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Existence And Uniqueness Of Weak Solution For p-Laplacian Problem In $\mathbb{R}^{N}$*

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Abstract

This paper shows the existence and uniqueness of a weak solution of a problem in $\mathbb{R}^N$, which involves the p-Laplacian through the Browder Theorem.

1 Introduction

The present paper is concerned with the elliptic problem:

$$(\mathcal{P}) \ \ -\Delta_p u + m(x) \mid u \mid^{p-2} u = f(x,u) \ \ in \ \ \mathbb{R}^N,$$

where $1 < p < N, \ N \geq 3$, Δ_p denotes the p-Laplacian defined by

$$\Delta_p u = div(\mid \nabla u \mid^{p-2} \nabla u).$$

We make the following assumptions.

(m_0) $m \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < m(x) < +\infty$.

Let $\gamma = \frac{p^*}{p^*-(q+1)}$ and $p^* = \frac{Np}{N-p}$. There exist $a \in L^{(p^*)'}(\mathbb{R}^N)$ and $b \in L^\infty(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$ such that

(f_1) f satisfies

$$\mid f(x,s) \mid \leq a(x) + b(x) \mid s \mid^q,$$

where

$$1 < q \leq p-1.$$

(f_2) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a carathéodory (CAR) function which is decreasing with respect to the second variable, i.e.,

$$f(x,s_1) \leq f(x,s_2)$$

for a.e. $x \in \Omega$ and $s_1, s_2 \in \mathbb{R}, s_1 \geq s_2$.

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The goal of this paper is to prove the following result:

THEOREM 1. Assume that (m_0) holds and $f \in CAR(\mathbb{R}^N \times \mathbb{R})$ satisfies (f_1) and (f_2). Then the problem $(\mathcal{P})$ has a unique weak solution.

When $p = 2$, the problem $(\mathcal{P})$ is a normal Schrodinger equation which has been extensively studied. There are several studies of the existence of solutions of $(\mathcal{P})$ on a bounded domain of $\mathbb{R}^N$. We mention the results obtained in [1, 2] and [6] for the case of bounded domains. In recent years, more and more attention is paid to the quasilinear elliptic setting on $\mathbb{R}^N$. The main difficulty in the study of p-Laplacian equations in R^N arises from the lack of compactness.

In the squeal, we recall some basic definitions and notations which will be used throughout the paper. Whereas, the last part of the article is dedicated to the demonstration of our main result.

DEFINITION 1.We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution of problem $(\mathcal{P})$ if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\mathbb{R}^N} m(x) |u|^{p-2} uv dx = \int_{\mathbb{R}^N} f(x,u) v dx$$

for all $v \in W^{1,p}(\mathbb{R}^N)$.

For simplicity let $X = W^{1,p}(\mathbb{R}^N)$. According to condition (m_0), we can introduce a new norm defined as follows

$$\| u \| = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} m(x) |u|^p dx \right)^{\frac{1}{p}}.$$

DEFINITION 2. Let K be a Banach space. An operator $A : K \to K$ verifies

$$\langle Au - Av, u - v \rangle \geq 0 \tag{1}$$

for any $u, v \in K$ is called a monotone operator. An operator A is called strictly monotone if for $u \neq v$ the strict inequality holds in (1). An operator A is called strongly monotone if there exists $C > 0$ such that

$$\langle Au - Av, u - v \rangle \geq C \| u - v \|^2$$

for any $u, v \in K$.

We recall Browder Theorem.

Theorem 3 (cf. [3]). Let A be a reflexive real Banach space. Moreover, let $A : X \to X^*$ be an operator which is: bounded, demicontinuous, coercive, and monotone on the space X . Then, the equation $A(u) = f$ has at least one solution $u \in X$ for each $f \in X^*$. If moreover, A is strictly monotone operator, then the equation $(\mathcal{P})$ has precisely one solution $u \in X$ for every $f \in X^*$.

We define the operator $A : X \to X^*$ by

$$A := I - F,$$

where the operators I and F are defined from X into X^* as

$$\langle I(u), v\rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\mathbb{R}^N} m(x) |u|^{p-2} uv dx$$

and

$$\langle F(u), v\rangle = \int_{\mathbb{R}^N} f(x, u) v dx,$$

for all $u, v \in X$.

By Definition 1, the main tool in searching the weak solutions of $(\mathcal{P})$ is to finding $u \in X$ which satisfies the operator equation $Au = 0$.

2 Proof of The Main Result

We denote by C and $C_i, i = 1, 2...$ the general positive constants which are the exact values may change from line to line.

PROOF OF THEOREM 1. In order to apply Browder Theorem, we split the proof in several steps,

Step1. We prove that A is bounded. We know that the functional

$$\psi(u) = \int_{\mathbb{R}^N} \frac{1}{p} \left(|\nabla u|^p + m(x) |u|^p\right) dx$$

is of class C^1 (cf. [5]) and I is the derivative operator of ψ in the weak sense, so it yields I is bounded and continuous. Let $u \in X$, such that $\| u \| < K$. Using Hölder's inequality, we obtain

$$\begin{aligned}
&\|F(x,u)\|_{X^*}\\
&= \sup_{\|v\|=1} | \langle F(x,u), v\rangle |\\
&\le \sup_{\|v\|=1} \int_{\mathbb{R}^N} a(x) |v| dx + \int_{\mathbb{R}^N} b(x) |u|^q |v| dx\\
&\le \sup_{\|v\|=1} \left\{ \left(\int_{\mathbb{R}^N} a^{(p^*)'} dx\right)^{\frac{1}{(p^*)'}} + \left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{\frac{q}{p^*}} \left(\int_{\mathbb{R}^N} (bv)^{\frac{p^*}{p^*-q}} dx\right)^{\frac{p^*-q}{p^*}} \right\}\\
&\le \sup_{\|v\|=1} \left\{ \left[\left(\int_{\mathbb{R}^N} a^{(p^*)'} dx\right)^{\frac{1}{(p^*)'}} + \left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{\frac{q}{p^*}} \left(\int_{\mathbb{R}^N} b^{\gamma}\right)^{\frac{1}{\gamma}}\right] \left(\int_{\mathbb{R}^N} |v|^{p^*} dx\right)^{\frac{1}{p^*}} \right\}\\
&\le C_3 \|a\|_{(p^*)'} + C_4 K^q \|b\|_{\gamma},
\end{aligned}$$

hence A is bounded.

Step 2. We prove that A is demicontinuous. It is well known that the functional

$$\psi(u) = \int_{\mathbb{R}^N} \left(\frac{1}{p} (|\nabla u|^p + m(x)\,|u|^p \right) dx$$

is of class C^1. Since I is the Fréchet derivative of ψ hence I is continuous. Now we check that F is completely continuous that is, if $u_n \rightharpoonup u$ then $F(u_n) \to F(u)$ and it is well be done. Let u_n is weakly convergent to u in X so u_n is bounded in X. Set

$$B_k = \left\{ x \in \mathbb{R}^N : |x| < k \right\},$$

so we have $|b|_{L^\gamma(\mathbb{R}^N \setminus B_k)}$ converges to zero as $n \to +\infty$. For all $v \in X$ we have

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_k} a(x)\,|v|\,dx &\leq \left(\int_{\mathbb{R}^N \setminus B_k} |v|^{p^*}\,dx \right)^{\frac{1}{p^*}} \left(\int_{\mathbb{R}^N \setminus B_k} |a|^{(p^*)'}\,dx \right)^{\frac{1}{(p^*)'}} \\ &\leq C\,\|v\| \left(\int_{\mathbb{R}^N \setminus B_k} |a|^{(p^*)'}\,dx \right)^{\frac{1}{(p^*)'}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_k} b(x)\,|u|^q\,|v|\,dx &\leq \left(\int_{\mathbb{R}^N \setminus B_k} |u|^{p^*}\,dx \right)^{\frac{q}{p^*}} \left(\int_{\mathbb{R}^N \setminus B_k} (b(x)\,|v|)^{\frac{p^*}{p^*-q}}\,dx \right)^{\frac{p^*-q}{p^*}} \\ &\leq \left(\int_{\mathbb{R}^N \setminus B_k} |u|^{p^*} \right)^{\frac{q}{p^*}} \left(\int_{\mathbb{R}^N \setminus B_k} |v|^{p^*} \right)^{\frac{1}{p^*}} \left(\int_{\mathbb{R}^N \setminus B_k} b^\gamma \right)^{\frac{1}{\gamma}} \\ &\leq C\,\|u\|^q\,\|v\| \left(\int_{\mathbb{R}^N \setminus B_k} b(x)^\gamma dx \right)^{\frac{1}{\gamma}}. \end{aligned}$$

According to previous inequalities we have,

$$\left| \int_{\mathbb{R}^N \setminus B_k} (f(x,u_n) - f(x,u))v dx \right| \leq C\,\|v\| \left(\int_{\mathbb{R}^N \setminus B_k} b(x)^\gamma dx \right)^{\frac{1}{\gamma}} + \|v\| \left(\int_{\mathbb{R}^N \setminus B_k} |a(x)|^{(p^*)'}\,dx \right)^{\frac{1}{(p^*)'}},$$

which yields that

$$\int_{\mathbb{R}^N \setminus B_k} (f(x,u_n) - f(x,u))v dx \to 0$$

for k sufficiently large. From the compact embedding $W^{1,p}(B_k) \hookrightarrow L^q(B_k)$, we can infer that

$$\int_{B_k} f(x,u_n)v dx \to \int_{B_k} f(x,u)v dx$$

and then we have

$$\begin{aligned}\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))v dx &= \int_{\mathbb{R}^N \setminus B_k} (f(x,u_n) - f(x,u))v dx \\ + \int_{B_k} (f(x,u_n) - f(x,u))v dx &\to 0.\end{aligned}$$

So F is completely continuous and then F is continuous.

Step 3. We prove that A is monotone. We recall the following inequality for $p \geq 2$, $x, y \in \mathbb{R}^N$ (see [4])

$$|y|^p \geq |x|^p + p\,|x|^{p-2}\,x(y-x) + \frac{|y-x|^p}{2^{p-1}-1}.$$

Let

$$\begin{aligned}\langle I(u) - I(v), u - v\rangle &= \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\,\nabla u dx - |\nabla v|^{p-2}\,\nabla v\right)(\nabla u - \nabla v)dx \\ &\quad + \int_{\mathbb{R}^N} m(x)(|u|^{p-2}\,u - |v|^{p-2}\,v)(u-v)dx.\end{aligned}$$

We obtain that

$$\begin{aligned}\langle I(u) - I(v), u - v\rangle &\geq \frac{2}{p2^{p-1}-1}\left[\int_{\mathbb{R}^N} |\nabla u - \nabla v|^p\,dx + \int_{\mathbb{R}^N} m(x)\,|u-v|^p\,dx\right] \\ &= C_p\,\|u-v\|^p\,. \end{aligned} \tag{2}$$

Therefore, A is strongly monotone. (see e.g. [7]). Further, since f is decreasing with respect to the second variable,

$$\langle F(u) - F(v), u - v\rangle = \int_{\mathbb{R}^N} (f(x,u) - f(x,v))(u-v)dx \leq 0.$$

It follows that A is strongly monotone.

Step 4. We prove that A is a coercive operator. We have

$$\begin{aligned}\frac{1}{\|u\|}\langle Au, u\rangle &= \frac{1}{\|u\|}\left[\int_{\mathbb{R}^N} \left(|\nabla u|^p + m(x)\,|u|^p\right)dx - \int_{\mathbb{R}^N} f(x,u)udx\right] \\ &\geq \frac{1}{\|u\|}\left[\|u\|^p - \int_{\mathbb{R}^N} (a(x)\,|u| + b(x)\,|u|^q\,|u|)dx\right] \\ &\geq \frac{1}{\|u\|}\left(\|u\|^p - C_1\,\|a\|_{(p^*)'}\,\|u\| - C_2^{q+1}\,\|u\|\,\|u\|_\gamma\right),\end{aligned}$$

which yields the coercivity of A for $1 < q < p-1$. In the case when $q = p-1$, since $X \hookrightarrow L^p(\mathbb{R}^N)$ with continuous embedding, then by a similar argument to that used in [1], A is coercive.

Step 5. From the previous steps, the assumptions of Theorem 3 are fulfilled. Therefore, problem $(\mathcal{P})$ has a weak solution. For the uniqueness of weak solution for problem ($\mathcal{P}$), suppose that u and v be a weak solutions of ($\mathcal{P}$) such that $u \neq v$. By (2) it follows that

$$0 = \langle Au - Av, u - v\rangle \geq C_p \|u - v\|^p \geq 0.$$

Then $u = v$ and the proof now is completed.

This solution cannot be trivial provided that we suppose $f(x, 0) \neq 0$, because in this case $A0 \neq 0$.

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References

[1] G. A, Afrouzi, S. Mahdavi, and Z. Naghizadeh, Existence and uniqueness of solution for p-Laplacian dirichlet problem, Int. J. Nonlinear Sci., 8(2009), 274–278.

[2] S. Khafagy, Existence and uniqueness of weak solution for weighted p-Laplacian Dirichlet problem, J. Adv. Res. Dyn. Control Syst., 3(2011), 41–49.

[3] J. Leray, J. L. Lions, Quelques resultats de Visik sur les problems elliptiques nonlineaires par les methodes de Minty Browder, Bull. Soc. Math. France, 93(1965), 97–107.

[4] P. Lindqvist, On the equation $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc., 109(1990), 157–164.

[5] M. Mihăilescu and V. Rădelescu, A multiplicity for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. Roy. Soc. London Ser. A, 462(2006), 2625–2641.

[6] J. S. W. Wong, On the generalized Emden-Fowler equation, SIAM Rev., 17(1975), 339–360.

[7] E. Zeidler, Nonlinear Functional Analysis and Its Applications, Vol. I, II/A, II/B, III and IV, Springer Verlag, Berlin, Heidelberg, New York, (1986).