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A Note On The Constructive Proof Of Kakutani's Fixed Point Theorem With Uniformly Locally At Most One Fixed Point Without Countable Choice*

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Abstract

We will prove Kakutani's fixed point theorem in an n-dimensional simplex for multi-functions which have uniformly closed graph and have *uniformly locally at most one fixed point* from the viewpoint of constructive mathematics à la Bishop without countable choice.

1 Introduction

In [7] we proved Kakutani's fixed point theorem in an n-dimensional simplex for multi-functions (multi-valued functions or correspondences) which are sequentially locally non-constant and have uniformly closed graph from the viewpoint of constructive mathematics à la Bishop ([2], [3] and [4]). But we used the so called *countable choice*. According to [5] countable choice is characterized as follows.

> Let $\mathbf{N}$ denote the set of natural numbers. For X a set and S a subset of $X \times \mathbf{N}$, consider the following two statements.

1. For all $n \in \mathbf{N}$ there exists $x \in X$ such that $(x, n) \in S$.
2. There exists a sequence of elements $x_n \in X$ such that $(x_n, n) \in S$ for all $n \in \mathbf{N}$.

> (2) implies (1). The axiom of countable choice says that (1) implies (2). This axiom asserts the existence of certain sequences in X.

Countable choice, however, is not considered sufficiently constructive. So, some authors such as [5] and [6] presented constructive analyses without countable choice. In this paper according to these studies we will prove Kakutani's fixed point theorem in an n-dimensional simplex for multi-functions which have uniformly closed graph and have *uniformly locally at most one fixed point* from the viewpoint of constructive mathematics à la Bishop without countable choice. The concept of uniformly locally

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at most one fixed point is defined by reference to the concept of uniformly at most one minimum in [6], and it is essentially equivalent to sequential local non-constancy in [7]. In [8] we present a proof of Brouwer's fixed point theorem for single valued functions without countable choice.

2 Proofs of Kakutani's Fixed Point Theorem

2.1 With Countable Choice

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S. Also in constructive mathematics compactness of a set means *total boundedness with completeness*. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \ldots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $x \in S$ there exists y in that ε-approximation with $\rho(x, y) < \varepsilon$ ($\rho(x, y)$ is the distance between x and y). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S.

Completeness of a set in constructive mathematics with countable choice means that every Cauchy sequence in the set converges.

Let us consider an n-dimensional simplex Δ as a compact metric space. About a totally bounded set, according to Corollary 2.2.12 in [4], we have the following result.

LEMMA 1. For each $\varepsilon > 0$ there exist totally bounded sets $H_1, \ldots, H_h$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^h H_i$.

Let $x = (x_0, x_1, \ldots, x_n)$ be a point in Δ with $n \geq 2$, and consider a function f from Δ into itself. If f is a uniformly continuous function from Δ into itself, according to [9] and [10] it has an approximate fixed point. This means

$$\text{For each } \varepsilon > 0 \text{ there exists } x \in \Delta \text{ such that } \rho(f(x), x) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{x \in \Delta} \rho(f(x), x) = 0.$$

By Lemma 2.1 we have $\cup_{i=1}^h H_i = \Delta$, where h is a finite number. Since H_i is totally bounded for each i, $\rho(f(x), x)$ has the infimum in H_i because of the uniform continuity of f and ρ. Thus, we can find $H_i (1 \leq i \leq h)$ such that the infimum of $\rho(f(x), x)$ in H_i is 0, that is,

$$\inf_{x \in H_i} \rho(f(x), x) = 0,$$

for some i such that $\cup_{i=1}^h H_i = \Delta$.

By reference to the notion that a function has at most one minimum in [6] we define the notion that a function has *uniformly locally at most one fixed point*. It is defined as follows.

DEFINITION 1. (A function has uniformly locally at most one fixed point) There exists $\bar{\varepsilon} > 0$ with the following property:

For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_h$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^h H_i$ and in at least one H_i such that $\inf_{x \in H_i} \rho(f(x), x) = 0$, and for any $\delta > 0$ and $x, y \in H_i$ there exists $\varepsilon > 0$ such that if $\rho(f(x), x) < \varepsilon$ and $\rho(f(y), y) < \varepsilon$, then $\rho(x, y) \leq \delta$.

Let F be a compact and convex valued multi-function from Δ to the collection of its inhabited subsets. Since Δ and $F(x)$ for $x \in \Delta$ are compact, $F(x)$ is located (see Proposition 2.2.9 in [4]), that is, $\rho(F(x), y) = \inf_{z \in F(x)} \rho(z, y)$ for $y \in \Delta$ exists. We define the notion that a multi-function has *uniformly locally at most one fixed point* as follows;

DEFINITION 2. (A multi-function has uniformly locally at most one fixed point) There exists $\bar{\varepsilon} > 0$ with the following property: For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_h$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^h H_i$ and in at least one H_i such that $\inf_{x \in H_i} \rho(F(x), x) = 0$, and for any $\delta > 0$ and $x, y \in H_i$ there exists $\varepsilon > 0$ such that if $\rho(F(x), x) < \varepsilon$ and $\rho(F(y), y) < \varepsilon$, then $\rho(x, y) \leq \delta$.

A graph of a multi-function F from Δ to the collection of its inhabited subsets is

$$G(F) = \cup_{x \in \Delta} \{x\} \times F(x).$$

If the following condition is satisfied, we say that F has a uniformly closed graph.

> For any x, x' and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\rho(x, x') < \delta$, then for any $y \in F(x)$ and some $y' \in F(x')$ $\rho(y, y') < \varepsilon$, that is, $\rho(y, F(x')) < \varepsilon$ for some $y' \in F(x')$, and for any $y' \in F(x')$ and some $y \in F(x)$ $\rho(y, y') < \varepsilon$, that is, $\rho(y', F(x)) < \varepsilon$ for some $y \in F(x)$.

A fixed point of a multi-function is defined as follows;

DEFINITION 3. x is a fixed point of a multi-function F if $x \in F(x)$.

A constructive proof of Kakutani's fixed point theorem with countable choice is as follows.

THEOREM 1. If F is a compact and convex valued multi-function with uniformly closed graph from an n-dimensional simplex Δ to the collection of its inhabited subsets and it has uniformly locally at most one fixed point, then it has a fixed point.

The proof of 2 of this theorem is based on Lemma 2 in [1].

PROOF.

1. Let Δ be an n-dimensional simplex, and consider m-th subdivision of Δ. Subdivision in a case of 2-dimensional simplex is illustrated in Figure 1. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned

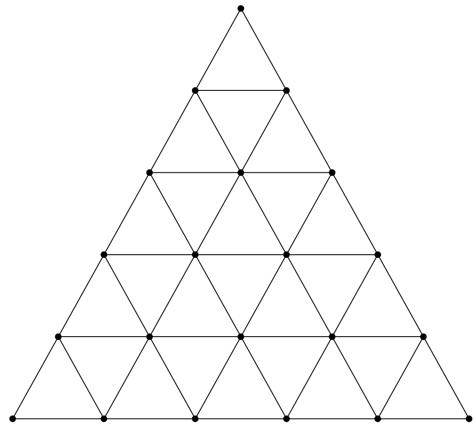

Figure 1: Subdivision of 2-dimensional simplex

into m^2 triangles. We consider subdivision of Δ inductively for cases of higher dimension.

Let us partition Δ sufficiently fine, and define a uniformly continuous function $f^m : \Delta \longrightarrow \Delta$ as follows. If x is a vertex of a simplex constructed by m-th subdivision of Δ, let $f^m(x) = y$ for some $y \in F(x)$. For other $x \in \Delta$ we define $f^m(x)$ by a convex combination of the values of F at vertices of a simplex x_0^m, x_1^m, $\dots$, x_n^m. Let $\sum_{i=0}^n \lambda_i = 1$, $\lambda_i \geq 0$,

$$f^m(x) = \sum_{i=0}^n \lambda_i f^m(x_i^m) \text{ with } x = \sum_{i=0}^n \lambda_i x_i^m.$$

Since f^m is clearly uniformly continuous, it has an approximate fixed point according to [9] and [10]. Let x^* be an approximate fixed point of f^m, then for each $\frac{\varepsilon}{2} > 0$ there exists $x^* \in \Delta$ which satisfies

$$\rho(x^*, f^m(x^*)) < \frac{\varepsilon}{2}.$$

Consider a sequence, $(\Delta_m)_{m\geq 1}$, of partition of Δ and a sequence of the distance between vertices of simplices constructed by partition $(\rho(x_i^m, x_j^m))_{m\geq 1},\ i \neq j$. Suppose $\rho(x_i^m, x_j^m) \longrightarrow 0$. Since F has a uniformly closed graph, for any $y_i^m \in F(x_i^m)$ and some $y_j^m \in F(x_j^m)$, $\rho(y_i^m, y_j^m) \longrightarrow 0$, and for any $y_j^m \in F(x_j^m)$ and some $y_i^m \in F(x_i^m)$, $\rho(y_i^m, y_j^m) \longrightarrow 0$. x^* is represented by $x^* = \sum_{i=0}^n \lambda_i x_i^m$. If $\rho(x_i^m, x_j^m) \longrightarrow 0$ for each pair of i and j ($j \neq i$), $\rho(x_i^m, x^*) \longrightarrow 0$. Thus, for any $y_i^m \in F(x_i^m)$ and some $y_i^* \in F(x^*)$, we have $\rho(y_i^m, y_i^*) < \frac{\varepsilon}{2}$. For different i, that is, different x_i^m, y_i^* may be different. But, the convexity of $F(x^*)$ implies

$$y^* = \sum_{i=0}^n \lambda_i y_i^* \in F(x^*).$$

Since, for sufficiently large m we have $\rho(y_i^m, y_i^*) < \frac{\varepsilon}{2}$ for each i, and

$$f^m(x^*) = \sum_{i=0}^{n} \lambda_i f^m(x_i^m) = \sum_{i=0}^{n} \lambda_i y_i^m,$$

we obtain $\rho(f^m(x^*), y^*) < \frac{\varepsilon}{2}$. From $\rho(x^*, f^m(x^*)) < \frac{\varepsilon}{2}$

$$\rho(x^*, y^*) < \varepsilon. \tag{1}$$

Since $y^* \in F(x^*)$, x^* is an approximate fixed point of F. ε is arbitrary, and so

$$\inf_{x^* \in \Delta} \rho(x^*, F(x^*)) = 0.$$

This means

$$\inf_{x^* \in H_i} \rho(F(x^*), x^*) = 0$$

in some H_i such that $\Delta = \cup_{i=1}^{h} H_i$.

2. Choose a sequence $(x_l)_{l \geq 1}$ in H_i such that $\rho(F(x_l), x_l) \longrightarrow 0$. Compute L such that $\rho(F(x_l), x_l) < \delta$ for all $l \geq L$. Then, for $l, l' \geq L$ we have $\rho(x_l, x_{l'}) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(x_l)_{l \geq 1}$ is a Cauchy sequence in H_i, and converges to a limit $\hat{x} \in H_i$. The uniformly closed graph property of F yields $\hat{x} \in F(\hat{x})$, and so $\hat{x}$ is a fixed point of F.

2.2 Without Countable Choice

Referring to [5] and [6] we investigate the proof of Kakutani's fixed point theorem for multi-functions with uniformly closed graph, which have uniformly locally at most one fixed point, without countable choice.

First we present the following lemma. It is based on Lemma 1 of [6].

LEMMA 2. Let Δ be an n-dimensional simplex, F be a multi-function from Δ to the collection of its inhabited subsets with uniformly closed graph, and g be a function from Δ to $\mathbb{R}$. Consider $\rho(F(x), x) = \inf_{y \in F(x)} \rho(y, x)$ for $x \in \Delta$. It is a function from Δ to $\mathbb{R}$. If $\inf_{x \in H_i} \rho(F(x), x) = 0$ in some H_i such that $\cup_{i=1}^{h} H_i = \Delta$, and F has uniformly locally at most one fixed point, then the following (a) and (b) are equivalent.

(a) For any $\delta > 0$ and $x, y \in H_i$ there exists $\varepsilon > 0$ such that if $\rho(F(x), x) < \varepsilon$ and $g(y) < \varepsilon$, then $\rho(x, y) \leq \delta$; and

(b) For any z and $\delta > 0$ and $x, y \in H_i$ there exists $\varepsilon > 0$ such that if $\rho(F(x), x) < \varepsilon$ and $g(y) < \varepsilon$, then $|\rho(x, z) - \rho(y, z)| \leq \delta$.

1. (b) is derived from (a) because $\rho(x, z) - \rho(y, z) \leq \rho(x, y)$ and $\rho(y, z) - \rho(x, z) \leq \rho(x, y)$.

2. Since F has uniformly locally at most one fixed point, for any δ, there exists η such that for any $x, z \in H_i$,

$$\text{if } \rho(F(x), x) < \eta \text{ and } \rho(F(z), z) < \eta,$$
$$\text{then } \rho(x, z) \leq \frac{\delta}{3}.$$

By (b) there exists ε such that for any $x, y \in H_i$,

$$\text{if } \rho(F(x), x) < \varepsilon \text{ and } g(y) < \varepsilon,$$
$$\text{then } |\rho(x, z) - \rho(y, z)| \leq \frac{\delta}{3}.$$

We can make $\varepsilon \leq \eta$. Then, we have

$$\begin{aligned} \rho(x, y) &\leq \rho(x, z) + \rho(y, z) \leq 2\rho(x, z) + |\rho(x, z) - \rho(y, z)| \\ &< \frac{2}{3}\delta + \frac{\delta}{3} = \delta. \end{aligned}$$

A location on H_i such that $\cup_{i=1}^{h} H_i = \Delta$ is a function $\Phi : \Delta \longrightarrow \mathbb{R}$ with $\inf_{x \in H_i} \Phi = 0$ and

$$\Phi(x) - \Phi(y) \leq \rho(x, y) \leq \Phi(x) + \Phi(y), \tag{2}$$

for all $x, y \in H_i$. It is equivalent to

$$\Phi(y) \geq |\Phi(x) - \rho(x, y)|.$$

Φ is nonnegative and uniformly continuous, and if $x \neq y$, that is, $\rho(x, y) > 0$, then either $\Phi(x) > 0$ or $\Phi(y) > 0$. Φ vanishes at most one point in H_i.

Let $\hat{H}_i$ be the set of locations on H_i. According to Theorem 3 of [5] we have the following result.

If Φ and Ψ are locations on H_i, then

$$\rho(\Phi, \Psi) = \sup_{y \in H_i} |\Phi(y) - \Psi(y)| = \inf_{x \in H_i} (\Phi(x) + \Psi(x))$$

exists and defines a metric on $\hat{H}_i$.

Every point $z \in H_i$ gives rise to the location $\hat{z}$ for each $x \in H_i$ defined by

$$\hat{z}(x) = \rho(z, x).$$

$\inf_{x \in H_i} \hat{z}(x) = 0$, and triangle inequality gives

$$\begin{aligned} \hat{z}(x) - \hat{z}(y) = \rho(z, x) - \rho(z, y) \leq \rho(x, y), \\ \rho(x, y) \leq \rho(z, x) + \rho(z, y) = \hat{z}(x) + \hat{z}(y) \end{aligned}$$

Call $\hat{z}$ as the image of z. We can identify a point in H_i with its image.

Let $\hat{w}$ be a location defined by $\hat{w}(x) = \rho(w, x)$ for some $w \neq z,\ w \in H_i$. It is the image of w. Since

$$\rho(z, w) = \inf_{x \in H_i} (\rho(z, x) + \rho(w, x)) = \inf_{x \in H_i} (\hat{z}(x) + \hat{w}(x)),$$

we have

$$\rho(z, w) = \rho(\hat{z}, \hat{w}).$$

Thus, the map from H_i into $\hat{H}_i$ is an isometry (distance preserving map). If $\Phi \in \hat{H}_i$, then we have

$$|\Phi(x) - \hat{z}(x)| = |\Phi(x) - \rho(z, x)| \leq \Phi(z)$$

for every x. $\Phi(z)$ can be arbitrarily small. Thus, the set of images of points of H_i is dense in $\hat{H}_i$, and if H_i is complete

$$\Phi = \hat{z}$$

for some $z \in H_i$. Let $h : H_i \longrightarrow \mathbb{R}$ and $a \in \mathbb{R}$ with

$$\inf_{x \in H_i} |h(x) - a| = 0.$$

For every $g : H_i \longrightarrow \mathbb{R}$,

$$\lim_{h(x) \longrightarrow a} g(x) = b$$

represents;

$$\begin{aligned} &\text{for any } \varepsilon \text{ there exists } \delta \text{ such that} \\ &|h(x) - a| < \delta \Rightarrow |g(x) - b| < \varepsilon. \end{aligned} \tag{3}$$

According to [6] a necessary and sufficient condition for the existence of a limit $\lim_{h(x) \longrightarrow a} g(x)$ of a function g is that for any ε there exists δ such that if $|h(x) - a| < \delta$ and $|h(y) - a| < \delta$, then $|g(x) - g(y)| < \varepsilon$. From (2) every location $\Phi \in \hat{H}_i$ satisfies

$$\Phi(x) = \lim_{\Phi(y) \longrightarrow 0} \rho(x, y).$$

A function $g : \Delta \longrightarrow \mathbb{R}$ extends to a mapping $\hat{g} : \hat{\Delta} \longrightarrow \hat{\mathbb{R}} = \mathbb{R}$ with

$$\hat{g}(\Phi)(r) = \lim_{\Phi(x) \longrightarrow 0} \rho(g(x), r)$$

for every $r \in \mathbb{R}$. $\hat{\Delta}$ is the set of locations on Δ (see Theorem 4 of [5]).

Now we show the following theorem. It is based on Lemma 2 and Theorem 5 of [6].

THEOREM 2.

1. Let F be a multi-function with uniformly closed graph from an n-dimensional simplex Δ to the collection of its inhabited subsets, and assume $\inf_{x \in H_i} \rho(F(x), x) = 0$ in some H_i such that $\sum_{i=1}^{h} H_i = \Delta$, and F has uniformly locally at most one fixed point. Then,

$$\Phi_\rho(x) = \lim_{\rho(F(y), y) \longrightarrow 0} \rho(x, y)$$

defines $\Phi_\rho \in \hat{H}_i$ with $\hat{\rho}(\Phi_\rho)(0) = 0$, where

$$\hat{\rho}(\Phi_\rho)(0) = \lim_{\Phi_\rho(x) \longrightarrow 0} \rho(\rho(F(x), x), 0).$$

2. f has a fixed point in H_i.

PROOF.

1. If $g(y) = \rho(F(y), y)$, the condition (a) of Lemma 2 is to say that F has uniformly locally at most one fixed point, and the condition (b) means that $\lim_{\rho(F(y),y) \longrightarrow 0} \rho(x, y)$ exists for every x. We show that Φ_ρ is a location on H_i. Since F has uniformly locally at most one fixed point, for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\rho(F(x), x) < \varepsilon \text{ and } \rho(F(y), y) < \varepsilon \Rightarrow \rho(x, y) < \delta.$$

Since $\inf_{x \in H_i} \rho(F(x), x) = 0$, for this ε there is x with $\rho(F(x), x) < \varepsilon$. If also $\rho(F(y), y) < \varepsilon$, then $\rho(x, y) < \delta$. Thus, we have the following result.

For any δ and $y \in H_i$ there exist ε and x such that
$\rho(F(y), y) < \varepsilon \ \Rightarrow \rho(x, y) < \delta.$

By triangle inequality we get

$$\Phi_\rho(y) - \Phi_\rho(z) = \lim_{\rho(F(x),x) \longrightarrow 0} (\rho(y, x) - \rho(z, x)) \leq \rho(y, z),$$

and

$$\rho(y, z) \leq \lim_{\rho(F(x),x) \longrightarrow 0} (\rho(y, x) + \rho(z, x)) = \Phi_\rho(y) + \Phi_\rho(z).$$

Thus, Φ_ρ is a location on H_i. Let us prove

$$\hat{\rho}(\Phi_\rho)(0) = \lim_{\Phi_\rho(x) \longrightarrow 0} \rho(\rho(F(x), x), 0) = \lim_{\Phi_\rho(x) \longrightarrow 0} \rho(F(x), x) = 0.$$

The last equality means that for any x and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\Phi_\rho(x) < \frac{\delta}{2}$, then $\rho(F(x), x) < \varepsilon$. We can make $\delta \leq \frac{\varepsilon}{3}$. Since F has uniformly closed graph, there exists δ such that for any x, y if $\rho(x, y) < \delta$, then $\rho(F(x), F(y)) < \frac{\varepsilon}{3}$, where

$$\rho(F(x), F(y)) = \inf_{x' \in F(x)} \rho(x', F(y)) = \inf_{x' \in F(x)} \inf_{y' \in F(y)} \rho(x', y').$$

Let x satisfy $\Phi_\rho(x) < \frac{\delta}{2}$. There exists $0 < \eta \leq \frac{\varepsilon}{3}$ such that for any y if $\rho(F(y), y) < \eta$, then $|\rho(x, y) - \Phi_\rho(x)| < \frac{\delta}{2}$. Since $\inf_{x \in H_i} \rho(F(x), x) = 0$, there exists y which satisfies $\rho(F(y), y) < \eta$. Then, we have

$$\rho(x, y) \leq \Phi_\rho(x) + |\rho(x, y) - \Phi_\rho(x)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

and

$$\begin{aligned} \rho(F(x), x) &\leq \rho(F(y), y) + \rho(x, y) + \rho(F(x), F(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

2. Since H_i is a closed subset of Δ, it is complete. Thus, Φ_ρ corresponds to some point z in H_i, that is, $\Phi_\rho = \hat{z}$ for some $z \in H_i$, and $\Phi_\rho(z) = \hat{z}(z) = 0$. Then,

$$\hat{\rho}(\Phi_\rho)(0) = \lim_{\Phi_\rho(x) \longrightarrow 0} \rho(F(x), x) = 0$$

means

$$\rho(F(z), z) = 0,$$

that is, $z \in F(z)$, and z is a fixed point of F.

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