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The Convergence Of Weak Reversible Splitting Of Matrix*

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Abstract

We discuss the convergence of weak nonnegative reversible splittings and weaker reversible splittings.

1 Introduction

Consider the linear system

$$Ax = b. \tag{1}$$

For the iterative solution of system (1) it is customary to represent the matrix A as

$$A = M - N.$$

If the matrix M is nonsingular, the iterative method is expressed in the form

$$x^{(n+1)} = M^{-1}Nx^{(n)} + M^{-1}b, \quad n \geqslant 0\,. \tag{2}$$

As is well known, the above iterative scheme converges to the unique solution $x = A^{-1}b$ of system (1) for each initial vector $x^{(0)}$ if, and only if, $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ is the spectral radius of the iteration matrix $M^{-1}N$ [2].

The theory of matrix splittings plays an important role in convergence analysis for the iterative scheme (2) [2, 4, 5, 6]. Some splittings, however, are not included in definitions presented by Varga [2], Miller and Neumann [3], Woźnicki [4, 5, 6], among others, so that we cannot discuss its convergence according to the known theories. Yang [1], however, introduced the concept of reversible splittings of matrix:

DEFINITION 1 ([1]). Let $A \in R^{n\times n}$. Then the decomposition $A = M - N$ is called a reversible splitting of matrix A if M and N are nonsingular, and $\lambda_i(M^{-1}A) > 0$ for $i = 1, 2, ..., n$.

DEFINITION 2 ([1]). Let $A \in R^{n\times n}$. Then the reversible splitting $A = M - N$ is called

(a) a regular reversible splitting of A if $M^{-1} \geq 0$ and $A \geq 0$;

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(b) a nonnegative reversible splitting of A if $M^{-1} \geq 0$, $M^{-1}A \geq 0$ and $AM^{-1} \geq 0$;

(c) a weak nonnegative reversible splitting of A if $M^{-1} \geq 0$ and either $M^{-1}A \geq 0$ (the first type) or $AM^{-1} \geq 0$ (the second type);

(d) a weak reversible splitting of A if $M^{-1}A \geq 0$ and $AM^{-1} \geq 0$; and

(e) a weaker reversible splitting of A if either $M^{-1}A \geq 0$ (the first type) or $AM^{-1} \geq 0$ (the second type).

Also, the author proved that the splittings defined in the first two items of Definition 2 are convergent if $N^{-1} \geqslant 0$ [1]. In this paper, we will discuss the conditions of convergence for weak (weaker) splitting. We need the following notations:

Let λ_i for $i = 1, 2, ..., n$ be the eigenvalues of $n \times n$ complex matrix A, $\sigma(A) = \min_{1 \leq i \leq n} |\lambda_i|$, and I be the identity matrix.

2 Convergence of Weak Reversible Splitting

In this section, we will discuss the convergence of splitting defined in the third item of Definition 2.

LEMMA 1 ([2]). If M is an $n \times n$ matrix with $\rho(M) < 1$, then $I - M$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \cdots ,$$

the series on the right converges; conversely, if the series on the right converges, then $\rho(M) < 1$.

LEMMA 2 ([1]). Let $A = M - N$ be a reversible splitting of A and $\rho(M^{-1}A) < 1$. Then $\rho(M^{-1}N) < 1$.

LEMMA 3 ([1]). Let $A = M - N$ be a reversible splitting of A and $\rho(M^{-1}A) < 1$. Then

$$\rho(M^{-1}N) = 1 - \sigma(M^{-1}A).$$

THEOREM 1. Let $A = M - N$ be a weak nonnegative reversible splitting of the first type. Then the following conditions are equivalent

(a) $N^{-1} \geq 0$;

(b) $N^{-1} \geq M^{-1}$;

(c) $N^{-1}M \geq 0$;

(d) $\rho(M^{-1}A) = \dfrac{\rho(N^{-1}M) - 1}{\rho(N^{-1}M)}$;

(e) $\rho(M^{-1}A) < 1$;

(f) $(I - M^{-1}A)^{-1} \geq 0$;

(g) $N^{-1}A \geq 0$;

(h) $N^{-1}A \geq M^{-1}A$;

(i) $\rho(M^{-1}A) = \dfrac{\rho(N^{-1}A)}{1+\rho(N^{-1}A)}$.

PROOF. (a)⇒(b): From $N^{-1} \geqslant 0$ and $M^{-1}A \geqslant 0$, we have

$$M^{-1}AN^{-1} = M^{-1}(M-N)N^{-1} = N^{-1} - M^{-1} \geqslant 0,$$

that is, $N^{-1} \geq M^{-1}$.

(b)⇒(c): Obvious.

(c)⇒(d): From $A = M - N = N(N^{-1}M - I)$, we have

$$M^{-1}A = M^{-1}N(N^{-1}M - I) = (N^{-1}M)^{-1}(N^{-1}M - I). \tag{3}$$

Since $N^{-1}M \geqslant 0$, for the eigenvalue $\rho(N^{-1}M) \neq 0$ there exists an eigenvector $x \geqslant 0$(see [2]) such that

$$N^{-1}Mx = \rho(N^{-1}M)x.$$

Now by equality (3), we have

$$M^{-1}Ax = \frac{\rho(N^{-1}M) - 1}{\rho(N^{-1}M)}x,$$

i.e., $\frac{\rho(N^{-1}M)-1}{\rho(N^{-1}M)}$ is an eigenvalue of $M^{-1}A$. Hence,

$$\rho(M^{-1}A) \geqslant \frac{\rho(N^{-1}M) - 1}{\rho(N^{-1}M)}. \tag{4}$$

On the other hand, since $M^{-1}A \geqslant 0$, for the eigenvalue $\rho(M^{-1}A)$, there exists an eigenvector $y \geqslant 0$ such that

$$M^{-1}Ay = \rho(M^{-1}A)y.$$

Now by equality (3), we have

$$M^{-1}Ay = (N^{-1}M)^{-1}(N^{-1}M - I)y,$$

i.e., $\rho(M^{-1}A)$ is an eigenvalue of $(N^{-1}M)^{-1}(N^{-1}M - I)$. Hence

$$\rho(M^{-1}A) \leqslant \frac{\rho(N^{-1}M) - 1}{\rho(N^{-1}M)}. \tag{5}$$

From inequalities (4) and (5) we obtain (d).

(d)⇒(e): Obvious.

(e)⇒(f): From Lemma 1, we have

$$(I - M^{-1}A)^{-1} = \sum_{i=0}^{\infty}(M^{-1}A)^i \geqslant 0.$$

(f)⇒(g): From $N = M - A$ and $M^{-1}A \geqslant 0$, we have

$$N^{-1}A = (M - A)^{-1}A = (I - M^{-1}A)^{-1} \cdot M^{-1}A \geqslant 0.$$

(g)⇔(h): From $A = M - N$, we have

$$N^{-1}A = N^{-1}M \cdot M^{-1}A = N^{-1}(N + A) \cdot M^{-1}A = M^{-1}A + N^{-1}A \cdot M^{-1}A.$$

Then

$$N^{-1}A - M^{-1}A = N^{-1}A \cdot M^{-1}A \geqslant 0$$

because $N^{-1}A \geqslant 0$ and $M^{-1}A \geqslant 0$.

The converse is trivial because $M^{-1}A \geqslant 0$.

(g)⇒(i): Similar to (c)⇒(d), we use

$$M^{-1}A = (N + A)^{-1}A = (I + N^{-1}A)^{-1} \cdot N^{-1}A$$

instead of equality (3).

(g)⇒(c): From $A = M - N$, we have

$$N^{-1}M = N^{-1}(N + A) = I + N^{-1}A \geqslant 0$$

because it is a sum of nonnegative matrices.

(f)⇒(a): From $N = M - A$, $M^{-1} \geqslant 0$ and $(I - M^{-1}A)^{-1} \geqslant 0$, we have

$$N^{-1} = (M - A)^{-1} = (I - M^{-1}A)^{-1} \cdot M^{-1} \geqslant 0.$$

REMARK 1. The above theorem also holds if we replace “first type" by “second type" and matrices $N^{-1}M$, $M^{-1}A$ and $N^{-1}A$ by MN^{-1}, AM^{-1} and AN^{-1} respectively.

REMARK 2. By Lemma 3 and items (d) and (i) of Theorem 1, the spectral radius $\rho(M^{-1}N)$ can be obtained as follows:

$$\rho(M^{-1}N) = \frac{1}{\sigma(N^{-1}M)} \tag{6}$$

or

$$\rho(M^{-1}N) = \frac{1}{1 + \sigma(N^{-1}A)}. \tag{7}$$

Theorem 1 provides some sufficient conditions for a weak nonnegative reversible splitting both of the first and the second type to be convergent, and we can see that, as pointed out in [1], the condition $N^{-1} \geqslant 0$ also plays an important role in the convergence of this type of reversible splitting, in fact, the splittings defined in first three items of Definition 2 are convergent if and only if $N^{-1} \geqslant 0$, which means that both conditions $N^{-1} \geqslant 0$ and $\rho(M^{-1}N) < 1$ are equivalent.

EXAMPLE 1. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = M - N$ where

$$M = \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix} \text{ and } N = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

It is a weak nonnegative reversible splitting of the first type [1], and

$$N^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \geqslant 0.$$

By Theorem 1, we know that the splitting is convergent. In fact, we have

$$M^{-1}A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \geqslant 0 \text{ and } N^{-1}A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \geqslant 0.$$

From $\sigma(M^{-1}A) = \dfrac{1}{3}$ and $\sigma(N^{-1}A) = \dfrac{1}{2}$, we obtain

$$\rho(M^{-1}N) = 1 - \sigma(M^{-1}A) = \frac{2}{3} < 1 \text{ or } \rho(M^{-1}N) = \frac{1}{1 + \sigma(N^{-1}A)} = \frac{2}{3} < 1\,.$$

As for weak and weaker splittings, the assumption $N^{-1} \geqslant 0$ is not a sufficient condition in order to ensure the convergence of a given splitting of matrix A; it is also possible to construct a convergent weak or weaker splitting even if $N^{-1} \ngeqslant 0$.

EXAMPLE 2. Let $A = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix} = M - N$ where

$$M = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} \text{ and } N = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Evidently, M, N are nonsingular,

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} \ngeqslant 0,\ N^{-1} = \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} \ngeqslant 0,$$

$$M^{-1}A = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \geqslant 0 \text{ and } AM^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \geqslant 0.$$

From Definition 2, it is a weak reversible splitting, and although $N^{-1} \ngeqslant 0$, the splitting is convergent because we have $\rho(M^{-1}N) = 1 - \sigma(M^{-1}A) = 1 - \dfrac{1}{3} = \dfrac{2}{3} < 1$.

EXAMPLE 3. Let $A = \begin{pmatrix} \frac{1}{2} & 2 \\ -\frac{1}{2} & -4 \end{pmatrix} = M - N$ where

$$M = \begin{pmatrix} \frac{3}{2} & 6 \\ -\frac{1}{4} & -2 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 4 \\ \frac{1}{4} & 2 \end{pmatrix}.$$

Evidently M, N are nonsingular,

$$M^{-1} = \begin{pmatrix} \frac{4}{3} & 4 \\ -\frac{1}{6} & -1 \end{pmatrix} \not\geqslant 0 \text{ and } N^{-1} = \begin{pmatrix} 2 & -4 \\ -\frac{1}{4} & 1 \end{pmatrix} \not\geqslant 0,$$

and

$$M^{-1}A = \begin{pmatrix} -\frac{4}{3} & -\frac{40}{3} \\ \frac{5}{12} & \frac{11}{3} \end{pmatrix} \not\geqslant 0 \text{ and } AM^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix} \geqslant 0.$$

It is a weaker reversible splitting of the second type. Here $N^{-1} \not\geqslant 0$, and the splitting does not converge because we can easily obtain that $\rho(M^{-1}N) = 1$.

EXAMPLE 4. Let $A = \begin{pmatrix} 4 & 0 & 1 \\ -1 & 0 & 2 \\ 1 & 3 & 3 \end{pmatrix} = M - N$ where

$$M = \begin{pmatrix} 12 & -36 & 7 \\ -3 & 9 & 14 \\ 3 & -\frac{9}{2} & 21 \end{pmatrix} \text{ and } N = \begin{pmatrix} 8 & -36 & 6 \\ -2 & 9 & 12 \\ 2 & -\frac{15}{2} & 18 \end{pmatrix}.$$

It is also a weaker reversible splitting of the first type [1], and although

$$N^{-1} = \begin{pmatrix} -\frac{14}{9} & -\frac{67}{18} & 3 \\ -\frac{10}{27} & -\frac{22}{27} & \frac{2}{3} \\ \frac{1}{54} & \frac{2}{27} & 0 \end{pmatrix} \not\geqslant 0,$$

the splitting is convergent (see Example 7).

Therefore we need to establish other sufficient conditions for a weaker reversible splitting.

3 Convergence of Weaker Reversible Splitting

We have

THEOREM 2. Let $A = M - N$ be a weaker reversible splitting of the first type. Then the following inequality holds if $N^{-1}A \geqslant 0$,

$$\rho(M^{-1}A) = \frac{\rho(N^{-1}A)}{1 + \rho(N^{-1}A)} < 1\,. \tag{8}$$

Conversely, if $\rho(M^{-1}A) < 1$, then $N^{-1}A \geqslant 0$.

PROOF. The proof of equality (8) under the condition $N^{-1}A \geqslant 0$ can be accomplished just as (g)$\Rightarrow$(i) of Theorem 2. Conversely, from $A = M - N$ we have

$$N = M - A = M(I - M^{-1}A).$$

If $\rho(M^{-1}A) < 1$, then from Lemma 1 and Definition 1, it follows that

$$\begin{aligned} N^{-1}A &= [M(I - M^{-1}A)]^{-1}A = (I - M^{-1}A)^{-1} \cdot M^{-1}A \\ &= \sum_{i=1}^{\infty}(M^{-1}A)^{i+1} \geqslant 0. \end{aligned}$$

The proof is complete.

REMARK 3. Theorem 2 also holds if we replace "first type" by "second type" and matrix $N^{-1}A$ by AN^{-1}.

As an immediate consequence of Lemma 2 and Theorem 2, we obtain the following result.

COROLLARY 1. Let $A = M - N$ be a weaker reversible splitting of the first (second) type. If $N^{-1}A \geqslant 0$ ($AN^{-1} \geqslant 0$), then $\rho(M^{-1}N) < 1$.

Similar to the proof (c)$\Rightarrow$ (d) of Theorem 2, we have the following

THEOREM 3. Let $A = M - N$ be a weaker reversible splitting of the first type. Then the following inequality holds if $N^{-1}M \geqslant 0$ and

$$\rho(M^{-1}A) = \frac{\rho(N^{-1}M) - 1}{\rho(N^{-1}M)} < 1.$$

Conversely, if $\rho(M^{-1}A) < 1$, then $N^{-1}M \geqslant 0$.

REMARK 4. Theorem 3 also holds if we replace "first type" by "second type" and matrix $N^{-1}M$ by MN^{-1}.

COROLLARY 2. Let $A = M - N$ be a weaker reversible splitting of the first (second) type. If $N^{-1}M \geqslant 0$ ($MN^{-1} \geqslant 0$), then $\rho(M^{-1}N) < 1$.

REMARK 5. For a weak reversible splitting defined in the fourth item of Definition 2, we can prove that it is convergent if $N^{-1}M \geqslant 0$ and $N^{-1}A \geqslant 0$, and from

$$N^{-1}M = N^{-1}(N + A) = I + N^{-1}A$$

we know that $N^{-1}M \geqslant 0$ if $N^{-1}A \geqslant 0$, which implies the following result.

COROLLARY 3. Let $A = M - N$ be a weak reversible splitting. If $N^{-1}A \geqslant 0$, then $\rho(M^{-1}N) < 1$.

EXAMPLE 5. In the weak reversible splitting given in Example 2, where

$$N^{-1}A = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} \geqslant 0,$$

by Corollary 3, we know that the splitting converges.

EXAMPLE 6. In the weaker reversible splitting of the second type given in Example 3, where

$$AN^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix} \ngeqslant 0 \text{ and } MN^{-1} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & -1 \end{pmatrix} \ngeqslant 0,$$

by Corollary 1 or Corollary 2, we know that the splitting does not converge.

EXAMPLE 7. In the weaker reversible splitting of the first type given in Example 4, where

$$N^{-1}M = \begin{pmatrix} \frac{3}{2} & 9 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{7}{6} \end{pmatrix} \geqslant 0 \text{ and } N^{-1}A = \begin{pmatrix} \frac{1}{2} & 9 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} \geqslant 0$$

by Corollary 1 or Corollary 2, we know that the splitting is convergent. In fact, from the above matrices we know that

$$\sigma(N^{-1}M) = \frac{7}{6} \text{ and } \sigma(N^{-1}A) = \frac{1}{6},$$

hence from equalities (6) and (7) we obtain

$$\rho(M^{-1}N) = \frac{1}{\sigma(N^{-1}M)} = \frac{6}{7} < 1 \text{ and } \rho(M^{-1}N) = \frac{1}{1+\sigma(N^{-1}A)} = \frac{6}{7} < 1.$$

4 Notes

According to Definition 2, we know that "if $M^{-1} \geqslant 0$ and $A \geqslant 0$, then $M^{-1}A \geqslant 0$", that is, if $A = M - N$ is a regular reversible splitting of matrix A, it must be a weak nonnegative reversible splitting of A, but the converse is not true. See Example 1, where $A = M - N$ is a weak nonnegative reversible splitting of the first type, and we know that any splittings of matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

are not regular reversible splitting because $A \ngeqslant 0$, so we cannot discuss the rates of convergence of the above two kinds of reversible splittings under the condition $A \ngeqslant 0$.

On the other hand, if $A \geqslant 0$, then the regular reversible splitting of matrix A is equivalent to the weak nonnegative reversible splitting, so they have the same rates of convergence.

In fact, we can compare the speed of convergence of two different (and evidently convergent) reversible splittings of the same type, for example, let

$$A = M_1 - N_1 = M_2 - N_2$$

be two regular reversible splittings of matrix A, then we can discuss which one of the two splittings will converge faster. The theories about them will be studied elsewhere.

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