Applied Mathematics E-Notes, 13(2013), 221-227 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Commuting Graphs Of Dihedral Type Groups*

Zahid Raza†, Shahzad Faizi‡

Received 4 July 2013

Abstract

For a non-abelian group G and a subset X of G, we define the commuting graph, denoted $\Gamma(X) = C(G, X)$, to be the graph whose vertex set is X with two distinct vertices $x, y \in X$ joined by an edge if and only if $xy = yx$.

In this short note, certain properties of commuting graphs constructed on the dihedral type groups D_{2n} with respect to some specific subsets are discussed. More precisely, the chromatic number and clique number of these commuting graphs are obtained.

1 Introduction

The study of algebraic structures, using the properties of graphs, has become an exciting research topic in the last twenty years, leading to many fascinating results and raising questions. For example, the study of zero-divisor graphs [6], total graph of commutative rings and commuting graph of groups has attracted many researchers towards this dimension. The concept of non-commuting graph has been studied in [2] and [16]. Recently, the commuting graphs of groups have been studied extensively, see for example [5, 15–17], and those of rings in [3–4]. In [15], Iranmanesh and Jafarzadeh conjectured that there is a universal upper bound on the diameter of a connected commuting graph for any finite nonabelian group. They determined that when the commuting graph of a symmetric or alternating group is connected and that the diameter is at most 5 in this case. The paper [17] proves that for all finite classical simple groups over a field of size at least 5, when the commuting graph of a group is connected then its diameter is at most 10.

If X is a conjugacy class of involutions of a group G, then $\Gamma(G, X)$ is called a commuting involution graph. Aschbacher [1] also showed a necessary condition on a commuting involution graph for the presence of a strongly embedded subgroup in G. The detailed study of commuting involution graphs can be found in [7–10, 13–15].

In this informative note, we discuss certain properties of commuting graphs constructed on the dihedral type groups D_{2n} with respect to some specific subsets. For ordinary dihedral group D_{2n}, the commuting graph of dihedral group D_{2n} was discussed in [11].

*Mathematics Subject Classifications: 05C12.

†Department of Mathematics, National University of Computer & Emerging Sciences Lahore Campus, B-Block, Faisal Town, Lahore, Pakistan.

‡Department of Mathematics, National University of Computer & Emerging Sciences Lahore Campus, B-Block, Faisal Town, Lahore, Pakistan.

2 Definitions and Notations

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph Γ, we denote the set of the vertices and edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree $\deg(v)$ of a vertex v in Γ is the number of edges incident to v. A graph Γ is regular if the degrees of all the vertices of Γ are same.

A subset X of the vertices of Γ is called a clique if the induced subgraph of X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $w(\Gamma)$.

Let $k > 0$ be an integer. A k-vertex coloring of a graph Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertices receive the same color. The chromatic number $\psi(\Gamma)$ of a graph Γ, is the minimum k for which Γ has a k-vertex coloring.

If u and v are vertices in Γ, then $d(u, v)$ denotes the length of the shortest path between u and v. The maximum distance between all pairs of the vertices of Γ is called the diameter of Γ, and is denoted by $\mathrm{diam}(\Gamma)$.

A matching or independent edge set in a graph is a set of edges without common vertices. It may also be an entire graph consisting of edges without common vertices.

A perfect matching is a matching which matches all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching.

A subset X of the vertices of Γ is called an independent set if the induced subgraph on X has no edge. The independent number of Γ is the maximum size of an independent set of vertices and is denoted by $\alpha(\Gamma)$.

A vertex cover of a graph is a set $S \subseteq V(\Gamma)$ that contains at least one endpoint of every edge. The minimum size of a vertex cover is denoted by $\beta(\Gamma)$.

3 Group Properties of D_{2n}

The dihedral type groups denoted by D_{2n}, are defined in terms of generators a, b and relations as: $D_{2n} =< a, b \mid b^n = a^2 = 1,\ ab = b^r a >$, where $n = 4k$ with $r = 2k - 1$ or $r = 2k + 1$ and any positive integer $k \geq 2$. In this section, we discuss some of the group theoretic properties of the group D_{2n}. Throughout this note a, b represent the abstract generators of the groups D_{2n} and n, m, d, r are always positive integers.

LEMMA 1. If a and b are the abstract generators of the groups D_{2n}, then we have the followings:

(i) $a(b^m a) = b^{mr}$ for all $m < n$,

(ii) $(b^m a)^{2l} = b^{(r+1)lm}$ for all $m < n$ and any positive integer l,

(iii) $(b^m a)^{2l-1} = b^{(l-1)mr+lm} a$ for all $m < n$ and any positive integer l.

Note: Lemma 1 shows that the order of $b^m a$ must be even. The following Lemmas are easy to proof, so we will give only statements:

LEMMA 2. If $n = 4k$ and $r = 2k \pm 1$, then $\gcd(n, r-1)$ is either 2, 4 or $\frac{n}{2}$.

LEMMA 3. The element b^i is in the center of the group D_{2n} if and only if $n \mid (r-1)i$, where $i \neq 1$.

LEMMA 4. Consider the group D_{2n}. We have the following results:

(i) If $\gcd(n, r-1) = d$, then there are d central elements of D_{2n}. More precisely,

$$Z(D_{2n}) = \{b^{\frac{n}{d}}, b^{\frac{2n}{d}}, b^{\frac{3n}{d}}, \dots, b^n\}.$$

(ii) The element $b^i a$ will commute with all the elements $b^j a$, where $1 \leq j \leq n-1$ if and only if $n \mid (r-1)(j-i)$ for fixed i.

(iii) If $\gcd(n, r-1) = d$, then $[D_{2n} : Z(D_{2n})] = \frac{2n}{d}$.

4 Commuting Graph of D_{2n}

Let $[G : Z(G)] = m$ and $T = \{1, x_1, x_2 \dots, x_{m-1}\}$ be a transversal of $Z(G)$ in a group G. It is clear that every two elements of the coset $x_i Z(G)$, $1 \leq i \leq m-1$, commute. Thus, every two elements of these coset are adjacent. Throughout this section, we shall denote $X_1 = \{b^1, b^2, b^3, \dots, b^n\}$ and $X_2 = \{b^1 a, b^2 a, b^3 a, \dots, b^n a\}$ two subsets of D_{2n}.

We associate to the commuting graph $\Gamma(D_{2n}) = C(D_{2n}, D_{2n})$ of the group D_{2n}, the induced subgraph $\Gamma^u(D_{2n})$ as follows: The vertex set of $\Gamma^u(D_{2n})$ is $U = T - \{1\} = \{x_1, x_2 \dots, x_{m-1}\}$, and two vertices x_i and x_j are adjacent if and only if $x_i x_j = x_j x_i$, where $1 \leq i, j \leq m-1$.

LEMMA 5. If X is any subset of D_{2n} and $\Gamma(X) = C(D_{2n}, X)$ is the commuting graph on X, then for any $a \in X$, $\deg(a) = \mid C_X(a) \mid -1$.

LEMMA 6. If $\Gamma(D_{2n}) = C(D_{2n}, D_{2n})$, then

$$(1)\ \deg(b^m a) = \begin{cases} 3 & \text{if } \gcd(n, r-1) = 2, \\ 7 & \text{if } \gcd(n, r-1) = 4, \\ n-1 & \text{if } \gcd(n, r-1) = \frac{n}{2}. \end{cases}$$

$$(2)\ \deg(b^i) = \begin{cases} 2n-1 & \text{if } n \mid (r-1)\, i, \\ n-1 & \text{otherwise.} \end{cases}$$

PROOF. (1) If $\gcd(n, r-1) = 2$, then there are two central elements e and $b^{\frac{n}{2}}$. Now, we have $C_{b^m a} = \{e, b^{\frac{n}{2}}, b^m a, b^{m+\frac{n}{2}} a\}$ for all $b^m a \in D_{2n}$. Hence by virtue of Lemma 5 we have, $\deg(b^m a) = 4 - 1 = 3$. The remaining case can be proof in a similar manner.

(2) If $n \mid (r-1)i$, then by Lemma 3, we have $C_{b^i} = D_{2n}$ and so $\deg(b^i) = 2n-1$. If $n \nmid (r-1)i$, then $C_{b^i} = \{b^j : 1 \leq j \leq n\}$ and so $\deg(b^i) = n-1$.

COROLLARY 1. If $X = D_{2n} \setminus Z(D_{2n})$ and $\Gamma(X) = C(D_{2n}, X)$ is the commuting graph on X, then

(1) $\deg(b^m a) = \begin{cases} 1 & \text{if } \gcd(n, r-1) = 2, \\ 3 & \text{if } \gcd(n, r-1) = 4, \\ \frac{n}{2} - 1 & \text{if } \gcd(n, r-1) = \frac{n}{2}. \end{cases}$

(2) $\deg(b^i) = \begin{cases} n-3 & \text{if } \gcd(n, r-1) = 2, \\ n-5 & \text{if } \gcd(n, r-1) = 4, \\ \frac{n}{2} - 1 & \text{if } \gcd(n, r-1) = \frac{n}{2}. \end{cases}$

(3) If $\Gamma(X) = C(D_{2n}, X)$, where $X = D_{2n} \setminus Z(D_{2n})$, then $diam(\Gamma(X)) = \infty$.

LEMMA 7. If X is any subset of D_{2n}, then $\Gamma(X) = K_n$ if and only if $X = X_1$.

PROOF. Suppose $X = \{b^i : 1 \leqslant i \leqslant n\}$. Then X is a cyclic subgroup of D_{2n} and so $\Gamma(X)$ is a complete graph of n vertices. Conversely, Suppose $\Gamma(X) = K_n$. Then by Lemma 5, we have $X = \{b^i : 1 \leqslant i \leqslant n\} = X_1$.

COROLLARY 2. There does not exist any subset X of D_{2n} such that $\Gamma(X)$ is n regular.

PROPOSITION 1. If $X = D_{2n}$, then the edges in the commuting graph $\Gamma(X)$ are:

$$|E(\Gamma(D_{2n}))| = \begin{cases} \frac{n(n+4)}{2} & \text{if } \gcd(n, r-1) = 2, \\ \frac{n(n+10)}{2} & \text{if } \gcd(n, r-1) = 4, \\ \frac{n(5n-4)}{4} & \text{if } \gcd(n, r-1) = \frac{n}{2}. \end{cases}$$

PROOF. Note that $X_1 \bigcap X_2 = \emptyset$ and $X_1 \bigcup X_2 = D_{2n}$. Suppose that $\gcd(n, r-1) = 2$. Then the subgraph induced by X_1 is complete and the subgraph induced by X_2 is $\frac{n}{2}K_2$. Therefore the number of edges in $\Gamma(D_{2n})$ is the sum of the number of edges in these subgraphs and the number of edges from the center elements b^n and $b^{\frac{n}{2}}$ to the set of vertices in the induced graph by X_2. Thus $E(\Gamma(D_{2n})) = \frac{n(n-1)}{2} + \frac{n}{2} + 2n = \frac{n(n+4)}{2}$. Similarly one can prove the remaining cases.

COROLLARY 3. If $\Gamma(X) = C(D_{2n}, X)$, where $X = D_{2n} \setminus Z(D_{2n})$, then

$$|E(\Gamma(D_{2n}))| = \begin{cases} \frac{n^2-4n+6}{2} & \text{if } \gcd(n, r-1) = 2, \\ \frac{n^2-6n+20}{2} & \text{if } \gcd(n, r-1) = 4, \\ \frac{3n(n-2)}{8} & \text{if } \gcd(n, r-1) = \frac{n}{2}. \end{cases}$$

THEOREM 1. There exists no subset X of D_{2n} such that $\Gamma(X) = C_4$.

PROOF. Let $\gcd(n, r-1) = 2$ and suppose $\Gamma(X) = C_4$ for some subset X of D_{2n}. Then $\Gamma(X)$ contains a complete graph of type K_n in X_1 and $\frac{n}{2}$ complete graphs of type K_2 in X_2. If X contains two vertices of X_2 from different K_2 and two vertices from X_1. Then this graph has no edge between the elements from different K_2, a contradiction.

Suppose X contains two elements from any one of K_2 in X_2 and two elements from X_1 and they could be either one or both central elements. Then in both cases, the degree of vertex of the central element is 3 and hence we get a contradiction. If X contains any 3 elements from X_1 and one element from any K_2 in X_2 then the graph contains a complete graph of type K_3 in X_1 and thus a contradiction. Similarly if $X \subset X_1$ then $\Gamma(X)$ is an induced subgraph of a complete graph $\Gamma(X_1)$ and hence itself complete, a contradiction again. Similarly one can prove the remaining cases.

The following Lemma can be proof in a similar fashion as Theorem 1:

LEMMA 8. There exist no subset X of D_{2n} such that $\Gamma(X) = P_4$.

THEOREM 2. If $\Gamma(D_{2n})$ is the commuting graph on D_{2n}, then $w(\Gamma(D_{2n})) = \psi(\Gamma(D_{2n})) = n$.

PROOF. Since $X_1 = \{b^1, b^2, b^3, \dots, b^n\} \subset D_{2n}$ and $\Gamma(X_1)$ is a maximal complete subgraph of $\Gamma(D_{2n})$. Hence $w(\Gamma(D_{2n})) = n$.

If $\gcd(n, r-1) = 2$, then we need n colors to color the induced subgraph $\Gamma(X_1) \subset \Gamma(D_{2n})$ and so $\psi(\Gamma(D_{2n})) \geq n$. Note that e and $b^{\frac{n}{2}}$ are two central elements in X_1 and they are adjacent to all vertices in $\Gamma(D_{2n})$ and so color assigned to these vertices cannot be assigned to any other vertices. The remaining $n-2$ vertices in X_1 are not adjacent to any of the remaining vertices in $\Gamma(D_{2n})$ and so these vertices can be colored by any one of the remaining $n-2$ colors. Hence $\psi(\Gamma(D_{2n})) = n$. Similarly one can prove the remaining cases.

COROLLARY 4. The following assertions (1)–(2) hold.

(1) If $\Gamma(X) = C(D_{2n}, X)$, where $X = D_{2n} \setminus Z(D_{2n})$ and $\gcd(n, r-1) = d$, then $w(\Gamma(D_{2n})) = \psi(\Gamma(D_{2n})) = n - d$.

(2) The commuting graph $\Gamma(D_{2n})$ has a perfect matching.

THEOREM 3. If $\gcd(n, r-1) = d$, then $\alpha(\Gamma^u(D_{2n})) = \frac{n}{d} + 1$.

PROOF. Since $\gcd(n, r-1) = d$, therefore $\mid Z(D_{2n}) \mid = d$ and the set $T = \{1, b, b^2, \dots, b^{\frac{n}{d}-1}, a, ba, b^2a, \dots, b^{\frac{n}{d}-1}a\}$ is a transversal of $Z(D_{2n})$ in D_{2n}, so we have the set $U = T - 1 = \{b, b^2, \dots, b^{\frac{n}{d}-1}, a, ba, b^2a, \dots, b^{\frac{n}{d}-1}a\}$. For any j, $1 \leq j \leq \frac{n}{d} - 1$, the set $A_j = \{b^j, a, ba, \dots, b^{\frac{n}{d}-1}a\}$ is an independent set of $\Gamma^u(D_{2n})$ and each two elements of the set $X = \{b^i, 1 \leq i \leq \frac{n}{d} - 1\}$ are adjacent. Thus for any j, $1 \leq j \leq \frac{n}{d} - 1$, $\mid A_j \mid = \frac{n}{d} + 1$. Hence $\alpha(\Gamma^u(G)) = \frac{n}{d} + 1$.

COROLLARY 5. As $\alpha(\Gamma(D_{2n})) = \alpha(\Gamma^u(D_{2n}))$. Hence $\alpha(\Gamma(D_{2n})) = \frac{n}{d} + 1$.

LEMMA 9. If $\gcd(n, r-1) = d$, then $\beta(\Gamma^u(D_{2n})) = \frac{n}{d} - 2$.

COROLLARY 6. The following assertions (1)–(5) hold.

(1) If $(\Gamma(D_{2n}))$ is the commuting graph on D_{2n}, then $\beta(\Gamma(D_{2n})) = \frac{n}{d} + (n-1)d - (n+1)$.

(2) The element a is always an involution and b^i of D_{2n} is involution if $i = \frac{n}{2}$.

(3) The elements $b^{2i}a$, $0 \leq i \leq \frac{n}{2} - 1$ of D_{2n} are involutions if $r = 2k - 1$.

(4) The element $b^{\frac{n}{2}}a$ of D_{2n} is involution if $8 \mid n$ and $r = 2k + 1$.

(5) The elements $b^{\frac{ni}{4}}a$, $1 \leq i \leq 4$ of D_{2n} are involutions if $8 \nmid n$ and $r = 2k + 1$.

COROLLARY 7. The following assertions (1)–(3) hold.

(1) If Y_1 is the set of all involutions of D_{2n} for $r = 2k - 1$, then $\Gamma(Y_1) = F(\frac{n}{2} + 1, \frac{n}{4})$ is a fan graph on $\frac{n}{2} + 1$ vertices and $\frac{n}{4}$ number of triangles.

(2) If Y_2 is the set of all involutions of D_{2n} for $r = 2k+1$ and $8 \mid n$, then $\Gamma(Y_2) = K_3$.

(3) If Y_3 is the set of all involutions of D_{2n} for $r = 2k + 1$ and $8 \nmid n$, then $\Gamma(Y_3) = F(5, 2)$.

COROLLARY 8. The following assertions (1)–(5) hold.

(1) In $\Gamma(Y_1)$; $\deg(b^{\frac{n}{2}}) = \frac{n}{2}$ and $\deg(b^{2i}a) = 2$.

(2) In $\Gamma(Y_2)$; $\deg(b^{\frac{n}{2}}) = \deg(b^{\frac{n}{2}}a) = \deg(b^n a) = 2$.

(3) In $\Gamma(Y_3)$; $\deg(b^{\frac{n}{2}}) = 4$ and $\deg(b^{\frac{ni}{4}}) = 2$ for $1 \leq i \leq 4$.

(4) $w(\Gamma(Y_j)) = 3 = \psi(\Gamma(Y_j))$ for $j = 1, 2, 3$.

(5) $\operatorname{diam}(\Gamma(Y_1)) = 2 = \operatorname{diam}(\Gamma(Y_3))$ and $\operatorname{diam}(\Gamma(Y_2)) = 1$.

Acknowledgment. We would like to thank anonymous referees for their valuable comments and suggestions.

References

[1] M. Aschbacher, A condition for the existence of a strongly embedded subgroup, Proc. Amer. Math. Soc., 38(1973), 509–511.

[2] A. Abdollahi, S. Akbary and H. R. Maimani, Non-commuting graph of a group, J. Algebra, 28(2006), 468–492.

[3] S. Akbari, H. Bidkhori and A. Mohammadian, Commuting graphs of matrix algebras, Comm. Algebra, 36(2008), 4020–4031.

[4] S. Akbari, M. Ghandehari, M. Hadian and A. Mohammadian, On commuting graphs of semisimple rings, Linear Algebra Appl., 390(2004), 345–355.

[5] S. Akbari, A. Mohammadian, H. Radjavi and P. Raja, On the diameters of commuting graphs, Linear Algebra Appl., 418(2006), 161–176.

[6] D. F. Anderson and P. Livingston, The zero divisor graph of a commutative ring, J. Algebra, 217(1999), 434–447.

[7] C. Bates, D. Bundy, S. Perkins and P. Rowley, Commuting involution graphs for symmetric groups, J. Algebra, 266(2003), 133–153.

[8] C. Bates, D. Bundy, S. Perkins and P. Rowley, Commuting involution graphs for finite Coxeter groups, J. Group Theory, 6(2003), 461–476.

[9] C. Bates, D. Bundy, S. Perkins and P. Rowley, Commuting involution graphs in special linear groups, Comm. Algebra, 32(2004), 4179–4196.

[10] C. Bates, D. Bundy, S. Hart and P. Rowley, Commuting involution graphs for sporadic simple groups, J. Algebra, 316(2007), 849–868.

[11] T. T. Chelvam, L. Selvakumar and S. Raja, Commuting graphs on dihedral group, TJMCS, 2(2011), 402–406.

[12] A. Everett and P. Rowley, Commuting involution graphs for 4-dimensional projective symplectic groups, http://eprints.ma.man.ac.uk/1564/; (preprint).

[13] B. Fischer, Finite groups generated by 3-transpositions, University of Warwick Lecture Notes, 1969.

[14] A. Iranmanesh and A. Jafarzadeh, On the commuting graph associated with the symmetric and alternating groups, J. Alg. Appl., 7(2008), 129–146.

[15] Z. Raza and S. Faizi, Non-commuting graph of finitely presented group, Sci. Int.(Lahore), 25(2013), 883–885.

[16] Y. Segev, The commuting graph of minimal nonsolvable groups, Geometriae Dedicata, 88(2001), 55–66.

[17] Y. Segev and G. Seitz, Anisotropic groups of type A_n and the commuting graph of finite simple groups, Pacific J. Math., 202(2002), 125–225.