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On Coefficient Determinants With Fekete-Szegö Parameter*

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Abstract

In this paper we introduce Hankel determinants involving the Fekete-Szegö parameter and other similar determinants for the coefficients of analytic functions in the open unit disk. We investigate bounds on such determinants for the class of functions of bounded turning.

1 Introduction

The Fekete-Szegö functional $|a_3 - \lambda a_2^2|$ for normalized univalent functions

$$f(z) = z + a_2 z^2 + \cdots$$

is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1932 conjecture of Littlewood and Parley that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in many subclasses of the family of univalent functions.

For integers $n \geq 1$ and $q \geq 1$, the q-th Hankel determinant, defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix},$$

which include the Fekete-Szegö functional as a special case ($\lambda = 1$), has also received the attention of many researchers for wide range of subclasses of functions. One natural consequence of the continuous investigations is that, for function classes defined by other function classes (for example the classes of close-to-star, close-to-convex, quasi-convex, α-starlike, α-convex, α-close-to-star, α-close-to-convex whose definitions involve other function classes), coefficient functionals of the form $|a_2 a_3 - \lambda a_4|$ and

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$|a_2a_4 - \lambda a_3^3|$ (and possibly more) for the defining function classes have frequently appeared to be resolved in the investigations of Hankel determinants for the desired classes of functions.

In this work, therefore, we are motivated by such emerging functionals to define what we call the Hankel determinants with Fekete-Szegö parameter as follows:

DEFINITION 1. Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the q-th Hankel determinants with Fekete-Szegö parameter λ, that is $H_q^\lambda(n)$, as

$$H_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \lambda a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

DEFINITION 2. Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the $B_q^\lambda(n)$ determinants as

$$B_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+q} & a_{n+q+1} & \cdots & a_{n+2q-1} \\ a_{n+2q} & a_{n+2q+1} & \cdots & a_{n+3q-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q(q-1)} & \cdots & \cdots & \lambda a_{n+q^2-1} \end{vmatrix}.$$

The investigation of the determinants $H_q^\lambda(n)$ and $B_q^\lambda(n)$ is of interest for many classes of functions. We investigate in this paper, the determinants

$$H_2^\lambda(2) = \begin{vmatrix} a_2 & \lambda a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - \lambda a_3^2|$$

and

$$B_2^\lambda(1) = \begin{vmatrix} 1 & a_2 \\ a_3 & \lambda a_4 \end{vmatrix} = |a_2a_3 - \lambda a_4|$$

for the class of functions whose derivatives have positive real parts, known as functions of bounded turning. Those are functions satisfying $\operatorname{Re} f'(z) > 0$ in the open unit disk, and are denoted by R.

In the next section we state the lemmas we shall use to establish the desired bounds in Section 3.

2 Preliminary Lemmas

Let P denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are regular in E and satisfy $\operatorname{Re} p(z) > 0$, $z \in E$. To prove the main results in the next section we shall require the following two lemmas.

LEMMA 1 ([2]). Let $p \in P$. Then $|c_k| \leq 2$, $k = 1, 2, ...$, and the inequality is sharp. Equality is realized by the Möbius function $L_0(z) = (1+z)/(1-z)$.

LEMMA 2 ([4,5]). Let $p \in P$. Then

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1}$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2 c_1(4 - c_1^2) + 2z(1 - |x|^2)(4 - c_1^2) \tag{2}$$

for some x, z such that $|x| \leq 1$ and $|z| \leq 1$.

3 Main Result

Our first result is the following.

THEOREM 1. Let $f \in R$. Then

$$\left|a_2 a_4 - \lambda a_3^2\right| \leq \begin{cases} \frac{1}{2} & \text{if} \quad \lambda = 0, \\ \frac{729 - 1152\lambda + 512\lambda^2}{144(9 - 8\lambda)} & \text{if} \quad 0 < \lambda \leq \frac{27}{32}, \\ \frac{4}{9}\lambda & \text{if} \quad \lambda \geq \frac{27}{32}. \end{cases}$$

The inequalities are sharp. For each λ, equality is attained by $f(z)$ given by

$$f(z) = \begin{cases} z + z^2 + \frac{2}{3}z^3 + \frac{1}{2}z^4 + \cdots & \text{if} \quad \lambda = 0, \\ z + \frac{1}{2}\sqrt{\frac{27-32\lambda}{9-8\lambda}}\, z^2 + \frac{2}{3}z^3 + \frac{81 - 64\lambda}{8\sqrt{(9-8\lambda)(27-32\lambda)}} z^4 + \cdots & \text{if} \quad 0 < \lambda \leq \frac{3}{4}, \\ z + \frac{1}{2}\sqrt{\frac{27-32\lambda}{9-8\lambda}}\, z^2 + \frac{2}{3}z^3 + \frac{32\lambda - 27}{18}\sqrt{\frac{27-32\lambda}{9-8\lambda}}\, z^4 + \cdots & \text{if} \quad \frac{3}{4} < \lambda \leq \frac{27}{32}, \\ z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \cdots & \text{if} \quad \lambda \geq \frac{27}{32}. \end{cases}$$

PROOF. Let $f \in R$. Then there exists a $p \in P$ such that $f'(z) = p(z)$. Equating coefficients of $f'(z)$ and $p(z)$, we find that $2a_2 = c_1$, $3a_3 = c_2$ and $4a_4 = c_3$. Thus we have

$$\left|a_2 a_4 - \lambda a_3^2\right| = \left|\frac{c_1 c_3}{8} - \lambda \frac{c_2^2}{9}\right|. \tag{3}$$

The trivial case $\lambda = 0$ is a consequence of Lemma 1. In this case the extremal is achieved by choosing $c_1 = c_2 = c_3 = 2$.

Now substituting for c_2 and c_3 using Lemma 2, we obtain

$$\begin{aligned}\left|a_2a_4-\lambda a_3^2\right| = \Bigg| &\frac{(9-8\lambda)c_1^4}{288}+\frac{(9-8\lambda)c_1^2(4-c_1^2)x}{144}-\frac{\lambda(4-c_1^2)^2x^2}{36}\\ &-\frac{c_1^2(4-c_1^2)x^2}{32}+\frac{c_1(4-c_1^2)(1-|x|^2)z}{16}\Bigg|. \qquad (4)\end{aligned}$$

By Lemma 1, $|c_1|\le 2$. Then letting $c_1=c$, we may assume without restriction that $c\in[0,2]$.

Now suppose $9-8\lambda$ is nonnegative. Then applying the triangle inequality on (4), with $\rho=|x|$, we obtain

$$\begin{aligned}\left|a_2a_4-\lambda a_3^2\right| &\le \frac{(9-8\lambda)c^4}{288}+\frac{c^2(4-c^2)}{16}+\frac{(9-8\lambda)c^2(4-c^2)\rho}{144}\\ &\quad+\frac{[(9-8\lambda)c^2-18c+32\lambda](4-c^2)\rho^2}{288}\\ &= F(\rho).\end{aligned}$$

Now we have

$$F'(\rho)=\frac{(9-8\lambda)c^2(4-c^2)}{144}+\frac{[(9-8\lambda)c^2-18c+32\lambda](4-c^2)\rho}{144}.$$

Observe that $F'(\rho)\ge F'(1)>0$ since the first term is nonnegative, so that

$$F'(\rho)\ge\frac{(4-c^2)[2(9-8\lambda)c^2-18c+32\lambda]}{144}\ge\frac{(4-c^2)(2-c)}{8}>0.$$

Thus $F(\rho)$ is increasing on $[0,1]$ so that $F(\rho)\le F(1)$. That is

$$F(\rho)\le\frac{4\lambda}{9}+\frac{(27-32\lambda)c^2}{72}-\frac{(9-8\lambda)c^4}{144}=G(c).$$

It is easy to see that, if $27-32\lambda$ is negative, that is $\lambda\ge 27/32$, then $G(c)$ is decreasing on $[0,2]$ so that $G(c)\le G(0)=4\lambda/9$. Otherwise (that is $\lambda\le 27/32$), we have

$$G'(c)=\frac{(27-32\lambda)c}{36}-\frac{(9-8\lambda)c^3}{36}.$$

Then the maximum of $G(c)$ on $[0,2]$ occurs at $c=\sqrt{(27-32\lambda)/(9-8\lambda)}$ and is given by

$$G\left(\sqrt{\frac{27-32\lambda}{9-8\lambda}}\right)=\frac{729-1152\lambda+512\lambda^2}{144(9-8\lambda)}.$$

By setting $c_1=c=\sqrt{(27-32\lambda)/(9-8\lambda)}$ and selecting $x=1$ in (1) we find that $c_2=2$. Then the admissible c_3 is given by $c_3=(81-64\lambda)/[2\sqrt{(9-8\lambda)(27-32\lambda)}]$ if $0<\lambda\le 3/4$ and for $3/4\le\lambda\le 27/32$ we find the admissible $c_3=[(32\lambda-$

$27)\sqrt{(27-32\lambda)/(9-8\lambda)}]/18$. Thus equality in these cases is attained by the function defined in the theorem.

Next we consider that $9-8\lambda$ is negative. Then we write (4) as

$$\begin{aligned} \left|a_2a_4-\lambda a_3^2\right| &= \left|\frac{(8\lambda-9)c_1^4}{288}+\frac{(8\lambda-9)c_1^2(4-c_1^2)x}{144}+\frac{\lambda(4-c_1^2)^2x^2}{36}\right. \\ &\quad \left.+\frac{c_1^2(4-c_1^2)x^2}{32}-\frac{c_1(4-c_1^2)(1-|x|^2)z}{16}\right|. \end{aligned}$$

Following the same argument as above, we have

$$\begin{aligned} \left|a_2a_4-\lambda a_3^2\right| &\leq \frac{(8\lambda-9)c^4}{288}+\frac{c^2(4-c^2)}{16}+\frac{(8\lambda-9)c^2(4-c^2)\rho}{144} \\ &\quad +\frac{[32\lambda-18c-(8\lambda-9)c^2](4-c^2)\rho^2}{288} \\ &= F(\rho). \end{aligned}$$

Furthermore,

$$F(\rho)\leq F(1)=\frac{4\lambda}{9}-\frac{c^2}{8}=G(c)$$

and $G(c)$ is decreasing on $[0,2]$ so that $G(c)\leq G(0)=4\lambda/9$.

In this case we set $c_1=0$ and selecting $x=1$ in (1) and (2), we find that $c_2=2$ and $c_3=0$ so that equality is attained by $f(z)$ defined in the theorem and the proof is complete.

COROLLARY 1 ([3]). Let $f\in R$. Then

$$\left|a_2a_4-a_3^2\right|\leq\frac{4}{9}.$$

The inequality is sharp. Equality is attained by

$$f(z)=\int_0^z\frac{1+t^2}{1-t^2}dt=z+\frac{2}{3}z^3+\frac{2}{5}z^5+\cdots.$$

Next we have

THEOREM 2. Let $f\in R$. Then

$$|a_2a_3-\lambda a_4|\leq\begin{cases}\frac{4-3\lambda}{6} & \text{if } 0\leq\lambda\leq\frac{2}{3}, \\ \frac{9\lambda-4}{18}\sqrt{\frac{9\lambda-4}{3(3\lambda-2)}} & \text{if } \frac{2}{3}<\lambda\leq\frac{4}{3}, \\ \frac{9\lambda-4}{54}\sqrt{\frac{2(9\lambda-4)}{\lambda}} & \text{if } \lambda\geq\frac{4}{3}.\end{cases}$$

The inequalities are sharp. For each λ, equality is attained by $f(z)$ given by

$$f(z) = \begin{cases} z + z^2 + \frac{2}{3}z^3 + \frac{1}{2}z^4 + \cdots & \text{if} \quad 0 \le \lambda \le \frac{2}{3}, \\ z + \frac{1}{2}\sqrt{\frac{9\lambda-4}{9\lambda-6}}z^2 + \frac{9\lambda-8}{18}z^3 - \frac{1}{4}\sqrt{\frac{9\lambda-4}{9\lambda-6}}z^4 + \cdots & \text{if} \quad \frac{2}{3} < \lambda \le \frac{4}{3}, \\ z + \frac{1}{2}\sqrt{\frac{2(9\lambda-4)}{9\lambda}}z^2 + \frac{9\lambda-8}{18}z^3 - \frac{1}{4}\sqrt{\frac{2(9\lambda-4)}{9\lambda}}z^4 + \cdots & \text{if} \quad \lambda \ge \frac{4}{3}. \end{cases}$$

PROOF. As is in the proof of Theorem 1, if $f \in R$, then $2a_2 = c_1$, $3a_3 = c_2$ and $4a_4 = c_3$. Thus we have

$$|a_2a_3 - \lambda a_4| = \left| \frac{c_1c_2}{6} - \lambda\frac{c_3}{4} \right|. \tag{5}$$

Substituting for c_2 and c_3 using Lemma 2, we obtain

$$\begin{aligned} |a_2a_3 - \lambda a_4| = \bigg| &\frac{(4-3\lambda)c_1^3}{48} + \frac{(4-6\lambda)c_1(4-c_1^2)x}{48} + \frac{\lambda c_1(4-c_1^2)x^2}{16} \\ &- \frac{\lambda(4-c_1^2)(1-|x|^2)z}{8} \bigg|. \end{aligned} \tag{6}$$

By Lemma 1, $|c_1| \le 2$. Then letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$.

Now suppose $4 - 6\lambda$ is nonnegative. Then applying the triangle inequality on (6), with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - \lambda a_4| &\le \frac{(4-3\lambda)c^3}{48} + \frac{\lambda(4-c^2)}{8} + \frac{(4-6\lambda)c(4-c^2)\rho}{48} + \frac{\lambda(c-2)(4-c^2)\rho^2}{16} \\ &= F(\rho). \end{aligned}$$

The extreme values of $F(\rho)$ are at $\rho = 0$, $\rho = 1$ and ρ such that

$$F'(\rho) = \frac{(4-6\lambda)c(4-c^2)}{48} + \frac{\lambda(c-2)(4-c^2)\rho}{8} = 0.$$

Now let

$$G_1(c) = F(0) = \frac{(4-3\lambda)c^3}{48} + \frac{\lambda(4-c^2)}{8},$$

$$G_2(c) = F(1) = \frac{(4-3\lambda)c}{12}$$

and

$$G_3(c) = F\left(\frac{(2-3\lambda)c}{3\lambda(2-c)}\right) = \frac{(4-3\lambda)c^3}{48} + \frac{\lambda(4-c^2)}{8} + \frac{c^2(c+2)(2-3\lambda)^2}{144}.$$

By elementary calculus, we find that $G_1(c) \le G_1(0) = \lambda/2$, $G_2(c) \le G_2(2) = (4-3\lambda)/6$ and $G_3(c) \le G_3(0) = \lambda/2$ for all admissible c. Hence $G(c) \le G_2(2) = (4-3\lambda)/6$.

By selecting $c_1 = c = 2$ and selecting $x = 1$ in (1) and (2) we find that $c_2 = c_3 = 2$. Thus the extremal function for this case $0 \le \lambda \le 2/3$ is the function given in the theorem.

Next we consider the case: $4 - 6\lambda$ is negative while $4 - 3\lambda$ is nonnegative, that is $2/3 < \lambda \le 4/3$. In this case we write (6) as

$$|a_2a_3 - \lambda a_4| = \left| \frac{(4-3\lambda)c_1^3}{48} - \frac{(6\lambda-4)c_1(4-c_1^2)x}{48} + \frac{\lambda c_1(4-c_1^2)x^2}{16} \right.$$
$$\left. - \frac{\lambda(4-c_1^2)(1-|x|^2)z}{8} \right|. \tag{7}$$

so that as in the first part with $c_1 = c \in [0,2]$ and $\rho = |x|$ we have

$$\begin{aligned} |a_2a_3 - \lambda a_4| &\le \frac{(4-3\lambda)c^3}{48} + \frac{\lambda(4-c^2)}{8} + \frac{(6\lambda-4)c(4-c^2)\rho}{48} + \frac{\lambda(c-2)(4-c^2)\rho^2}{16} \\ &= F(\rho). \end{aligned}$$

Using the same extreme value technique, we find that the extreme value of $F(\rho)$ yielding the best possible bound for functional is

$$G(c) = F(1) = \frac{(2-3\lambda)c^3}{12} + \frac{(9\lambda-4)c}{12}$$

and that

$$G'(c) = \frac{(2-3\lambda)c^2}{4} + \frac{(9\lambda-4)}{12}.$$

Thus the maximum of $G(c)$ on $[0,2]$ occurs at $c = \sqrt{(9\lambda-4)/(9\lambda-6)}$ and is given by

$$G\left(\sqrt{\frac{9\lambda-4}{3(3\lambda-2)}}\right) = \frac{9\lambda-4}{18}\sqrt{\frac{9\lambda-4}{3(3\lambda-2)}}.$$

By setting $c_1 = c = \sqrt{(9\lambda-4)/(9\lambda-6)}$ and selecting $x = 1$ in (2) we find that $c_3 = c_1$. The choice $c_3 = -c_1$ is also appropriate as the context so admits. Then the admissible c_2 is given by $c_2 = (9\lambda-8)/6$. Thus for $2/3 < \lambda \le 4/3$, the given function in the theorem attains the equality.

Finally we suppose $4 - 3\lambda$ is negative, that is $\lambda \ge 4/3$. Then we write (6) as

$$|a_2a_3 - \lambda a_4| = \left| \frac{(3\lambda-4)c_1^3}{48} + \frac{(6\lambda-4)c_1(4-c_1^2)x}{48} - \frac{\lambda c_1(4-c_1^2)x^2}{16} \right.$$
$$\left. + \frac{\lambda(4-c_1^2)(1-|x|^2)z}{8} \right|. \tag{8}$$

As we have shown above, with $c_1 = c \in [0,2]$ and $\rho = |x|$, we have

$$\begin{aligned} |a_2a_3 - \lambda a_4| &\le \frac{(3\lambda-4)c^3}{48} + \frac{\lambda(4-c^2)}{8} + \frac{(6\lambda-4)c(4-c^2)\rho}{48} + \frac{\lambda(c-2)(4-c^2)\rho^2}{16} \\ &= F(\rho). \end{aligned}$$

Using the same extreme value technique, we find that

$$G(c) = F(1) = \frac{(9\lambda - 4)c}{12} - \frac{\lambda c^3}{8}$$

and

$$G'(c) = \frac{(9\lambda - 4)}{12} - \frac{3\lambda c^2}{8}.$$

Thus the maximum of $G(c)$ on $[0, 2]$ occurs at $c = \sqrt{2(9\lambda - 4)/9\lambda}$ and is given by

$$G\left(\sqrt{\frac{2(9\lambda - 4)}{9\lambda}}\right) = \frac{9\lambda - 4}{54}\sqrt{\frac{2(9\lambda - 4)}{\lambda}}.$$

Finally setting $c_1 = c = \sqrt{2(9\lambda - 4)/(9\lambda)}$ and selecting $x = 1$ in (2) we find that $c_3 = c_1$. Again the choice $c_3 = -c_1$ is also appropriate as the context so admits. Thus the admissible c_2 is given by $c_2 = (9\lambda - 8)/6$. Thus for $2/3 < \lambda \leq 4/3$, the given function in the theorem is admitted in the equality and the proof is now complete.

COROLLARY 2 ([1]). Let $f \in R$. Then

$$|a_2 a_3 - a_4| \leq \frac{5}{18}\sqrt{\frac{5}{3}}.$$

The inequality is sharp. Equality is attained by

$$f(z) = z + \frac{1}{2}\sqrt{\frac{5}{3}}z^2 + \frac{1}{18}z^3 - \frac{1}{4}\sqrt{\frac{5}{3}}z^4 + \cdots .$$

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