Applied Mathematics E-Notes, 13(2013), 208-211 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Long Cycle Theorem Involving Fan-Type Degree Condition And Neighborhood Intersection*

Bo Ning†

Received 23 February 2013

Abstract

In this short note we give a new sufficient condition for the existence of long cycles in 2-connected graphs involving Fan-type degree condition and neighborhood intersection.

1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph, H a subgraph of G, and v a vertex of G. We use $N_H(v)$ to denote the set of neighbors of v in H, and call $d_H(v) = |N_H(v)|$ the *degree* of v in H. For $x, y \in V(G)$, an (x, y)-*path* is a path P connecting x and y, and vertices x, y will be called the *end-vertices* of P. If $x, y \in V(H)$, the *distance* between x and y in H, denoted by $d_H(x, y)$, is the length of a shortest (x, y)-path in H. When there is no danger of ambiguity, we will use $N(v)$, $d(v)$ and $d(x, y)$ instead of $N_G(v)$, $d_G(v)$ and $d_G(x, y)$, respectively.

Let G be a graph and G' be a subgraph of G. We call G' an *induced subgraph* of G if G' contains every edge $xy \in E(G)$ with $x, y \in V(G')$. A graph isomorphic to $K_{1,3}$ is called a *claw*. A *modified claw* is a graph isomorphic to one obtained by attaching an edge to one vertex of a triangle. We say that G is *claw-free* if G contains no induced subgraph isomorphic to a claw.

In 1984, Fan [5] gave the following long cycle theorem involving the maximum degree of every pair of vertices at distance two in a 2-connected graph.

THEOREM 1 ([5]). Let G be a 2-connected graph such that $\max\{d(u), d(v)\} \geq c/2$ for each pair of vertices u and v at distance 2. Then G contains either a Hamilton cycle or a cycle of length at least c.

In [3], Bedrossian, Chen and Schelp gave an improvement of Fan's theorem. They further weakened the restriction on the pair of vertices u and v in graphs: they must be vertices of an induced claw or an induced modified claw at distance two.

*Mathematics Subject Classifications: 05C38, 05C45.

†Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P. R. China

THEOREM 2 ([3]) Let G be a 2-connected graph such that $\max\{d(u), d(v)\} \geq c/2$ for each pair of nonadjacent vertices u and v that are vertices of an induced claw or an induced modified claw of G. Then G contains either a Hamilton cycle or a cycle of length at least c.

On the other hand, Shi [7] gave the following sufficient condition for the existence of Hamilton cycles in claw-free graphs.

THEOREM 3 ([7]) Let G be a 2-connected claw-free graph. If $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u, v) = 2$, then G contains a Hamilton cycle.

In this short note we give a new sufficient condition for the existence of long cycles involving Fan-type degree condition and neighborhood intersection. It can be seen as a generalization of Theorem 3.

THEOREM 4 Let G be a 2-connected graph such that $\max\{d(u), d(v)\} \geq c/2$ for each pair of nonadjacent vertices u and v in an induced claw, and $|N(x) \cap N(y)| \geq 2$ for each pair of nonadjacent vertices x and y in an induced modified claw. Then G contains either a Hamilton cycle or a cycle of length at least c.

2 Proof of Theorem 4

Before our proof, we give some additional useful terminology and notation. A v_m*-path* is a path which has v_m as one end-vertex. If a v_m-path is a longest path among all paths, then we call it a v_m*-longest path*. Let $P = v_1v_2 \ldots v_m$ be a path and denote by $t = t(P) = \max\{j : v_1v_j \in E(G)\}$.

The proof of Theorem 4 is motivated by [3]. It is mainly based on two lemmas below.

LEMMA 1 ([1]) Let G be a 2-connected graph and P be a longest path with two end-vertices x and y. Then G contains a Hamilton cycle or a cycle of length at least $d(x) + d(y)$.

LEMMA 2 Let G be a non-Hamiltonian 2-connected graph satisfying the condition of Theorem 4. Let $P = v_1v_2 \ldots v_m (v_m = v)$ be a longest path of G. Then there exists a v-longest path such that the other end-vertex of the path has degree at least $c/2$.

PROOF. Suppose not. Now we choose a path P_1 such that $t' = t(P_1)$ is as large as possible among all v_m-longest paths of G. Without loss of generality, we still denote $\overrightarrow{P_1} = v_1v_2 \ldots v_m$. To complete this proof, we divide it into four steps.

Step1. We prove that $t' \leq m - 1$. If $v_1v_m \in E(G)$, then G is Hamiltonian or G has a non-Hamilton cycle including all vertices of P_1. Since G is 2-connected, G has a Hamilton cycle or a path longer than P_1, which is a contradiction.

Step2. We prove that $\{v_1, v_{t'-1}, v_{t'}, v_{t'+1}\}$ induces a modified claw. By the fact that $t' \leq m - 1$ and the choice of t', $v_{t'+1}$ exists and $v_1v_{t'+1} \notin E(G)$. By the connectedness of G and the choice of P_1, $t' \geq 3$. Assume that $v_1v_{t'-1}, v_{t'-1}v_{t'+1} \notin$

$E(G)$. Then $\{v_1, v_{t'-1}, v_{t'}, v_{t'+1}\}$ induces a claw. By the condition of Theorem 4 and the hypothesis that $d(v_1) < c/2$, we have $d(v_{t'-1}) \geq c/2$ and $d(v_{t'+1}) \geq c/2$. Let $P_1^{'} = v_{t'-1}\overleftarrow{P_1}v_1v_{t'}\overrightarrow{P_1}v_m$. Then $P_1^{'}$ is a v_m-longest path such that the other end-vertex $v_{t'-1}$ has degree at least $c/2$, a contradiction. If $v_{t'-1}v_{t'+1} \in E(G)$, then $P_1' = v_{t'-1}\overleftarrow{P_1}v_1v_t'\overrightarrow{P_1}v_m$ is a v_m-longest path with $t(P') \geq t' + 1$, a contradiction.

By the assumption of Theorem 4, we have $|N(v_1) \cap N(v_{t'+1})| \geq 2$. By the definition of t', there is a vertex $v_i \in N(v_1) \cap N(v_{t'+1})$, where $2 \leq i \leq t' - 2$.

Step 3. We prove that $\{v_1, v_i, v_{i+1}, v_{t'+1}\}$ induces a modified claw. We have $v_1v_{i+1} \notin E(G)$, since otherwise $P_1^{'} = v_i\overleftarrow{P_1}v_1v_{i+1}\overrightarrow{P_1}v_m$ is a v_m-longest path such that $t(P_1^{'}) \geq t' + 1$, a contradiction. If $v_{i+1}v_{t'+1} \notin E(G)$, then $\{v_1, v_i, v_{i+1}, v_{t'+1}\}$ induces a claw. By the condition of Theorem 4 and our hypothesis that $d(v_1) < c/2$, we have $d(v_{i+1}) \geq c/2$. Let $P_1^{'} = v_{i+1}\overrightarrow{P_1}v_{t'}v_1\overrightarrow{P_1}v_iv_{t'+1}\overrightarrow{P_1}v_m$. Then $P_1^{'}$ is a v_m-longest path with the other end-vertex of degree at least $c/2$, a contradiction. Thus, we have $v_{i+1}v_{t'+1} \in E(G)$ and the proof of this step is complete.

Step 4. Now, we consider the same path $P_1^{'} = v_{i+1}\overrightarrow{P_1}v_{t'}v_1\overrightarrow{P_1}v_iv_{t'+1}\overrightarrow{P_1}v_m$. Then $P_1^{'}$ is a v_m-longest path such that $t(P_1^{'}) \geq t' + 1$, a contradiction. This completes the proof of Lemma 2.

PROOF OF THEOREM 4. Suppose that G contains no Hamilton cycles. By using lemma 2 twice, we obtain a longest path with both end-vertices having the degree at least $c/2$. Then by Lemma 2, we can find a cycle of length at least c.

3 Concluding remarks

In 1989, Zhu, Li and Deng [8] proposed the definition of implicit degree. Based on this definition, many theorems on long cycles including Theorems 1 and 2, are largely improved. For details, see [4,6,8]. It may be of interest to find a version of Theorem 4 under the condition of implicit degree.

Acknowledgements. This work is supported by NSFC (No. 11271300) and the Doctorate Foundation of Northwestern Polytechnical University (cx201326). The author thanks the anonymous referee for helpful comments on an earlier version of this note.

References

[1] J. A. Bondy, Large cycles in graphs, Discrete Math., 1(1971), 121–132.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan London and Elsevier, New York, 1976.

[3] P. Bedrossian, G. Chen and R. H. Schelp, A generalization of Fan's condition for Hamiltonicity, pancyclicity, and Hamiltonian connectedness, Discrete Math., 115(1993), 39–50.

[4] B. Chen and S. Zhang, An implicit degree condition for long cycles in 2-connected graphs, Appl. Math. Lett., 19(2006), 1148–1151.

[5] G. Fan, New sufficient conditions for cycles in graphs, J. Combin. Theory Ser. B 37(1984), 221–227.

[6] H. Li, W. Ning and J. Cai, An implicit degree condition for Hamiltonian graphs, Discrete Math., 312(2012), 2190–2196.

[7] R. Shi, 2-Neighborhoods and Hamiltonian conditions, J. Graph Theory, 16(1992), 267–271.

[8] Y. Zhu, H. Li and X. Deng, Implicit-degrees and circumferences, Graphs Combin., 5(1989), 283–290.