Applied Mathematics E-Notes, 13(2013), 141-147 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

E-Valued Meromorphic Functions With Maximal Deficiency Sum*

Zhaojun Wu†, Yuxian Chen‡

Received 08 April 2013

Abstract

The purpose of this paper is to discuss the relationship between the characteristic function of an E-valued meromorphic function f and and that of its derivative f'. Consequently, for a finite order E-valued meromorphic function f with maximal deficiency sum, we obtain that $T(r, f') \sim (2 - \delta(\widehat{\infty}, f))T(r, f)$ as $r \to +\infty$. Results are obtained to extend the related results for meromorphic scalar valued function of Weitsman Allen in [Weitsman, Allen. Meromorphic functions with maximal deficiency sum and a conjecture of F. Nevanlinna. Acta Math. 123(1969), 115–139].

1 Introduction of E-Valued Meromorphic Function

In 2006, Hu and Hu [4] introduced the fundamental notations and established the Nevanlinna's theorems for an E-valued meromorphic function from the complex plane $\mathbb{C}$ to an infinite-dimensional Banach spaces E with a Schauder basis. In 2010, Xuan and Wu [11] established the Nevanlinna's theorems for an E-valued meromorphic function from a generic domain $D \subseteq \mathbb{C}$ to E. In 2011, Hu [3] investigated the application of the Nevanlinna theory of E-valued meromorphic functions in infinite-dimensional spaces. Therefore, any results of the Nevanlinna theory of E-valued meromorphic functions has potential application. In this paper, we shall continue to study the Nevanlinna theory of E-valued meromorphic functions.

In 2012, Wu and Xuan [9] investigated the characteristic functions and Borel exceptional values of E-valued meromorphic functions, Wu and Xuan [10] discussed the deficiency of E-valued meromorphic functions. In this paper, we shall discuss the relation between the characteristic function and the deficiency of E-valued meromorphic functions. In fact, we shall investigate the relationship between the characteristic function of an E-valued meromorphic function f and that of its derivative f' when f has maximal deficiency sum.

The structure of this paper is as follows. In Section 1, we introduce the basic notations and fundamental results of E-valued meromorphic function, see [1, 3, 4, 9, 10, 11]. In Section 2, we establish the main results of this paper and give their proof.

*Mathematics Subject Classifications: 30D35, 30G30, 46G20.

†School of Mathematics and Statistics, Hubei University of Science and Technology, Xianning, Hubei, 437100, P. R. China

‡School of Mathematics and Computer Science, Xinyu University, Xinyu, Jiangxi, 338004, P. R. China

Assume that $(E, \|\cdot\|)$ is a complex Banach space with Schauder basis $\{e_j\}$ and the norm $\|\cdot\|$. Then the elements of E are called vectors and are usually denoted by letters from the alphabet: $a, b, c, \cdots$. The symbol 0 denotes the zero vector of E. The symbols $\widehat{\infty}, \infty$ and $+\infty$ denote the vector infinity, complex number infinity and the norm infinity, respectively.

An E-valued meromorphic function $f(z)$ defined in $\mathbb{C}$ can be written as $f(z) = (f_1(z), f_2(z), ..., f_k(z), ...) \in E$, where $f_1(z), f_2(z), ..., f_k(z), ...$, are the component functions of $f(z)$. A point $z_0 \in \mathbb{C}$ is called a pole of $f(z) = (f_1(z), f_2(z), ..., f_k(z), ...)$ if z_0 is a pole of at least one of the component functions of $f(z)$. A point $z_0 \in \mathbb{C}$ is called a zero of $f(z) = (f_1(z), f_2(z), ..., f_k(z), ...)$ if z_0 is a common zero of all the component functions of $f(z)$. The j-th $(j = 1, 2, ...)$ derivative of $f(z)$ are defined by

$$f^{(j)}(z) = (f_1^{(j)}(z), f_2^{(j)}(z), ..., f_k^{(j)}(z), ...),$$

and assume that $f^{(0)}(z) = f(z)$.

Suppose that $f(z)$ is an E-valued meromorphic function in $\mathbb{C}$ and $a \in E$ is a vector. We define the volume function of $f(z)$ by

$$V(r, \widehat{\infty}) = V(r, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi)\| dx \wedge dy, \quad \xi = x + iy$$

and

$$V(r, a) = V(r, f - a) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi) - a\| dx \wedge dy, \quad \xi = x + iy.$$

Let $n(r, f)$ or $n(r, \widehat{\infty})$ denote the number of poles of $f(z)$ in $|z| \leq r$, and $n(r, a)$ denote the number of a-points of $f(z)$ in $|z| \leq r$, counting with multiplicities. We define the counting function of finite or infinite a-points by

$$N(r, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt,$$

$$N(r, \widehat{\infty}) = n(0, \widehat{\infty}) \log r + \int_0^r \frac{n(t, \widehat{\infty}) - n(0, \widehat{\infty})}{t} dt,$$

and

$$N(r, a) = n(0, a) \log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt,$$

respectively. Next, we define

$$m(r, \widehat{\infty}) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\| f(re^{i\theta}) \right\| d\theta,$$

$$m(r, a) = m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|f(re^{i\theta}) - a\|} d\theta,$$

and

$$T(r, f) = m(r, f) + N(r, f),$$

where $T(r, f)$ is called the Nevanlinna characteristic function. The order and the lower order of $f(z)$ are defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \text{ and } \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

We call the E-valued meromorphic function f admissible if

$$\limsup_{r\to+\infty} \frac{T(r, f)}{\log r} = +\infty.$$

Let E_n be an n-dimensional projective space of E with the basis $\{e_j\}_1^n$. The projective operator $P_n : E \to E_n$ is a realization of E_n associated to the basis. An E-valued meromorphic function $f(z)$ in $\mathbb{C}$ is of compact projection, if for any given $\varepsilon > 0$, $\|P_n(f(z)) - f(z)\| < \varepsilon$ holds for suficiently large n in any fixed compact subset $D \subset \mathbb{C}$.

Suppose that $f(z)$ is an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ and $a \in E$. It follows from [3] and [10], we define the number $\delta(a) = \delta(a, f)$ by putting

$$\delta(a) = \delta(a, f) = \liminf_{r\to+\widehat{\infty}} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r\to+\infty} \frac{V(r, a) + N(r, a)}{T(r, f)}$$

and

$$\delta(\widehat{\infty}) = \delta(\widehat{\infty}, f) = \liminf_{r\to+\infty} \frac{m(r, f)}{T(r, f)} = 1 - \limsup_{r\to+\infty} \frac{N(r, f)}{T(r, f)}.$$

Then the set $\{a \in E \cup \{\widehat{\infty}\}, \delta(a) > 0\}$ is at most countable and summing over all such points we have

$$\sum_a \delta(a) \le 2.$$

If $\sum\limits_a \delta(a) = 2$, then we say that $f(z)$ has maximum deficiency sum or maximal defect.

2 Main Results

Let $f(z)$ be a meromorphic scalar valued function in the Gaussian complex plane $\mathbb{C}$. The characteristic function of the derivative of $f(z)$ with maximum defect has been studied by Shan, Singh, Kulkarni, Edrei and Weitsman. For example, Shan and Singh [5, 6] have proved

THEOREM A. Let $f(z)$ be a transcendental meromorphic function of finite order and assume $\sum\limits_{a\in\mathbb{C}} \delta(a) = 2$. Then

$$T(r, f') \sim 2T(r, f), \ r \to +\infty.$$

Edrei [2] and Weitsman [7] have proved

THEOREM B. Let $f(z)$ be a transcendental meromorphic scalar valued function of finite order and assume $\sum_a \delta(a) = 2$. Then

$$\lim_{r\to+\infty} \frac{T(r,f')}{T(r,f)} = 2 - \delta(\infty) \text{ and } \lim_{r\to+\infty} \frac{N\left(r,\frac{1}{f'}\right)}{T(r,f')} = 0.$$

It is natural to consider whether there exists a similar results, if meromorphic scalar valued function $f(z)$ is replaced by E-valued meromorphic function $f(z)$. In this paper, the main contribution is to extend the above theorem to E-valued meromorphic function by referring the method of [4], [7–10].

THEOREM 1. Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ of finite order and assume $\sum_a \delta(a) = 2$. Then

$$\lim_{r\to+\infty} \frac{T(r,f')}{T(r,f)} = 2 - \delta(\widehat{\infty})$$

and

$$\lim_{r\to+\infty} \frac{N(r,0,f') + V(r,0,f')}{T(r,f')} = 0.$$

Consequently,

$$\delta(0,f') = 1.$$

PROOF. Let $\{a^{[j]}\}$ be a sequence of distinct vectors in E containing all the vectors of $\delta\left(a^{[j]}\right) > 0$. Given $\varepsilon > 0$, we choose $q \geq 2$ sufficiently large so that

$$\sum_{j=1}^{q} \delta\left(a^{[j]}\right) + \delta(\widehat{\infty}) > 2 - \varepsilon. \tag{1}$$

It follows from [10] that

$$\frac{N(r,0,f') + V(r,0,f')}{T(r,f')} + \frac{T(r,f)}{T(r,f')} \left(\frac{\sum\limits_{\mu=1}^{q} m(r,a^{[\mu]})}{T(r,f)} - o(1) \right) \leq 1, \quad r \to +\infty, \tag{2}$$

and

$$\limsup_{r\to+\infty} \frac{T(r,f')}{T(r,f)} \leq 2 - \delta(\widehat{\infty}). \tag{3}$$

Therefore, we can derive the following inequality from (1)-(3),

$$\begin{aligned}
1 &\geq \limsup_{r\to+\infty}\left[\frac{N(r,0,f')+V(r,0,f')}{T(r,f')}+\frac{T(r,f)}{T(r,f')}\left(\frac{\sum_{\mu=1}^{q}m(r,a^{[\mu]})}{T(r,f)}-o(1)\right)\right]\\
&\geq \limsup_{r\to+\infty}\frac{N(r,0,f')+V(r,0,f')}{T(r,f')}+\liminf_{r\to+\infty}\frac{T(r,f)}{T(r,f')}\left(\frac{\sum_{\mu=1}^{q}m(r,a^{[\mu]})}{T(r,f)}-o(1)\right)\\
&\geq \limsup_{r\to+\infty}\frac{N(r,0,f')+V(r,0,f')}{T(r,f')}+\liminf_{r\to+\infty}\frac{T(r,f)}{T(r,f')}\liminf_{r\to+\infty}\frac{\sum_{\mu=1}^{q}m(r,a^{[\mu]})}{T(r,f)}\\
&\geq \limsup_{r\to+\infty}\frac{N(r,0,f')+V(r,0,f')}{T(r,f')}+\frac{2-\delta(\widehat{\infty},f)-\varepsilon}{2-\delta(\widehat{\infty},f)}.
\end{aligned}$$

Thus, we deduce

$$\limsup_{r\to+\infty}\frac{N(r,0,f')+V(r,0,f')}{T(r,f')}\leq\frac{\varepsilon}{2-\delta(\widehat{\infty},f)}.$$

Since $\varepsilon>0$ is arbitrary, we have

$$\limsup_{r\to+\infty}\frac{N(r,0,f')+V(r,0,f')}{T(r,f')}=0.$$

So

$$\delta(0,f')=1.$$

On the other hand, by (2), we have

$$\begin{aligned}
1 &\geq \limsup_{r\to+\infty}\frac{T(r,f)}{T(r,f')}\left(\frac{\sum_{\mu=1}^{q}m(r,a^{[\mu]})}{T(r,f)}-o(1)\right)\\
&\geq \limsup_{r\to+\infty}\frac{T(r,f)}{T(r,f')}\liminf_{r\to+\infty}\left(\frac{\sum_{\mu=1}^{q}m(r,a^{[\mu]})}{T(r,f)}-o(1)\right)\\
&\geq \limsup_{r\to+\infty}\frac{T(r,f)}{T(r,f')}\liminf_{r\to+\infty}\frac{\sum_{\mu=1}^{q}m(r,a^{[\mu]})}{T(r,f)}\\
&\geq \limsup_{r\to+\infty}\frac{T(r,f)}{T(r,f')}[2-\delta(\widehat{\infty},f)-\varepsilon].
\end{aligned}$$

Thus

$$\limsup_{r\to+\infty}\frac{T(r,f)}{T(r,f')}\le\frac{1}{2-\delta(\widehat{\infty},f)-\varepsilon}.$$

Therefore, by (3) we can get

$$2-\delta(\widehat{\infty},f)-\varepsilon\le\liminf_{r\to+\infty}\frac{T(r,f')}{T(r,f)}\le\limsup_{r\to+\infty}\frac{T(r,f')}{T(r,f)}\le 2-\delta(\widehat{\infty}).$$

Since $\varepsilon>0$ were arbitrary, we have

$$\lim_{r\to+\infty}\frac{T(r,f')}{T(r,f)}=2-\delta(\widehat{\infty}).$$

From Theorem 1, we have

COROLLARY 1. Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ of finite order and assume $\sum_{a\neq\widehat{\infty}}\delta(a)=2$. Then

$$T(r,f')\sim 2T(r,f),\quad r\to+\infty.$$

COROLLARY 2. Let $f(z)$ be an admissible E-valued meromorphic function of compact projection in $\mathbb{C}$ of finite order and assume $\sum_{a\neq\widehat{\infty}}\delta(a)=\eta\ge 1$ and $\delta(\widehat{\infty})=2-\eta$. Then

$$T(r,f')\sim \eta T(r,f),\quad r\to+\infty,$$

and

$$\lim_{r\to+\infty}\frac{N\left(r,0,f'\right)}{T(r,f')}=0.$$

Acknowledgment. This research was partly supported by the National Natural Science Foundation of China (Grant No. 11201395), NSF of Jiangxi Province (Grant 20122BAB201006) and by the Science Foundation of Educational Commission of Hubei Province (Grant no. Q20132801).

References

[1] S. S. Bhoosnurmath and V. L. Pujari, E-valued borel exceptional balues of meromorphic functions, Int. J. Math. Anal., 4(2010), 2089–2099.

[2] A. Edrei, Sums of deficiencies of meromorphic functions II, J. Analyse Math., 19(1967), 53–74.

[3] C. G. Hu, The Nevanlinna theory and its related Paley problems with applications in infinite-dimensional spaces, Complex Var. Elliptic Equ., 56(2011), 299–314.

[4] C. G. Hu and Q. J. Hu, The Nevanlinna's theorems for a class, Complex Var. Elliptic Equ., 51(2006), 777–791.

[5] S. M. Shah and S. K. Singh, Borel's theorem on a-points and exceptional values of entire and meromorphic functions, Math. Z., 59(1953), 88–93.

[6] S. M. Shah and S. K. Singh, On the derivative of a meromorphic function with maximum defect, Math. Zeitschr, 65(1956), 171–174.

[7] A. Weitsman, Meromorphic functions with maximal deficiency sum and a conjecture of F. Nevanlinna, Acta Math., 123(1969), 115–139.

[8] Z. J. Wu and Y. X. Chen, An inequality of meromorphic vector functions and its application, Abstr. Appl. Anal., (2011), Art. ID 518972, 13 pp.

[9] Z. J. Wu and Z. X. Xuan, Characteristic functions and Borel exceptional values of E-valued meromorphic functions, Abstr. Appl. Anal., (2012), Art. ID 260506, 15 pp.

[10] Z. J. Wu and Z. X. Xuan, Deficiency of E-valued meromorphic functions, Bull. Belg. Math. Soc. Simon Stevin, 19(2012), 703–715.

[11] Z. X. Xuan and N. Wu, On the Nevanlinna's theory for vector-valued mappings, Abstr. Appl. Anal., 2010, Art. ID 864539, 15 pp.