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Maximum Norm Analysis For Nonlinear Two-Point Boundary Value Problems.*

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Abstract

In this paper, we propose a new approach for the finite difference approximation on non-uniform mesh of the nonlinear two-point boundary value problem $-(p(x)u')' = f(x,u)$, $a < x < b$; $u(a) = u(b) = 0$. Under a realistic assumption on the nonlinearity and a $C^{3,1}[a,b]$ regularity of the solution, we show that the approximation is $O(h^2)$ accurate in the maximum norm, making use of the Banach's fixed point principle.

1 Introduction

We are interested in the error estimate of the finite difference approximation of the nonlinear boundary value problem

$$\begin{cases} -(p(x)u')' = f(x,u), \ a < x < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

and its discretization by finite difference methods (FDM) on (nonuniform) nodes

$$\triangle : a = x_0 < x_1 < ... < x_n < x_{n+1} < b,$$
$$h_i = x_i - x_{i+1}, \ h = \max_i h_i.$$

Throughout the paper, we assume that $p(x) \in C^1[a,b]$, $p(x) \geq p_* > 0$ with a constant p_*, f is a nonlinearity satisfying the following condition:

$$0 \leq \frac{\partial f}{\partial u} \leq K \tag{2}$$

where K is a positive constant, and put

$$\mathcal{D} = \left\{u \in C^2[a,b] \text{ such that } u(a) = u(b) = 0\right\}. \tag{3}$$

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The usual discretization of (1) by the finite difference method is

$$\begin{cases} L_h U_i = -\dfrac{1}{\omega_i}\left[p_{i+\frac{1}{2}}\dfrac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}}\dfrac{U_i - U_{i-1}}{h_i}\right] = f(x_i, U_i), \\ U_0 = U_{n+1} = 0,\ 1 \le i \le n, \end{cases} \tag{4}$$

where

$$\omega_i = \tfrac{1}{2}(h_i + h_{i+1}),\ x_{i+\frac{1}{2}} = \tfrac{1}{2}(x_i + x_{i+1}),\ p_{i+\frac{1}{2}} = p(x_{i+\frac{1}{2}})$$

and U_i denote the approximation of the exact value $u(x_i)$. In a matrix form, problem (4) reads as follows

$$HAU = F(U) \tag{5}$$

where

$$H = diag\left(\frac{1}{\omega_1}, ...,\frac{1}{\omega_n}\right),$$

$$U = (U_1, ..., U_n)^t,\ F(U) = \left[f(x_1, U_1), ..., f(x_n, U_n)\right]^t,$$

$$A = \begin{bmatrix} a_1 + a_2 & -a_2 & & \\ -a_2 & a_2 + a_3 & -a_3 & \\ & \ddots & \ddots & -a_n \\ & & -a_n & a_n + a_{n+1} \end{bmatrix} \text{ and } a_i = \frac{1}{h_i} p_{i-\frac{1}{2}}.$$

Thanks to Yamamoto [5], the inverse matrix of A enjoys the following interesting properties:

$$A^{-1} = G(x_i, x_j) \tag{6}$$

where $G(x, \xi)$ denotes the Green's function for the operator

$$\mathcal{L} = -(pu')' : \mathcal{D} \to C[a, b]$$

and

$$|G(x_i, x_j)| \le M \tag{7}$$

where M is a is a constant independent of h.

Error estimates of the approximation by finite difference methods for problem (1) have been extensively discussed by several authors since the early sixties (cf.,e.g., references [1–6]).

In this paper, we propose a new and simple approach to derive error estimate of the above finite difference approximation. Indeed, combining assumption (2) with the properties (6) and (7), we characterize the solution of the discrete problem (5) as the fixed point of a contraction. As a result of this, we also show that, if the exact solution is in $C^{3,1}[a, b]$, then the approximation is $O(h^2)$ accurate in the maximum norm, that is

$$\max_{1 \le i \le n} |U_i - u_i| \le Ch^2$$

where $u_i = u(x_i)$ and C is a constant independent of h.

For existence and uniqueness of solutions to the continuous and discrete problems (1) and (5), we refer to [5]. Next, we shall characterize the solution of the discrete problem (5) as the unique fixed point of a contraction.

2 Characterization of the Discrete Solution as a Fixed Point of a Contraction

Consider the mapping

$$T : \mathbb{R}^n \to \mathbb{R}^n$$
$$V \to TV = \eta$$

where η is the unique solution of the linear system

$$HA\eta = F(V). \tag{8}$$

LEMMA 1. Let (6) and (7) hold. Then for h sufficiently small, we have $\rho = K \left\|A^{-1}\right\|_\infty \cdot \left\|H^{-1}\right\|_\infty < 1$.

PROOF. We know that

$$H^{-1} = diag\{\omega_1, ..., \omega_n\}.$$

Then

$$\left\|H^{-1}\right\|_\infty \leq h.$$

Since $A^{-1} = G(x_i, y_j)$ is tridiagonal and $|G(x_i, y_j)| \leq M$, we observe that

$$\left\|A^{-1}\right\|_\infty \leq 3M.$$

So, choosing h sufficiently small, we get the desired result.

THEOREM 1. Let conditions of Lemma 1 hold. Then T is a contraction with rate of contraction equal to ρ.

PROOF. Let U, V in $\mathbb{R}^n$ and $\xi = TU, \eta = TV$ be the respective solutions of equation (8) and $W = U - V$. In order to simplify notations, we denote $f(x_i, U_i)$ by $f(U_i)$ and $\dfrac{\partial f}{\partial u}$ by f'. Then, we have

$$F(U) - F(V) = [f(U_1) - f(V_1), ..., f(U_n) - f(V_n)]^t$$

and

$$\frac{1}{W_i}[f(U_i) - f(V_i)] = \int_0^1 f'(V_i + \theta(U_i - V_i))d\theta.$$

So

$$|f(U_i) - f(V_i)| \leq K |U_i - V_i|$$

and

$$\max_i |f(U_i) - f(V_i)| \leq K \max_i |U_i - V_i|$$

or

$$\|F(U) - F(V)\|_\infty \leq K \|U - V\|_\infty .$$

On the other hand, we have

$$\begin{aligned}
\|TU - TV\|_\infty &= \|\xi - \eta\|_\infty \\
&= \left\|A^{-1}H^{-1}\left(F(U) - F(V)\right)\right\|_\infty \\
&\leq \left\|A^{-1}\right\|_\infty \left\|H^{-1}\right\|_\infty \|F(U) - F(V)\|_\infty \\
&\leq \left\|A^{-1}\right\|_\infty \cdot \left\|H^{-1}\right\|_\infty K \|U - V\|_\infty \\
&\leq \rho \|U - V\|_\infty .
\end{aligned}$$

Thus, making use of Lemma 1, T is a contraction.

3 Error Estimate

Let $\bar{U} = T[u]$ be the unique solution of the linear system

$$HA\bar{U} = F([u]) \tag{9}$$

where u is the solution of problem (1), $[u] = [u_1, ..., u_n]^t$, and

$$F([u]) = [f(x_1, u_1), ..., f(x_n, u_n)]^t .$$

Then we have the following error estimate.

LEMMA 2. Assume $u \in C^{3,1}[a, b]$. Then, there exists a constant C independent of h such that

$$\left\|\bar{U} - [u]\right\|_\infty \leq Ch^2 \tag{10}$$

where

$$\left\|\bar{U} - [u]\right\|_\infty = \max_{1 \leq i \leq n} \left|\bar{U}_i - u_i\right| .$$

For more details regarding the regularity assumption $C^{3,1}[a, b]$ of u, we refer to [5].

PROOF. Since $F([u])$ is the restriction of $f(u)$ to nodes $\{x_i\}$, then problem (9) is the discrete counterpart of (1). So, making use of [6], we get the desired result.

THEOREM 2. Let u and U be the solutions of (1) and (5), respectively. Then there exists a constant C independent of h such that

$$\|[u] - U\|_\infty \leq Ch^2.$$

PROOF. Since $U = TU$ and $\bar{U} = T[u]$, making use of (10) and the fact that T is a contraction, we get

$$\begin{aligned}
\|[u] - U\|_\infty &\leq \|[u] - T[u]\|_\infty + \|T[u] - U\|_\infty \\
&\leq \left\|\bar{U} - [u]\right\|_\infty + \|T[u] - TU\|_\infty \\
&\leq Ch^2 + \rho \|[u] - U\|_\infty .
\end{aligned}$$

So,

$$\|[u] - U\|_\infty \leq \frac{Ch^2}{1-\rho}.$$

References

[1] T. N. Tikhonov and A. A. Samarskii, Homogeneous difference schemes of a high degree of accuracy nets, U.S.S.R. Comp. Math. Phys., (1961), 1465–1486.

[2] F. de Hoog and D. Jackett, On the rate of convergence of finite difference schemes on nonuniform grids, J. Austral. Math. Soc. Ser. B, 26(1985), 247–256.

[3] H. O. Kreiss, T. A. Manteuffel, B. Swarts, B. Wendroff and A. B. White, Jr., Supra-convergent schemes on irregular grids, Math. Comp., (1986), 537–554.

[4] T. A. Manteuffel and A. B. White, Jr., The numerical solution of a second order boundary value problem on nonuniform meshes, Math. Comp., 47(1986), 511–535.

[5] T. Yamamoto and S. Oishi, A Mathematical theory for numerical treatment of non-linear two-point boundary value problems, Japan J. Indust. Appl. Math., 23(2006), 31–62.

[6] T. T. Yamamoto, Discretization principles for linear two-point boundary value problems, Numer. Funct. Anal. & Optimiz., 28(2007), 149–172.

[7] F. A. M. Al Kharousi, Mathematical and Numerical Analysis of Nonlinear Two-Point Boundary Value Problems, Master Thesis, Sultan Qaboos University, (2008).