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On Mild Solutions Of Nonlocal Semilinear Impulsive Functional Integro-Differential Equations*

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Abstract

In the present paper, we investigate the existence, uniqueness and continuous dependence on initial data of mild solutions of first order nonlocal semilinear functional impulsive integro-differential equations of more general type with finite delay in Banach spaces. Our analysis is based on semigroup theory and Banach contraction theorem.

1 Introduction

Impulsive equations arise in many different real processes and phenomena which appears in physics, population dynamics, medicine, economics etc. The study of impulsive functional differential equations is linked to their utility in stimulating processes and phenomena subject to short time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. That is the reason for the perturbations to be considered as taking place instantaneously in the form of impulses. Impulsive differential equations in recent years have been the object of investigation with increasing interest. For more information see the monographs, Lakshmikantham [14], Samoilenko and Perestyuk [14], the research papers [5, 16] and the references cited therein.

Also, the problems of existence, uniqueness and other qualitative properties of solutions for semilinear differential equations in Banach spaces has been studied extensively in the literature for last many years, see [1, 3, 4, 6, 7, 9–13, 15, 17]. On the other hand, differential equations with nonlocal condition have been studied by many researchers as they are more precise to describe natural phenomena than differential equations with classical initial condition, for example see [1, 4, 6, 9, 10, 12] and the references cited therein.

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Recently, Li et. al [17] studied fractional integro-differential equations of the type:

$$D^q x(t) = Ax(t) + f(t, x(t) + \int_0^t k(t,s)h(t,s,x(s)ds),$$
$$x(0) = g(x) + x_0,$$

using measure of noncompactness and fixed point theorem of condensing maps. Also, Radhakrishnan and Bhalchandran [2] investigated controllability results for the impulsive integro-differential evolution system with nonlocal condition of the form:

$$x'(t) = A(t)x(t) + Bu(t) + f\left(t, x(t), \int_0^t a(t,s,x(s)ds\right),\ t \in J,\ t \neq t_i,$$
$$x(0) = g(x) + x_0,$$
$$\Delta x|_{t=t_i} = I_i x(t_i),\ i = 1, 2, .., m.$$

Ahmed et. al [8] obtained existence results of mild solutions of integro-differential equations:

$$u'(t) = Au(t)) + f\left(t, u(t), \int_0^t H(t,s)u(s)ds\right),\ t \in [0, K],$$
$$u(0) = g(u) + u_0,$$
$$\Delta u(t_i) = I_i u(t_i), i = 1, 2, .., m,$$

using measure of noncompactness and new fixed point theorem. In the present paper we consider semilinear functional impulsive integro-differential equation of first order of the type:

$$x'(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)h(s,x_s)ds\right),\ t \in (0, T],\ t \neq \tau_k,\ k = 1, 2, .., m, \quad (1)$$

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t),\ \ -r \leq t \leq 0, \quad (2)$$

$$\Delta x(\tau_k) = I_k x(\tau_k),\ k = 1, 2, ..., m, \quad (3)$$

where $0 < t_1 < t_2 < ... < t_p \leq T$, $p \in \mathbb{N}$, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ and $I_k (k = 1, 2, ..., m)$ are the linear operators acting in a Banach space X. The functions f, h, g, k and ϕ are given functions satisfying some assumptions. The impulsive moments τ_k are such that $0 \leq \tau_0 < \tau_1 < \tau_2 < ... < \tau_m < \tau_{m+1} \leq T$, $m \in \mathbb{N}$ and $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0)$ where $x(\tau_k + 0)$ and $x(\tau_k - 0)$ are, respectively, the right and the left limits of x at τ_k.

Equations of the form (1)-(3) or their special forms arise in some physical applications as a natural generalization of the classical initial value problems. Our aim of the present paper is to generalize and extend results reported in [5, 16]. The models in this paper are studied for impulse effect. We study the existence, uniqueness and continuous dependence of mild solution of nonlocal IVP problem for an impulsive functional integro-differential equation. The main tool used in our analysis is based on an application of the Banach contraction theorem and semigroup theory.

As usual, in the theory of impulsive differential equations at the points of discontinuity τ_i of the solution $t \to x(t)$, we assume that $x(\tau_i) \equiv x(\tau_i - 0)$. It is clear that, in general the derivatives $x'(\tau_i)$ do not exist. On the other hand, from (1), there exist the limits $x'(\tau_i \pm 0)$. According to the above convention, we assume $x'(\tau_i) = x'(\tau_i - 0)$.

This paper is organized as follows. Section 2 presents the preliminaries and hypotheses. In Section 3, we prove existence and uniqueness of mild solution. In section 4, we prove continuous dependence of solutions on initial data. Finally, in section 5, we give application based on our result .

2 Preliminaries and Hypotheses

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r, 0], X)$, $0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm $\|\psi\|_C = \sup\{\|\psi(t)\| : -r \le t \le 0\}$ and B denote the set $\{x : [-r, T] \to X : x(t)$ is continuous at $t \neq \tau_k$, left continuous at $t = \tau_k$, and the right limit $x(\tau_k + 0)$ exists for $k = 1, 2, ..., m\}$. Clearly, B is a Banach space with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : t \in [-r, T] \backslash \{\tau_1, \tau_2, ..., \tau_m\}\}$. For any $x \in B$ and $t \in [0, T] \backslash \{\tau_1, \tau_2, ..., \tau_m\}$, we denote x_t the element of C given by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$ and ϕ is a given element of C.

In this paper, we assume that, there exist positive constant $K \ge 1$ such that $\|T(t)\| \le K$ for every $t \in [0, T]$. Also assume $k : [0, T] \times [0, T] \to \mathbb{R}$ is continuous function and as the set $[0, T] \times [0, T]$ is compact, there exists a constant $L > 0$ such that $|k(t, s)| \le L$ for $0 \le s \le t \le T$.

DEFINITION 2.1. A function $x \in B$ satisfying the equations

$$\begin{aligned} x(t) &= T(t)\phi(0) - T(t)(g(x_{t_1}, ..., x_{t_p}))(0) \\ &\quad + \int_0^t T(t-s) f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, x_\tau) d\tau\right) ds \\ &\quad + \sum_{0<\tau_k<t} T(t-\tau_k) I_k x(\tau_k) \end{aligned}$$

for $t \in (0, T]$, and

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t) \text{ for } -r \le t \le 0,$$

is said to be the mild solution of the initial value problem (1)-(3).

REMARK. A mild solution of equations (1)-(3) satisfies (2) and (3). However, a mild solution may not be differentiable at zero.

The following inequality will be useful while proving our result.

LEMMA 2.2 ([15, p.12]). Let a nonnegative piecewise continuous function $u(t)$ satisfies the inequality

$$u(t) \leq D + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0<\tau_i<t} \beta_i u(\tau_i)$$

for $t \geq t_0$ where $D \geq 0$, $\beta_i \geq 0$, $v(t) > 0$ and τ_i are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq D \prod_{t_0<\tau_i<t} (1+\beta_i) \exp\left(\int_{t_o}^{t} v(s)ds\right).$$

We list the following hypotheses for our convenience.

(H_1) Let $f : [0,T] \times C \times X \to X$ such that for every $w \in B$, $x \in X$ and $t \in [0,T]$, $f(\cdot, w_t, x) \in B$ and there exists a constant $F > 0$ such that

$$\|f(t,\psi,x) - f(t,\phi,y)\| \leq F(\|\psi - \phi\|_C + \|x - y\|),\ \phi,\psi \in C,\ x,y \in X.$$

(H_2) Let $h : [0,T] \times C \to X$ such that for every $w \in B$ and $t \in [0,T]$, $h(\cdot, w_t) \in B$ and there exists a constant $H > 0$ such that

$$\|h(t,\psi) - h(t,\phi)\| \leq H\|\psi - \phi\|_C,\ \phi,\psi \in C.$$

(H_3) Let $g : C^p \to C$ such that there exists a constant $G \geq 0$ satisfying

$$\|(g(x_{t_1}, x_{t_2}, ..., x_{t_p}))(t) - (g(y_{t_1}, y_{t_2}, ..., y_{t_p}))(t)\| \leq G\|x - y\|_B,\ t \in [-r, 0].$$

(H_4) Let $I_k : X \to X$ are functions such that there exists constants L_k satisfying

$$\|I_k(v)\| \leq L_k\|v\|,\ v \in X,\ k = 1, 2, ..., m.$$

3 Existence and Uniqueness

We have the following

THEOREM 3.1. Suppose that the hypotheses (H_1)–(H_4) are satisfied and $\Gamma < 1$ where

$$\Gamma = KG + KF[1 + LHT]T + K \sum_{0<\tau_k<t} L_k.$$

Then the initial-value problem (1)-(3) has a unique mild solution x on $[-r, T]$.

PROOF. We introduce an operator $\mathcal{F}$ on a Banach space B as follows

$$(\mathcal{F}x)(t) = \begin{cases} \phi(t) - (g(x_{t_1}, ..., x_{t_p}))(t) \text{ if } -r \le t \le 0, \\ T(t)[\phi(0) - g(x_{t_1}, ..., x_{t_p})(0)] + \int_0^t T(t-s) f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, s_\tau) d\tau\right) ds \\ + \sum_{0<\tau_k<t} T(t-\tau_k) I_k x(\tau_k) \text{ if } t \in (0, T]. \end{cases}$$

It is easy to see that $\mathcal{F} : B \to B$. Now we will show that $\mathcal{F}$ is a contraction on B. Let $x, y \in B$. Then

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| = \|g(x_{t_1}, ..., x_{t_p})(t) - g(y_{t_1}, ..., y_{t_p})(t)\| \le G\|x - y\|_B \tag{4}$$

for $t \in [-r, 0]$ and

$$\begin{aligned} \|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| &\le \Big\|T(t)\left[g(x_{t_1}, ..., x_{t_p})(0) - g(y_{t_1}, ..., y_{t_p})(0)\right] \\ &+ \int_0^t \left[T(t-s) f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, x_\tau) d\tau\right)\right. \\ &\left. - f\left(s, y_s, \int_0^s k(s,\tau) h(\tau, y_\tau) d\tau\right)\right] ds \\ &+ \sum_{0<\tau_k<t} T(t-\tau_k)[I_k x(\tau_k) - I_k y(\tau_k)]\Big\| \\ &\le \|T(t)\| \|g(x_{t_1}, ..., x_{t_p})(0) - g(y_{t_1}, ..., y_{t_p})(0)\| \\ &+ \int_0^t \|T(t-s)\| \left\| f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, x_\tau) d\tau\right)\right. \\ &\left. - f\left(s, y_s, \int_0^s k(s,\tau) h(\tau, y_\tau) d\tau\right)\right\| ds \\ &+ \sum_{0<\tau_k<t} \|T(t-\tau_k)\| \|[I_k x(\tau_k) - I_k y(\tau_k)]\| \\ &\le KG\|x - y\|_B + J_1 + J_2 \end{aligned} \tag{5}$$

for $t \in (0, T]$ where

$$\begin{aligned} J_1 &= \int_0^t \|T(t-s)\| \| \left\| f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, x_\tau) d\tau\right)\right. \\ &\left. - f\left(s, y_s, \int_0^s k(s,\tau) h(\tau, y_\tau) d\tau\right)\right\| ds \\ &\le K \int_0^t F[\|x_s - y_s\|_C + \int_0^s |k(s,\tau)| \|h(\tau, x_\tau) - h(\tau, y_\tau)\| d\tau] ds \\ &\le KF \int_0^t [\|x_s - y_s\|_C + L \int_0^s H\|x_\tau - y_\tau\|_C d\tau] ds \\ &\le KF \int_0^t [\|x - y\|_B + LHT\|x - y\|_B] ds \\ &\le KF \int_0^t [1 + LHT]\|x - y\|_B ds \\ &\le KF[1 + LHT]T\|x - y\|_B \end{aligned} \tag{6}$$

and

$$
\begin{aligned}
J_2 &= \sum_{0<\tau_k<t} \|T(t-\tau_k)\| \|I_k x(\tau_k) - I_k y(\tau_k)\| \\
&\leq \sum_{0<\tau_k<t} K \|I_k x(\tau_k) - I_k y(\tau_k)\| \\
&\leq K \sum_{0<\tau_k<t} L_k \|x(\tau_k) - y(\tau_k)\| \\
&\leq K \sum_{0<\tau_k<t} L_k \|x - y\|_B. \qquad (7)
\end{aligned}
$$

Using (6)-(7), inequality (5) becomes

$$
\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \leq KG\|x-y\|_B + \left(KF[1+LHT]T + K \sum_{0<\tau_k<t} L_k \right) \|x-y\|_B \quad (8)
$$

for $t \in [0, T]$. As $K \geq 1$, in view of inequality (4) and (8), we can say that inequality (8) holds good for $t \in [-r, T]$. Therefore, for $t \in [-r, T]$,

$$
\begin{aligned}
\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| &\leq KG\|x-y\|_B + \left(KF[1+LHT]T + K \sum_{0<\tau_k<t} L_k \right) \|x-y\|_B \\
&\leq \left(KG + KF[1+LHT]T + K \sum_{0<\tau_k<t} L_k \right) \|x-y\|_B,
\end{aligned}
$$

which implies

$$
\|\mathcal{F}x - \mathcal{F}y\|_B \leq \Gamma \|x-y\|_B
$$

where

$$
\Gamma = KG + KF[1+LHT]T + K \sum_{0<\tau_k<t} L_k.
$$

Since $\Gamma < 1$, the operator $\mathcal{F}$ satisfies all the assumptions of Banach contraction theorem and therefore $\mathcal{F}$ has unique fixed point in the space B and clearly it is the mild solution of nonlocal IVP problem (1)-(3) with impulse effect. This completes the proof.

4 Continuous Dependence on Initial Data

We have the following

THEOREM 4.1. Suppose that hypotheses (H$_1$)-(H$_4$) are satisfied and $\Gamma < 1$. Then for each ϕ_1, $\phi_2 \in C$ and for the corresponding mild solutions x_1 and x_2 of the problems

$$
x'(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)h(s,x_s)ds\right), \ t \in (0, T], \quad (9)
$$

$$
\Delta x(\tau_k) = I_k x(\tau_k), \ k = 1, 2, ..., m, \quad (10)
$$

$$
x(t) + g(x_{t_1}, ..., x_{t_p})(t) = \phi_i(t), \ i = 1, 2, \ t \in [-r, 0], \quad (11)
$$

the following inequality holds

$$\|x_1 - x_2\|_B \leq \frac{K \prod\limits_{0<\tau_k<t}(1+KL_k)\exp(KFT)}{\left(1 - \Lambda \prod\limits_{0<\tau_k<t}(1+KL_k)\exp(KFT)\right)} \times \|\phi_1 - \phi_2\|_C \tag{12}$$

where

$$\Lambda = GK + KFLHT^2.$$

PROOF. Let $\phi_1, \phi_2 \in B$ be arbitrary functions and let x_1 and x_2 be the mild solutions of the problem(9)-(11). Then we have

$$\begin{aligned} x_1(t) - x_2(t) = {} & T(t)[\phi_1(0) - \phi_2(0)] - T(t)\left[g(x_{1_{t_1}}, ..., x_{1_{t_p}})(0) - g(x_{2_{t_1}}, ..., x_{2_{t_p}})(0)\right] \\ & + \int_0^t T(t-s)\left[f\left(s, x_{1s}, \int_0^s k(s,\tau)h(s, x_{1s})d\tau\right)\right. \\ & \left. - f\left(s, x_{2_s}, \int_0^s k(s,\tau)h(s, x_{2_s})d\tau\right)\right] ds \\ & + \sum_{0<\tau_k<t} T(t-\tau_k)(I_k x_1(\tau_k) - I_k x_2(\tau_k)) \end{aligned} \tag{13}$$

for $t \in (0, T]$ and

$$x_1(t) - x_2(t) = \phi_1(t) - \phi_2(t) - [g(x_{1_{t_1}}, ..., x_{1_{t_p}})(t) - g(x_{2_{t_1}}, ..., x_{2_{t_p}})(t)] \tag{14}$$

for $t \in [-r, 0]$. From (13) and using hypothesis (H_1)-(H_4), we get

$$\begin{aligned} \|x_1(t) - x_2(t)\| \leq {} & K\|\phi_1 - \phi_2\|_C + GK\|x_1 - x_2\|_B + K\int_0^t F\left[\|x_{1s} - x_{2s}\|_C\right. \\ & \left. + HL\int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\right] ds + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\| \\ \leq {} & K\|\phi_1 - \phi_2\|_C + GK\|x_1 - x_2\|_B + K\int_0^t F\|x_{1s} - x_{2s}\|_C ds \\ & + FKLHT^2\|x_1 - x_2\|_B + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\| \\ \leq {} & K\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + KF\int_0^t \|x_{1s} - x_{2s}\|_C ds \\ & + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|. \end{aligned} \tag{15}$$

Simultaneously, by (14) and hypothesis (H_3),we get

$$\|x_1(t) - x_2(t)\| \leq \|\phi_1 - \phi_2\|_C + G\|x_1 - x_2\|_B, \; t \in [-r, 0]. \tag{16}$$

Since $K \geq 1$, the inequalities (15) and(16) imply, for $t \in [-r, T]$

$$\|x_1(t) - x_2(t)\| \leq K\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + KF\int_0^t \|x_{1s} - x_{2s}\|_C ds$$
$$+ K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0 < \tau_k \leq t, \quad t \in [0, T]. \tag{17}$$

Define the function $z : [-r, T] \to \mathbb{R}$ by $z(t) = \sup\{\|x_1(s) - x_2(s)\| : -r \leq s \leq t\}, t \in [0, T]$. Let $t^* \in [-r, t]$ be such that $z(t) = \|x_1(t^*) - x_2(t^*)\|$. If $t^* \in [0, t]$, then from inequality (17), we have

$$\begin{aligned} z(t) &= \|x_1(t^*) - x_2(t^*)\| \\ &\leq K\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + KF\int_0^{t^*} \|x_{1s} - x_{2s}\|_C ds \\ &+ K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\| \end{aligned}$$

for $0 < \tau_k \leq t^*$ and

$$z(t) \leq K\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + KF\int_0^t z(s)ds + K\sum_{0<\tau_k<t} L_k z(\tau_k) \tag{18}$$

Now applying Lemma 2.2 to the inequality (18), we get

$$z(t) \leq (K\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B)\prod_{0<\tau_k<t}(1 + KL_k)\exp(KFT).$$

Hence, we get

$$\|x_1 - x_2\|_B \leq (K\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B)\prod_{0<\tau_k<t}(1 + KL_k)\exp(KFT).$$

The inequality given by (12) is the immediate consequence of the above inequality. This completes the proof.

5 Application

To illustrate the application of our result proved in section 3, consider the following semilinear partial functional differential equation of the form

$$\frac{\partial}{\partial t}w(u, t) = \frac{\partial^2}{\partial u^2}w(u, t) + H\left(t, w(u, t - r), \int_0^t k(t, s)P(s, w(s - r))ds\right), \tag{19}$$

$$w(0, t) = w(\pi, t) = 0, \ 0 \leq t \leq T, \tag{20}$$

$$w(u, \zeta) + \sum_{i=1}^{p} w(u, t_i + \zeta) = \phi(u, t), \tag{21}$$

$$\Delta w(u, \tau_k) = I_k(w(u, \tau_k)), \ k = 1, 2, ..., m, \tag{22}$$

for $0 \le u \le \pi$, $0 \le t \le T$ and $-r \le \zeta \le 0$ where $0 < t_1 \le t_2 \le t_p \le T$, the functions $H : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $P : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $I_k : \mathbb{R} \to \mathbb{R}$ are continuous . We assume that the functions H, P and I_k satisfy the following conditions: For every $t \in [0,T]$ and $u, v, x, y \in \mathbb{R}$, there exists positive constants l, p, c_k and d such that

$$|H(t,u,x) - H(t,v,y)| \le l(|u-v| + |x-y|),$$

$$|P(t,u) - P(t,v)| \le p(|u-v|),$$
$$|I_k(x)| \le c_k\,|x|\,,\;\; k = 1,2,...,m,$$
$$\sum_{i=1}^{p} |w(u,t_i+t)| \le d,$$

and

$$Kd + Kl(1+LpT)T + K \sum_{0<\tau_k<t} c_k < 1.$$

Let us take $X = L^2[0,\pi]$. Define the operator $A : X \to X$ by $Az = z^{''}$ with domain $D(A) = \{z \in X : z,\, z^{'}$ are absolutely continuous, $z^{''} \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator A can be written as

$$Az = \sum_{n=1}^{\infty} -n^2(z,z_n)z_n,\;\; z \in D(A)$$

where $z_n(u) = (\sqrt{2/\pi})\sin nu$, $n = 1,2,...$ is the orthogonal set of eigenvectors of A and A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \ge 0$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} \exp(-n^2 t)(z,z_n)z_n,\;\; z \in X.$$

Now, the analytic semigroup $T(t)$ being compact, there exists constant K such that

$$|T(t)| \le K,\;\text{ for each } t \in [0,T].$$

Define the functions $f : [0,T] \times C \times X \to X$, $h : [0,T] \times C \to X$, $I_k : X \to X$ as follows

$$f(t,\psi,x)(u) = H(t,\psi(-r)u, x(u)),$$
$$h(t,\phi)(u) = P(t,\phi(-r)u),$$

for $t \in [0,T]$, $\psi, \phi \in C$, $x \in X$ and $0 \le u \le \pi$. With these choices of the functions the equations (19)-(22) can be formulated as an abstract integro-differential equation in Banach space X:

$$x'(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)h(s,x_s)ds\right),\;\; t \in [0,T],$$

$$x(t) + (g(x_{t_1},...,x_{t_p}))(t) = \phi(t),\;\; t \in [-r,0].$$

Since all the hypotheses of the theorem 3.1 are satisfied, the theorem 3.1, can be applied to guarantee the existence of mild solution $w(u,t) = x(t)u$, $t \in [0,T]$ and $u \in [0,\pi]$, of the semilinear partial integro-differential equation (19)-(22).

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