

2. $m = n \geq 5$

   By applying the above said procedure to color equitably, we color each of $\{y_i : 1 \leq i \leq n\} \cup \{z\}$ by c_2. Hence the remaining number of vertices to be colored is $3n$. These $3n$ vertices should be divided into two sets, each containing $\frac{3n}{2}$ and $\left(\frac{3n}{2} + 1\right)$ vertices.

   Since $\frac{3n}{2} \neq n + 1$and $\left(\frac{3n}{2} + 1\right) \neq n + 1$for $n \geq 5$, we conclude that the partition of $V(\mu(K_{m,n}))$into three sets satisfying $||V_i| - |V_j|| \leq 1$for $i \neq j$is not possible and hence $\chi_{=}(\mu(K_{m,n})) \geq 4$. We use the following partition to color the vertices of $\mu(K_{m,n})$for $m = n \geq 5$equitable in the following:

$$\begin{aligned} V_1 &= \{x_i : 1 \leq i \leq n\} \cup \{z\}, \\ V_2 &= \{y_i : 1 \leq i \leq n\}, \\ V_3 &= \{x_i' : 1 \leq i \leq n\}, \\ V_4 &= \{y_i' : 1 \leq i \leq n\}. \end{aligned}$$

   Clearly $||V_i| - |V_j|| \leq 1$ for $i \neq j$. Hence $\chi_{=}(\mu(K_{m,n})) \leq 4$ and so

$$\chi_{=}(\mu(K_{m,n})) = 4.$$

3. $2m = n$

   Since there exists an odd cycle $x_1 y_1' z x_1' y_1 x_1$ in $\mu(K_{m,n})$,

$$\chi_{=}(\mu(K_{m,n})) \geq 3 \text{ for } 2m = n,$$

$$\begin{aligned} V_1 &= \{x_i, x_i' : 1 \leq i \leq m\}, \\ V_2 &= \{y_i : 1 \leq i \leq n\} \cup \{z\}, \\ V_3 &= \{y_i' : 1 \leq i \leq n\}. \end{aligned}$$

   Clearly $||V_i| - |V_j|| \leq 1$ for $i \neq j$. Hence $\chi_{=}(\mu(K_{m,n})) \leq 3$ and so $\chi_{=}(\mu(K_{m,n})) = 3$.

4. $\left\lceil \frac{m+n}{\min(m,n)} \right\rceil$

   $|V(\mu(K_{m,n}))| = 2m+2n+1$ without loss of generality assume that $m < n$. Then by the construction of equitable coloring, all the vertices of

$$X = \{x_i : 1 \leq i \leq m\} \cup \{x_i' : 1 \leq i \leq m\}$$

   receive the colors according to any one of the following cases:

   (a) If X receives the same color 1. In this case the color 1, appears at the maximum of $2m$ times. The other colors $2, 3, \ldots$ are used at the maximum of $2m+1$ times to color the vertices of $\{y_j : 1 \leq j \leq n\} \cup \{y_j' : 1 \leq j \leq n\} \cup \{z\}$. Therefore each color appear at most $2m+1$ times and at least $2m$ times.

   (b) If X receives the same color 1 with an another color 3, then the color 1, appears at the maximum of $2m-1$ times and other colors $2, 3, \ldots$ are used at the maximum of $2m$ times to color the vertices of $\{y_j : 1 \leq j \leq n\} \cup \{y_j' : 1 \leq j \leq n\} \cup \{z\}$. Therefore in this case each color appear at most $2m$ times and at least $2m-1$ times.

Therefore we conclude from the subcases (a) and (b) that the minimum number of colors to be used to get equitable coloring of $\mu(K_{m,n})$ is either $\left\lceil \frac{2m+2n+1}{2m} \right\rceil$ or $\left\lceil \frac{2m+2n+1}{2m+1} \right\rceil$. (i.e.) $\left\lceil \frac{m+n}{m} \right\rceil$.
In general $\left\lceil \frac{m+n}{\min(m,n)} \right\rceil$.
In such a coloring, out of $\left\lceil \frac{m+n}{\min(m,n)} \right\rceil$ colors,
$(2m+2n+1) - \left(\left\lfloor \frac{2m+2n+1}{\left\lceil \frac{m+n}{\min(m,n)} \right\rceil} \right\rfloor \times \left\lceil \frac{m+n}{\min(m,n)} \right\rceil \right)$ colors appear $\frac{2m+2n+1}{\left\lceil \frac{m+n}{\min(m,n)} \right\rceil} + 1$ times and

$$\left(\left\lfloor \frac{2m+2n+1}{\left\lceil \frac{m+n}{\min(m,n)} \right\rceil} \right\rfloor + 1 \right) \times \left\lceil \frac{m+n}{\min(m,n)} \right\rceil - (2m+2n+1)$$

colors appear $\left\lfloor \frac{2m+2n+1}{\left\lceil \frac{m+n}{\min(m,n)} \right\rceil} \right\rfloor$ times. Hence $\chi_{=}(\mu(K_{m,n})) = \left\lceil \frac{m+n}{\min(m,n)} \right\rceil$. This completes the proof of theorem 2.

5 Summary

This paper establishes an optimal polynomial time solution to the equitable chromatic number of Mycielskian of wheels and bigraphs. A further study on equitable coloring and Mycielskian graph families would result in some nice contributions to the literature and applications in this area.

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