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Asymptotic Expansions Of Iterates Of Some Classical Functions*

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Abstract

Asymptotic expansions of iterates of five functions, namely, the logarithmic function, the inverse tangent function, the inverse hyperbolic sine function, the hyperbolic tangent function and the Fresnel integral are derived with explicit parameters using a refinement of a 1994 method of Bencherif and Robin.

1 Introduction

As mentioned at the beginning of [2, Chapter 8], many problems in asymptotic analysis can be stated as follows: let $f(x) : \mathbb{R} \to \mathbb{R}$ and $u_0 \in \mathbb{R}$ be given. Define the sequence $(u_n)_{n\geq 0}$ by

$$u_1 = f(u_0),\ u_n = f(u_{n-1}) = f_2(u_{n-2}) = f(f(u_{n-2})) = \cdots = f_n(u_0) \quad (n \geq 1).$$

The problem is to find an asymptotic expansion of u_n as $n \to \infty$. In [3, Problem 173], the case of $f(x) = \sin x$ was considered. Writing $u_1 = \sin x$ and $u_n := \sin_n x$ $(n \geq 1)$, the problem is to show that $\lim_{n\to\infty} \sqrt{n/3}\sin_n x = 1$. In the book [2, Section 8.6], de Bruijn improved this result by showing that

$$\sin_n x = \sqrt{\frac{3}{n}}\left\{1 - \frac{3}{10}\frac{\log n}{n} - \frac{C(x)}{2n} + \frac{\alpha\log^2 n + \beta\log n + \gamma}{n^2} + O\left(\frac{\log^3 n}{n^3}\right)\right\} \ (n \to \infty),$$

where $\alpha,\ \beta,\ \gamma$ are explicit parameters depending on $C(x)$, which in turn depends on x, but is independent of n. In 1994, Bencherif and Robin, [1], generalized this result by deriving an asymptotic expansion for iterates of a general continuous function f and applied their result to the function $f(x) = \sin x$ $(x \in (0, \pi))$ to obtain an even more precise result.

In the present work, we refine Bencherif-Robin's 1994 work to obtain as many explicit parameters as possible, and apply them to derive asymptotic expansions of the iterates of five important classical functions $\log(1+x)$, $\tan^{-1} x$, $\sinh^{-1} x$, $\int \cos\left(\pi t^2/2\right) dt$ and $\tanh x$.

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2 Preliminaries

Our main tool is the following result of Bencherif and Robin, [1, Theorem 2].

PROPOSITION 1. For $k \geq 2$, let $\{a_1, a_2, \ldots, a_k, t\}$ be a set of real numbers with $a_1 < 0,\ t > 0$, let $\lambda = -1/ta_1$, and let $b_1 = \left(1 + t - 2a_2/a_1^2\right)/2t$. Then there exists a sequence of $k+1$ polynomials $(P_m)_{0 \leq m \leq k}$ with coefficients in $\mathbb{Q}(a_1, a_2, \ldots, a_{m+1}, t)[X]$ satisfying the differential-difference equation

$$P'_{m+1} = b_1 P'_m + (tm+1)P_m \quad (0 \leq m \leq k-1)$$

with the following property: if (u_n) is a real positive sequence converging to 0 and satisfies

$$u_{n+1} = u_n + \sum_{m=1}^{k} a_m u_n^{mt+1} + O(u_n^{(k+1)t+1}) \quad (n \to \infty),$$

then it has an asymptotic expansion of the form

$$u_n = \left(\frac{\lambda}{n}\right)^{1/t} \left\{1 + \sum_{m=1}^{k} P_m\left(-\frac{1}{t}(b_1 \log n - C)\right)\frac{1}{n^m} + O\left(\frac{1}{n^k}\right)\right\} \qquad (n \to \infty), \quad (1)$$

where $C = \lim_{n\to\infty}\left(\lambda u_n^{-t} - n - b_1 \log n\right),\ P_0 = 1, P_1 = X$, and

$$P_k \in \mathbb{Q}(a_1, a_2, \ldots, a_k, t)[X].$$

Since our main objective is to derive asymptotic results with explicit parameters, following Bencherif-Robin, consider $v_n = \lambda/u_n^t$. The sequence (v_n) was shown in [1] to have the following properties.

I. As $n \to \infty$, we have

$$v_{n+1} - v_n = 1 + \sum_{j=1}^{k-1} \frac{b_j}{v_n^j} + O\left(\frac{1}{v_n^k}\right) \qquad (k \geq 2) \quad (2)$$

$$\log\left(\frac{v_{n+1}}{v_n}\right) = \sum_{j=1}^{k-1} \frac{a_{0j}}{v_n^j} + O\left(\frac{1}{v_n^k}\right) \qquad (k \geq 2) \quad (3)$$

$$\frac{1}{v_{n+1}^i} - \frac{1}{v_n^i} = \sum_{j=i+1}^{k-1} \frac{a_{ij}}{v_n^j} + O\left(\frac{1}{v_n^k}\right) \qquad (1 \leq i \leq k-2,\ k \geq 3). \quad (4)$$

II. For $k \geq 2$, the limit $\lim_{n\to\infty}(v_n - n - b_1 \log n) = C$ exists.

III. For $k \geq 3$, there is a unique family of real numbers

$$(c_m)_{1 \leq m \leq k-2},\ c_m \in \mathbb{Q}(b_1, b_2, \ldots, b_{m+1})$$

for which the function

$$\Psi(y) = y - (b_1 \log y + C) + \frac{c_1}{y} + \cdots + \frac{c_{k-2}}{y^{k-2}} \tag{5}$$

is well-defined in the neighborhood of infinity, and the inverse function Ψ^{-1} exists and satisfies

$$v_n = \Psi^{-1}(n) + O\left(n^{-k+1}\right) (k \geq 3). \tag{6}$$

To apply Proposition 1, consider a function f continuous in a neighborhood of the origin and is of the form

$$f(x) = x + \sum_{m=1}^{k} a_m x^{mt+1} + O(x^{(k+1)t+1}) \qquad (x \to 0)$$

with $a_1 < 0$ and $t > 0$. For a given sufficiently small $u_0 = x_0 \in \mathbb{R}$, define $u_{n+1} = f(u_n)$ $(n \geq 1)$. Thus,

$$u_{n+1} = u_n + \sum_{m=1}^{k} a_m u_n^{mt+1} + O(u_n^{(k+1)t+1}). \tag{7}$$

With $v_n = \lambda/u_n^t$, to derive asymptotic estimates for v_n, Bencherif and Robin, [1, Proposition 8], showed that if f is increasing over $(0, \delta]$, then the limiting function (in the right hand expression of (1))

$$C(x_0) = \lim_{n\to\infty} \left(\lambda f_n(x_0)^{-t} - n - b_1 \log n\right)$$

exists, is continuous over $(0, \delta]$ and satisfies the asymptotic expansion

$$C(x) = \lambda x^{-t} - b_1 \log\left(\lambda x^{-t}\right) + d_1 x^t + \cdots + d_{k-2} x^{(k-2)t} + O(x^{(k-1)t}) \;\; (x \to 0,\; k \geq 3)$$

where $d_i = c_i \lambda^{-i}$ $(i = 1, \ldots, k-2)$. The numbers c_i defined in (5), satisfy ([1, Lemma 1])

$$\sum_{i=0}^{j-1} a_{ij} c_i = -b_j \qquad (1 \leq j \leq k-1) \tag{8}$$

where the parameters b_j and a_{ij} are defined via (2), (3) and (4). The next three lemmas give explicit shapes of b_j and a_{ij}.

LEMMA A. For $1 \leq j \leq k-1$ $(k \geq 2)$, we have:

$$b_j = \lambda^{j+1} \sum_{s=0}^{j} \frac{(-t)_{j+1-s}}{(j+1-s)!} \sum\nolimits_{(k,j+1)} \binom{j+1-s}{n_1, n_2, \ldots, n_k}_* a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}, \tag{9}$$

where $(-t)_r := (-t)(-t-1)\cdots(-t-r+1)$, the sum $\sum_{(k,j+1)}$ is over nonnegative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = j+1$ and

$$\binom{j+1-s}{n_1, n_2, \ldots, n_k}_* = \begin{cases} \frac{(j+1-s)!}{n_1! n_2! \cdots n_k!} & \text{if } \; n_1 + n_2 + \cdots + n_k = j+1-s \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Using $v_{n+1} = \lambda/u_{n+1}^t$ and (7), we get

$$
\begin{aligned}
v_{n+1} &= \frac{\lambda}{u_n^t}\left(1+\sum_{m=1}^{k} a_m u_n^{mt} + O\left(u_n^{(k+1)t}\right)\right)^{-t} \\
&= \frac{\lambda}{u_n^t}\left\{\left(1+\sum_{m=1}^{k} a_m u_n^{mt}\right)^{-t} + O\left(u_n^{(k+1)t}\right)\right\} \\
&= \frac{\lambda}{u_n^t}\left\{1+\sum_{p=1}^{k} \frac{(-t)_p}{p!}\left(\sum_{m=1}^{k} a_m u_n^{mt}\right)^{p} + O\left(u_n^{(k+1)t}\right)\right\} \\
&= \frac{\lambda}{u_n^t}\left\{1+\sum_{m=1}^{k}\sum_{s=0}^{m-1} \frac{(-t)_{m-s}}{(m-s)!}\sum\nolimits_{(k,m)}\binom{m-s}{n_1,n_2,\ldots,n_k}_{*}(a_1u_n^t)^{n_1}\cdots(a_ku_n^{kt})^{n_k}\right. \\
&\quad \left. + O(u_n^{(k+1)t})\right\} \\
&= v_n\left\{1+\sum_{m=1}^{k}\lambda^m\left(\sum_{s=0}^{m-1}\frac{(-t)_{m-s}}{(m-s)!}\sum\nolimits_{(k,m)}\binom{m-s}{n_1,n_2,\ldots,n_k}_{*} a_1^{n_1}a_2^{n_2}\cdots a_k^{n_k}\right)\frac{1}{v_n^m}\right. \\
&\quad \left. + O\left(v_n^{-k-1}\right)\right\}
\end{aligned}
$$

The formula for b_j follows at once by comparing with (2).

LEMMA B. For $1 \le j \le k-1$ $(k \ge 2)$, we have

$$
a_{0j} = \sum_{s=0}^{j-1}\frac{(-1)^{j-1-s}}{j-s}\sum\nolimits_{(k,j)}\binom{j-s}{n_1,n_2,\ldots,n_k}_{*} 1^{n_1}b_1^{n_2}b_2^{n_3}\cdots b_{k-1}^{n_k} \tag{10}
$$

where the sums $\sum_{(k,j)}$ and $\binom{j-s}{n_1,n_2,\ldots,n_k}_{*}$ are similarly defined as in Lemma A.

PROOF. From (2), we obtain

$$
\begin{aligned}
\log\left(\frac{v_{n+1}}{v_n}\right) &= \log\left(1+\frac{1}{v_n}+\sum_{j=1}^{k-1}\frac{b_j}{v_n^{j+1}} + O\left(v_n^{-k-1}\right)\right) \\
&= \sum_{p=1}^{k-1}\frac{(-1)^{p+1}}{p}\left(\frac{1}{v_n}+\sum_{j=1}^{k-1}\frac{b_j}{v_n^{j+1}}\right)^{p} + O\left(v_n^{-k}\right) \\
&= \sum_{m=1}^{k-1}\left(\sum_{s=0}^{m-1}\frac{(-1)^{m-1-s}}{m-s}\sum\nolimits_{(k,m)}\binom{m-s}{n_1,n_2,\ldots,n_k}_{*}1^{n_1}b_1^{n_2}\cdots b_{k-1}^{n_k}\right)\frac{1}{v_n^m} \\
&\quad + O\left(v_n^{-k}\right)
\end{aligned}
$$

The formula for a_{0j} follows at once by comparing with (3).

Explicitly, the first four a_{0j}-terms are

$$a_{01}=1,\ a_{02}=b_1-1/2,\ a_{03}=b_2-b_1+1/3 \text{ and } a_{04}=-b_1^2/2+b_3-b_2+b_1-1/4.$$

LEMMA C. Let $k\geq 3$. For $i+1\leq j\leq k-1$, we have

$$a_{ij}=\sum_{s=0}^{j-i-1}\frac{(-i)_{j-i-s}}{(j-i-s)!}\sum\nolimits_{(k,j-i)}\binom{j-i-s}{n_1,n_2,\ldots,n_k}_* 1^{n_1}b_1^{n_2}b_2^{n_3}\cdots b_{k-1}^{n_k} \tag{11}$$

where the sum $\sum_{(k,j-i)}$ and $\binom{j-i-s}{n_1,n_2,\ldots,n_k}_*$ are similarly defined as in Lemma A.

PROOF. Using (2), we get

$$\begin{aligned}
\frac{1}{v_{n+1}^i}-\frac{1}{v_n^i}&=\frac{1}{v_n^i}\left(\frac{v_n^i}{v_{n+1}^i}-1\right)\\
&=\frac{1}{v_n^i}\left\{-1+\left(1+\frac{1}{v_n}+\sum_{j=1}^{k-1}\frac{b_j}{v_n^{j+1}}+O\left(\frac{1}{v_n^{k+1}}\right)\right)^{-i}\right\}\\
&=\frac{1}{v_n^i}\left\{-1+\left(1+\frac{1}{v_n}+\sum_{j=1}^{k-1}\frac{b_j}{v_n^{j+1}}\right)^{-i}+O\left(\frac{1}{v_n^{k+1}}\right)\right\}\\
&=\frac{1}{v_n^i}\left\{\sum_{p=1}^{k-1}\frac{(-i)_p}{p!}\left(\frac{1}{v_n}+\sum_{j=1}^{k-1}\frac{b_j}{v_n^{j+1}}\right)^p+O\left(\frac{1}{v_n^k}\right)\right\}\\
&=\sum_{m=1}^{k-1-i}\left(\sum_{s=0}^{m-1}\frac{(-i)_{m-s}}{(m-s)!}\sum\nolimits_{(k,m)}\binom{m-s}{n_1,n_2,\ldots,n_k}_* 1^{n_1}b_1^{n_2}b_2^{n_3}\cdots b_{k-1}^{n_k}\right)\frac{1}{v_n^{m+i}}\\
&+O\left(\frac{1}{v_n^k}\right)
\end{aligned}$$

The formula for a_{ij} follows by comparing with (4).

The first four a_{ij}-terms are

$$a_{i,i+1}=-i,\ a_{i,i+2}=\frac{i^2}{2}+\left(\frac{1}{2}-b_1\right)i,$$

$$a_{i,i+3}=-\frac{i^3}{6}+\left(b_1-\frac{1}{2}\right)i^2+\left(b_1-b_2-\frac{1}{3}\right)i$$

and

$$a_{i,i+4}=\frac{i^4}{24}+\left(-\frac{b_1}{2}+\frac{1}{4}\right)i^3+\left(\frac{b_1^2}{2}+b_2-\frac{3b_1}{2}+\frac{11}{24}\right)i^2+\left(-b_3+\frac{b_1^2}{2}+b_2-b_1+\frac{1}{4}\right)i.$$

Using (9), (10) and (11), we solve for c_i from the system (8) to get

$$c_0 = -b_1 \text{ and } c_i = \frac{1}{i}\left(b_{i+1} + \sum_{t=0}^{i-1} a_{t,i+1}c_t\right) \qquad (1 \le i \le k-2,\ k \ge 3). \tag{12}$$

LEMMA D. We have, for $k \ge 3$,

$$\Psi^{-1}(y) = y + T_1 + \sum_{m=1}^{k-2} \frac{T_{m+1}}{y^m} + O\left(\frac{1}{y^{k-1}}\right) \quad (y \to \infty) \tag{13}$$

and

$$u_n = \left(\frac{\lambda}{n}\right)^{1/t} \left\{1 + \sum_{m=1}^{k-1} \frac{1}{n^m} \sum_{s=0}^{m-1} \frac{(-1/t)_{m-s}}{(m-s)!} \right.$$
$$\left. \times \sum\nolimits_{(k-1,m)} \binom{m-s}{n_1, n_2, \ldots, n_{k-1}}_* T_1^{n_1} T_2^{n_2} \cdots T_{k-1}^{n_{k-1}} \right\} + O\left(\frac{1}{n^{k+1/t}}\right) \tag{14}$$

for $n \to \infty$, where $T_1 = X$, $T_2 = b_1X - c_1$,

$$T_3 = -b_1X^2/2 + (b_1^2 + c_1)X - b_1c_1 - c_2,$$

$$X = b_1 \log y + C,$$

$$C := \lambda x^{-t} - b_1 \log\left(\lambda x^{-t}\right) + c_1\lambda^{-1}x^t + \cdots + c_{k-2}\lambda^{-(k-2)}x^{(k-2)t} + O(x^{(k-1)t})$$

and, in general, for $m \ge 2$

$$T_{m+1} = b_1 \sum_{s=0}^{m-1} \frac{(-1)^{m-1-s}}{m-s} \sum\nolimits_{(k-1,m)} \binom{m-s}{n_1, n_2, \ldots, n_{k-1}}_* T_1^{n_1} T_2^{n_2} \cdots T_{k-1}^{n_{k-1}} - c_m$$
$$- \sum_{e=1}^{m-1} c_e \sum_{d=1}^{m-e} \frac{(-e)_d}{d!} \sum\nolimits_{(k-1,m-e)} \binom{d}{m_1, m_2, \ldots, m_{k-1}}_* T_1^{m_1} T_2^{m_2} \cdots T_{k-1}^{m_{k-1}},$$

with the two sums and the special multinomial symbols being similarly defined as in Lemma A.

PROOF. Using (5) and $\Psi^{-1}(\Psi(n)) = n$, we get

$$\Psi^{-1}\left(n - (b_1 \log n + C) + \frac{c_1}{n} + \cdots + \frac{c_{k-2}}{n^{k-2}}\right) = n,$$

which yields

$$\Psi^{-1}(y) = y + T_1 + \sum_{m=1}^{k-2} \frac{T_{m+1}}{y^m} + O\left(y^{-k+1}\right) \quad (y \to \infty).$$

To determine T_m $(1 \leq m \leq k-2)$, we use $\Psi\left(\Psi^{-1}(y)\right) = y$ and (5) to get

$$
\begin{aligned}
y &= y + T_1 - b_1 \log y - C + \sum_{m=1}^{k-2} \frac{T_{m+1}}{y^m} - b_1 \log\left(1 + \sum_{m=0}^{k-2} \frac{T_{m+1}}{y^{m+1}} + O\left(\frac{1}{y^k}\right)\right) \\
&+ \frac{c_1}{y}\left(1 + \sum_{m=0}^{k-2} \frac{T_{m+1}}{y^{m+1}} + O\left(\frac{1}{y^k}\right)\right)^{-1} \\
&+ \cdots + \frac{c_{k-2}}{y^{k-2}}\left(1 + \sum_{m=0}^{k-2} \frac{T_{m+1}}{y^{m+1}} + O\left(\frac{1}{y^k}\right)\right)^{-(k-2)}. \qquad (15)
\end{aligned}
$$

Substituting

$$
\begin{aligned}
&\log\left(1 + \sum_{m=0}^{k-2} \frac{T_{m+1}}{y^{m+1}} + O\left(\frac{1}{y^k}\right)\right) \\
&= \sum_{m=1}^{k-1} \left(\sum_{s=0}^{m-1} \frac{(-1)^{m-1-s}}{m-s} \sum\nolimits_{(k-1,m)} \binom{m-s}{n_1, n_2, \ldots, n_{k-1}}_{*} T_1^{n_1} T_2^{n_2} \cdots T_{k-1}^{n_{k-1}}\right) \frac{1}{y^m} \\
&+ O\left(\frac{1}{y^k}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{p=1}^{k-2} \frac{c_p}{y^p}\left(1 + \sum_{m=0}^{k-2} \frac{T_{m+1}}{y^{m+1}} + O\left(\frac{1}{y^k}\right)\right)^{-p} \\
&= \sum_{p=1}^{k-2} \frac{c_p}{y^p}\left(1 + \sum_{m=0}^{k-2} \frac{T_{m+1}}{y^{m+1}}\right)^{-p} + O\left(\frac{1}{y^{k-1}}\right) \\
&= \sum_{p=1}^{k-2} \frac{c_p}{y^p}\left\{1 + \sum_{s=1}^{k-2} \frac{(-p)_s}{s!}\left(\sum_{m=0}^{k-2} \frac{T_{m+1}}{y^{m+1}}\right)^{s}\right\} + O\left(\frac{1}{y^{k-1}}\right) \\
&= \frac{c_1}{y} + \sum_{m=2}^{k-2}\left\{c_m + \sum_{e=1}^{m-1} c_e \sum_{d=1}^{m-e} \frac{(-e)_d}{d!}\right. \\
&\quad \left.\times \sum\nolimits_{(k-1,m-e)} \binom{d}{m_1, \ldots, m_{k-1}}_{*} T_1^{m_1} \cdots T_{k-1}^{m_{k-1}}\right\} \frac{1}{y^m} + O\left(\frac{1}{y^{k-1}}\right),
\end{aligned}
$$

into (15) we get

$$
\begin{aligned}
y &= y + (T_1 - b_1 \log y - C) + (T_2 - b_1 T_1 + c_1)\,\frac{1}{y} \\
&+ \sum_{m=2}^{k-2} \Bigg\{ T_{m+1} - b_1 \sum_{s=0}^{m-1} \frac{(-1)^{m-1-s}}{m-s} \sum\nolimits_{(k-1,m)} \binom{m-s}{n_1,\ldots,n_{k-1}}_{*} T_1^{n_1}\cdots T_{k-1}^{n_{k-1}} \\
&+ c_m + \sum_{e=1}^{m-1} c_e \sum_{d=1}^{m-e} \frac{(-e)_d}{d!} \sum\nolimits_{(k-1,m-e)} \binom{d}{m_1,\ldots,m_{k-1}}_{*} T_1^{m_1}\cdots T_{k-1}^{m_{k-1}} \Bigg\} \frac{1}{y^m} \\
&+ O\left(\frac{1}{y^{k-1}}\right).
\end{aligned}
$$

The shape of T_m follows from comparing the coefficient of $1/y^m$ on both sides. Next, using (6) and (13), we get

$$
\begin{aligned}
u_n &= \left(\frac{\lambda}{v_n}\right)^{1/t} = \lambda^{1/t}\left(\Psi^{-1}(n) + O\left(\frac{1}{n^{k-1}}\right)\right)^{-1/t} \\
&= \left(\frac{\lambda}{n}\right)^{1/t} \left\{ \left(1 + \sum_{m=0}^{k-2} \frac{T_{m+1}}{n^{m+1}}\right)^{-1/t} + O\left(n^{-k}\right) \right\} \\
&= \left(\frac{\lambda}{n}\right)^{1/t} \left\{ 1 + \sum_{p=1}^{k-1} \frac{(-1/t)_p}{p!} \left(\sum_{m=0}^{k-2} \frac{T_{m+1}}{n^{m+1}}\right)^{p} + O\left(n^{-k}\right) \right\} \\
&= \left(\frac{\lambda}{n}\right)^{1/t} \Bigg\{ 1 + \sum_{m=1}^{k-1} \frac{1}{n^m} \sum_{s=0}^{m-1} \frac{(-1/t)_{m-s}}{(m-s)\,!} \sum\nolimits_{(k-1,m)} \binom{m-s}{n_1,\ldots,n_{k-1}}_{*} T_1^{n_1}\cdots T_{k-1}^{n_{k-1}} \\
&+ O\left(n^{-k}\right) \Bigg\}.
\end{aligned}
$$

As a useful by-product of the explicit forms so derived above, we use them to derive combinatorial identities which seems difficult to prove by other means.

PROPOSITION 2. Let $s,\ k\ (\geq 2) \in \mathbb{N}$. Then, for $j = 1, 2, \ldots, k-1$, we have

$$
\sum_{s=0}^{j} (-1)^s \sum\nolimits_{(k,j+1)} \binom{j+1-s}{n_1, n_2, \ldots, n_k}_{*} = 0,
$$

$$
\sum_{s=0}^{j-1} \frac{(-1)^{j-1-s}}{j-s} \sum\nolimits_{(k,j)} \binom{j-s}{n_1, n_2, \ldots, n_k}_{*} 1^{n_1} 0^{n_2} \cdots 0^{n_k} = \frac{(-1)^{j+1}}{j}
$$

and

$$
\sum_{s=0}^{j-i-1} \frac{(-i)_{j-i-s}}{(j-i-s)!} \sum\nolimits_{(k,j-i)} \binom{j-i-s}{n_1, n_2, \ldots, n_k}_{*} 1^{n_1} 0^{n_2} \cdots 0^{n_k} = \frac{(-i)_{j-i}}{(j-i)!}.
$$

PROOF. Consider a rational function of the form

$$f(x) = \frac{x}{1+Ax} = x - Ax^2 + A^2x^3 + \cdots + (-A)^k x^{k+1} + O\left(x^{k+1}\right) \qquad (A > 0,\ x \in (0,1)).$$

By direct computation, its iterates are $u_0 = x_0 \in (0,1),\ u_1 = f(u_0) = \frac{x_0}{1+Ax_0}$,

$$u_n = f(u_{n-1}) = \frac{x_0}{1+nAx_0} = \frac{1}{nA} - \frac{1}{(nA)^2 x_0} + \cdots + \frac{(-1)^k}{(nA)^{k+1} x_0^k} + O\left(n^{-k-2}\right) \quad (n, k \geq 2).$$

Referring to the notation of (7) and Lemma A, from

$$u_{n+1} = f(u_n) = u_n + \sum_{m=1}^{k} (-A)^m u_n^{m+1} + O\left(u_n^{k+2}\right) \quad (n \to \infty),$$

we have $t = 1,\ a_j = (-A)^j \quad (1 \leq j \leq k),\ \lambda = 1/A$. Since

$$v_{n+1} - v_n = \frac{1}{Au_{n+1}} - \frac{1}{Au_n} = \frac{1}{A}\left(\frac{1}{u_{n+1}} - \frac{1}{u_n}\right) = \frac{1}{A}\left(\frac{1+(n+1)Ax_0}{x_0} - \frac{1+nAx_0}{x_0}\right) = 1,$$

comparing with (2), we get $b_j = 0$ $(1 \leq j \leq k-1)$. Putting these values of b_j into (9), the first identity follows. Next, using

$$\log\left(\frac{v_{n+1}}{v_n}\right) = \log\left(1 + \frac{1}{v_n}\right) = \sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{j\ v_n^j} + O\left(v_n^{-k}\right),$$

and comparing with (3), we get $a_{0j} = (-1)^{j+1}/j \quad (1 \leq j \leq k-1)$. Substituting these values of a_{0j} into (10), we get the second identity. Since

$$\frac{1}{v_{n+1}^i} - \frac{1}{v_n^i} = \frac{1}{v_n^i}\left(\left(1 + \frac{1}{v_n}\right)^{-i} - 1\right) = \sum_{j=i+1}^{k-1} \frac{(-1)_{j-i}}{(j-i)!}\frac{1}{v_n^j} + O\left(v_n^{-k}\right),$$

comparing with (4), we get

$$a_{ij} = (-i)_{j-i}/(j-i)! \quad (1 \leq i \leq k-2,\ i+1 \leq j \leq k-1).$$

Putting these values of a_{ij} into (11), we get the third identity.

3 Asymptotic Formulas

Our asymptotic expansions of the five classical functions are:

THEOREM 1. Let $k \in \mathbb{N},\ k \geq 3$.
I. For $f(x) = \log(1+x)$ $(x \in (0,1))$, we have

$$f_n(x) = \frac{2}{n}\left\{1 + \frac{1}{n}\left[\left(-\frac{1}{3}\log n + C(x)\right) + \frac{1}{n^2}\left(\frac{1}{9}\log^2 n + \left(-\frac{2}{3}C(x) - \frac{2}{9}\right)\log n \right.\right.\right.$$
$$\left.\left.\left. + C^2(x) + \frac{2C(x)}{3} + \frac{1}{9}\right] + O\left(\frac{\log^3 n}{n^3}\right)\right\} \quad (n \to \infty) \qquad (16)$$

with

$$C(x)=\frac{2}{x}+\frac{1}{3}\log\left(\frac{2}{x}\right)+\frac{x}{36}+\frac{191x^2}{2160}+\cdots+c_{k-2}^{(\log)}\left(\frac{x}{2}\right)^{k-2}+O(x^{k-1}).$$

II. For $f(x)=\arctan x$ $(x\in(0,1))$, we have

$$f_n(x)=\sqrt{\frac{3}{2n}}\Bigg\{1+\left(\frac{3\log n}{40}-\frac{C(x)}{2}\right)\frac{1}{n}+\left(\left(\frac{27\log n}{3200}-\frac{9}{80}C(x)-\frac{9}{800}\right)\log n\right.$$
$$\left.+\frac{3C^2(x)}{8}+\frac{3C(x)}{40}+\frac{47}{5600}\right)\frac{1}{n^2}+O\left(\frac{\log^3 n}{n^3}\right)\Bigg\}\quad(n\to\infty)\tag{17}$$

with

$$C(x)=\frac{3}{2x^2}+\frac{3}{20}\log\left(\frac{3}{2x^2}\right)+\frac{47x^2}{4200}+\frac{x^4}{12000}+\cdots+c_{k-2}^{(\arctan)}\left(\frac{2x^2}{3}\right)^{k-2}+O(x^{2(k-1)}).$$

III. For $f(x)=\sinh^{-1}x$ $(x\in(0,1))$, we have

$$f_n(x)=\sqrt{\frac{3}{n}}\Bigg\{1+\left(\frac{3\log n}{10}-\frac{C(x)}{2}\right)\frac{1}{n}+\left(\left(\frac{27\log n}{200}-\frac{9C(x)}{20}-\frac{9}{50}\right)\log n\right.$$
$$\left.+\frac{3C^2(x)}{8}+\frac{3C(x)}{10}+\frac{79}{700}\right)\frac{1}{n^2}+O\left(\frac{\log^3 n}{n^3}\right)\Bigg\}\quad(n\to\infty)$$

with

$$C(x)=\frac{3}{x^2}+\frac{3}{5}\log\left(\frac{3}{x^2}\right)+\frac{79x^2}{1050}-\frac{11567x^4}{459}+\cdots+c_{k-2}^{(\mathrm{arc\,sinh})}\left(\frac{x^2}{3}\right)^{k-2}+O(x^{2(k-1)}).$$

IV. For $f(x)=\int_0^x\cos\left(\pi t^2/2\right)dt$ $(x\in(0,1))$, we have

$$f_n(x)=\sqrt[4]{\frac{10}{\pi^2 n}}\left\{1-\frac{X}{4}\cdot\frac{1}{n}+\left(\frac{5X^2}{32}-\frac{55X}{432}+\frac{127}{2992}\right)\frac{1}{n^2}+O\left(\frac{\log^3 n}{n^3}\right)\right\}\quad(n\to\infty),$$

where $X=\frac{55\log n}{108}+C(x)$,

$$C(x)=\frac{10}{\pi^2x^4}-\frac{55}{108}\log\left(\frac{10}{\pi^2x^4}\right)+\frac{127\pi^2x^4}{7480}+\frac{416\pi^4x^8}{228500}+\cdots+c_{k-2}^{(\mathrm{Fresnel})}\left(\frac{\pi^2x^4}{10}\right)^{k-2}+O(x^{4(k-1)}).$$

V. For $f(x)=\tanh x$ $(x\in(0,\pi/2))$, we have

$$f_n(x)=\sqrt{\frac{3}{2n}}\Bigg\{1-\frac{b_1\log n+C(x)}{2n}+\left(\left(\frac{3\log^2 n}{8}-\frac{\log n}{2}-\frac{1}{2}\right)b_1^2\right.$$
$$\left.+\left(\frac{3C(x)\log n}{4}-\frac{C(x)}{2}+\frac{1}{4}\right)b_1+\frac{b_2}{2}+\frac{3C^2(x)}{8}\right)\frac{1}{n^2}+O\left(\frac{\log^3 n}{n^3}\right)\Bigg\}\quad(n\to\infty),$$

where

$$C(x) = \frac{3}{2x^2} - b_1 \log\left(\frac{3}{2x^2}\right) + \frac{1}{2}\left(-b_1^2 + \frac{b_1}{2} + b_2\right)x^2 + \frac{1}{4}\left(2b_1^3 - b_1^2 + \frac{b_1}{3} - 4b_1b_2 \right.$$
$$\left. +2b_2 + 2b_3\right)x^4 + \cdots + c_{k-2}^{(\tanh)}\left(\frac{2x^2}{3}\right)^{k-2} + O(x^{2(k-1)}),$$

$$b_j = \left(\frac{3}{2}\right)^{j+1}\sum_{s=0}^{j}\frac{(-2)_{j+1-s}}{(j+1-s)!}\sum\nolimits_{(k,j+1)}\binom{j+1-s}{n_1, n_2, \ldots, n_k}_* a_1^{n_1}\cdots a_k^{n_k}$$

and

$$a_j = \frac{2^{2(j+1)}\left(2^{2(j+1)}-1\right)B_{2(j+1)}}{(2(j+1))!} \quad (1 \le j \le k-1)$$

with B_k being Bernoulli numbers.

PROOF. I. Here,

$$f(x) = \log(1+x) = \sum_{m\ge 0}\frac{(-1)^m x^{m+1}}{m+1} \quad (x \in (0,1)),$$

choose x_0 sufficiently small in $(0,\delta)$ with $0 < \delta < 1$, and define $u_0 = x_0$, $u_n = f(u_{n-1})$ $(n \ge 1)$. Since $0 < \log(1+x_0) < x_0$, by induction, we see that $0 < u_n < 1$ and $u_1 > u_2 > \cdots$. Thus, $\lim_{n\to\infty} u_n = 0$, and

$$u_{n+1} = \log(1+u_n) = u_n + \sum_{m=1}^{k}\frac{(-1)^m u_n^{m+1}}{m+1} + O\left(u_n^{k+2}\right).$$

Comparing with (7), we have here

$$a_j = (-1)^j/(j+1) \quad (1 \le j \le k), \quad t = 1$$

and so $\lambda = 2$. By Lemma A, the first four explicit b_j-terms are

$$b_1 = -1/3, b_2 = 1/3, b_3 = 3/10, b_4 = 3/5.$$

By Lemma B, the first four a_{0j}-terms are

$$a_{01} = 1,\ a_{02} = -5/6,\ a_{03} = 1,\ a_{04} = -121/180.$$

By Lemma C, we have

$$a_{i,i+1} = -i,\ a_{i,i+2} = \frac{i}{3} + \frac{(-i)(-i-1)}{2!}$$

and

$$a_{i,i+3} = \frac{-i}{3} - \frac{2}{3}\frac{(-i)(-i-1)}{2!} + \frac{(-i)(-i-1)(-i-2)}{3!}.$$

Using (12), we get

$$c_0 = 1/3,\ c_1 = 1/18,\ c_2 = 191/540.$$

From Lemma D, with $k \geq 3$, $T_1 = X$, $T_2 = -X/3 - 1/18$, $T_3 = X^2/6 + X/6 - 181/540$, we have

$$\begin{aligned}\Psi^{-1}(n) &= n\left\{1 + \frac{X}{n} + \frac{-X/3 - 1/18}{n^2} + \frac{X^2/6 + X/6 - 181/540}{n^3}\right.\\ &\quad \left. + \cdots + \frac{T_{k-2}}{n^{k-2}} + O\left(n^{-k+1}\right)\right\}\end{aligned}$$

with

$$\begin{aligned}X &= -\frac{\log n}{3} + C(x_0),\\ C(x_0) &= \frac{2}{x_0} + \frac{1}{3}\log\left(\frac{2}{x_0}\right) + \frac{x_0}{36} + \frac{191x_0^2}{2160} + \cdots + c_{k-2}\left(\frac{x_0}{2}\right)^{k-2} + O(x_0^{k-1}).\end{aligned}$$

The shape of $u_n = f_n(x)$ as stated in (16) follows from (14) in Lemma D.

Since the remaining four asymptotic expansions are derived via similar arguments as in Case I, we simply list their explicit expressions for records.

II. For

$$f(x) = \arctan x = \sum_{m\geq 0}(-1)^m x^{2m+1}/(2m+1),$$

we have

$$t = 2,\ \lambda = 3/2,$$

$$u_{n+1} = \arctan\ u_n = u_n + \sum_{m=1}^{k}\frac{(-1)^m u_n^{2m+1}}{2m+1} + O\left(u_n^{2k+3}\right),$$

$$a_j = (-1)^j/(2j+1)\ \ (1 \leq j \leq k),$$

$$b_1 = -3/20,\ b_2 = 4/35,\ b_3 = -19/175,\ b_4 = 222/1925,$$

$$a_{01} = 1,\ a_{02} = -13/20,\ a_{03} = 251/420,\ a_{04} = -3551/5600,$$

$$a_{i,i+1} = -i,\ a_{i,i+2} = \frac{10i^2 + 13i}{20},\ a_{i,i+3} = -\frac{i^3}{6} - \frac{13i^2}{20} - \frac{251i}{420},$$

$$c_0 = 3/20,\quad c_1 = 47/2800,\quad c_2 = 3/16000,$$

$$T_1 = X,\ T_2 = -\frac{3X}{20} - \frac{47}{2800},\ T_3 = \frac{3X^2}{40} + \frac{11X}{280} + \frac{261}{112000},$$

$$\begin{aligned}C(x_0) &= \frac{3}{2x_0^2} + \frac{3}{20}\log\left(\frac{3}{2x_0^2}\right) + \frac{47x_0^2}{4200} + \frac{x_0^4}{12000} + \cdots + c_{k-2}\left(\frac{2x_0^2}{3}\right)^{k-2}\\ &\quad + O(x_0^{2(k-1)}),\end{aligned}$$

$$X = -\frac{3\log n}{20} + C(x_0)$$

and

$$\Psi^{-1}(n) = n\left\{1 + \frac{X}{n} + \left(-\frac{3X}{20} - \frac{47}{2800}\right)\frac{1}{n^2} + \left(\frac{3X^2}{40} + \frac{11X}{280} + \frac{261}{112000}\right)\frac{1}{n^3} + \cdots + \frac{T_{k-2}}{n^{k-2}} + O\left(\frac{1}{n^{k-1}}\right)\right\}.$$

III. For

$$f(x) = \sinh^{-1} x = \sum_{m=0}^{\infty} \frac{(-1)^m (2m)!}{2^{2m}(m!)^2} \cdot \frac{x^{2m+1}}{2m+1},$$

we have

$$t = 2,\ \lambda = 3,$$

$$u_{n+1} = \sinh^{-1}\ u_n = u_n + \sum_{m=1}^{k} \frac{(-1)^m (2m)!}{2^{2m}(m!)^2} \cdot \frac{u_n^{2m+1}}{2m+1} + O\left(u_n^{2k+3}\right),$$

$$a_j = \frac{(-1)^j (2j)!}{2^{2j}(j!)^2(2j+1)} \quad (1 \le j \le k),$$

$$b_1 = -3/5,\ b_2 = 31/35,\ b_3 = -6317/55,\ b_4 = -34101/205,$$

$$a_{01} = 1, \quad a_{02} = -11/10, \quad a_{03} = 191/105, \quad a_{04} = -8641/74,$$

$$a_{i,i+1} = -i, \quad a_{i,i+2} = \frac{10i^2 + 22i}{20},\ a_{i,i+3} = -\frac{i^3}{6} - \frac{11i^2}{10} - \frac{191i}{105},$$

$$c_0 = 3/5,\ c_1 = 79/350,\ c_2 = -11567/51,$$

$$T_1 = X, \quad T_2 = -\frac{3X}{5} - \frac{79}{350}, \quad T_3 = \frac{3X^2}{10} + \frac{41X}{70} + \frac{7489}{33},$$

$$C(x_0) = \frac{3}{x_0^2} + \frac{3}{5}\log\left(\frac{3}{x_0^2}\right) + \frac{79x_0^2}{1050} - \frac{11567x_0^4}{459} + \cdots + c_{k-2}\left(\frac{x_0^2}{3}\right)^{k-2} + O(x_0^{2(k-1)}),$$

$$X = -\frac{3\log n}{5} + C(x_0)$$

and

$$\Psi^{-1}(n) = n\left\{1 + \frac{X}{n} + \left(-\frac{3X}{5} - \frac{79}{350}\right)\frac{1}{n^2} + \left(\frac{3X^2}{10} + \frac{41X}{70} + \frac{7489}{33}\right)\frac{1}{n^3} + \cdots + \frac{T_{k-2}}{n^{k-2}} + O\left(\frac{1}{n^{k-1}}\right)\right\}.$$

IV. For

$$f(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{\pi}{2}\right)^{2m}}{(2m)!(4m+1)} x^{4m+1},$$

we have

$$\lambda = 10/\pi^2,\ t = 4,$$

$$u_{n+1} = u_n + \sum_{m=1}^{k} \frac{(-1)^m \left(\frac{\pi}{2}\right)^{2m}}{(2m)!(4m+1)} u_n^{4m+1} + O\left(u_n^{4k+5}\right),$$

$$a_j = \frac{(-1)^j \left(\frac{\pi}{2}\right)^{2j}}{(2j)!(4j+1)} \quad (1 \le j \le k),$$

$$b_1 = 55/108,\ b_2 = 245/1404,\ b_3 = 89/12747,$$

$$a_{01} = 1,\ a_{02} = 1/108,\quad a_{03} = -1/702,\ a_{04} = -411/10835,$$

$$a_{i,i+1} = -i,\ a_{i,i+2} = \frac{i^2}{2} - \frac{i}{108},\ a_{i,i+3} = -\frac{i^3}{6} + \frac{i^2}{108} + \frac{i}{702},$$

$$c_0 = -55/108,\ c_1 = 127/748,\ c_2 = 416/2285,$$

$$T_1 = X,\quad T_2 = \frac{55X}{108} - \frac{127}{748},\quad T_3 = -\frac{55X^2}{216} + \frac{657X}{1531} - \frac{714}{2659},$$

$$\begin{aligned} C(x_0) \quad = \quad & \frac{10}{\pi^2 x_0^4} - \frac{55}{108}\log\left(\frac{10}{\pi^2 x_0^4}\right) + \frac{127\pi^2 x_0^4}{7480} + \frac{416\pi^4 x_0^8}{228500} + \cdots + c_{k-2}\left(\frac{\pi^2 x_0^4}{10}\right)^{k-2} \\ & + O(x_0^{4(k-1)}), \end{aligned}$$

$$X = \frac{55\log n}{108} + C(x_0)$$

and

$$\begin{aligned} \Psi^{-1}(n) \quad = \quad & n\left\{1 + \frac{X}{n} + \left(\frac{55X}{108} - \frac{127}{748}\right)\frac{1}{n^2} + \left(-\frac{55X^2}{216} + \frac{657X}{1531} - \frac{714}{2659}\right)\frac{1}{n^3}\right. \\ & \left. + \cdots + \frac{T_{k-2}}{n^{k-2}} + O\left(\frac{1}{n^{k-1}}\right)\right\}. \end{aligned}$$

V. For

$$f(x) = \tanh x = \sum_{m=0}^{\infty} \frac{2^{2(m+1)}\left(2^{2(m+1)} - 1\right) B_{2(m+1)} x^{2m+1}}{(2(m+1))!},$$

we have

$$t = 2,\ \lambda = 3/2,$$

$$u_{n+1} = \tanh u_n = u_n + \sum_{m=1}^{k} \frac{2^{2(m+1)}\left(2^{2(m+1)} - 1\right) B_{2(m+1)} u_n^{2m+1}}{(2(m+1))!} + O\left(u_n^{2k+3}\right),$$

$$a_j = \frac{2^{2(j+1)}\left(2^{2(j+1)} - 1\right) B_{2(j+1)}}{(2(j+1))!} \quad (1 \le j \le k),$$

$$b_1 = -\frac{126B_6}{5} + 675B_4^2,\ b_2 = -\frac{153B_8}{14} + 1134B_4B_6 - 13500B_4^3,$$

$$b_3 = -\frac{1023B_{10}}{350} + \frac{11907B_6^2}{25} + \frac{6885B_4B_8}{14} - 34020B_4^2B_6 + 253125B_4^4,$$
$$a_{01} = 1,\ a_{02} = b_1 - 1/2,\ a_{03} = b_2 - b_1 + 1/3,$$
$$a_{i,i+1} = -i,\ a_{i,i+2} = \frac{i^2}{2} + \left(\frac{1}{2} - b_1\right) i,$$
$$a_{i,i+3} = -\frac{i^3}{6} + \left(b_1 - \frac{1}{2}\right) i^2 + \left(b_1 - b_2 - \frac{1}{3}\right) i,$$
$$c_0 = -b_1,\ c_1 = -b_1^2 + \frac{b_1}{2} + b_2,\ c_2 = 2b_1^3 - b_1^2 + \frac{b_1}{3} - 4b_1b_2 + 2b_2 + 2b_3,$$
$$T_1 = X,\quad T_2 = b_1X - c_1,$$

$$\begin{aligned} C(x_0) \quad = \quad & \frac{3}{2x_0^2} - b_1 \log\left(\frac{3}{2x_0^2}\right) + \frac{1}{2}\left(-b_1^2 + \frac{b_1}{2} + b_2\right) x_0^2 + \frac{1}{4}\left(2b_1^3 - b_1^2 + \frac{b_1}{3}\right. \\ & \left. -4b_1b_2 + 2b_2 + 2b_3\right) x_0^4 + \cdots + c_{k-2}\left(\frac{2x_0^2}{3}\right)^{k-2} + O(x_0^{2(k-1)}), \end{aligned}$$

$$X = \left(-\frac{126B_6}{5} + 675B_4^2\right) \log n + C(x_0)$$

and

$$\Psi^{-1}(n) = n\left\{1 + \frac{X}{n} + \left(b_1X - b_2 + b_1^2 - \frac{b_1}{2}\right)\frac{1}{n^2} + \cdots + \frac{T_{k-2}}{n^{k-2}} + O\left(\frac{1}{n^{k-1}}\right)\right\}.$$

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