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Lorentzian Bobillier Formula*

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Abstract

In this paper, Lorentzian Euler-Savary Formula (giving the relation between the curvatures of the trajectory curves drawn by the points of the moving plane in the fixed plane) during one parameter Lorentzian planar motion is taken into consideration. By using an original geometrical interpretation of Lorentzian Euler-Savary Formula, Lorentzian Bobillier Formula is established.

However, another presentation is made in this paper without using the Euler-Savary Formula. Then the Lorentzian Euler-Savary Formula will appear as a particular case of Bobillier Formula and as a result of the direct way chosen, this new Lorentzian formula (Bobillier) can be considered as a fundamental law in a planar Lorentzian motion in place of Euler-Savary's.

1 Introduction

In 1988, M. Fayet presented a new formula relative to the curvatures in an one parameter planar Euclidean motion [1]. This formula is called Bobillier's Formula which analytically solves the problem that Bobillier's construction solved graphically as given in [2] and [3]. Bobillier's well known theorem on the centers of curvature was the first theorem concerning second order properties of general motion.

In [4] it is proved that the Bobillier Formula may also be obtained without using the Euler-Savary Formula which is derived by Euler in 1765 and Savary in 1845 and this relation (Euler-Savary Formula) is well documented in the literature [5] and [6]. In [7] the Bobillier Formula is proved and also illustrated by elementary tasks. Also in [8] Bobillier Formula which is concerned with second order properties of one parameter planar motions in the complex plane is established.

By taking the Lorentzian plane instead of the Euclidean plane, Ergin introduced one parameter planar motion and gave the relations between the velocities, accelerations and pole curves of this Lorentzian motion. In the Lorentzian plane Euler-Savary Formula is studied in references [9-13].

To the best of authors' knowledge Bobillier Formula for the Lorentzian planar motion is not studied yet. Thus, the study is proposed to serve such a need.

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2 Lorentzian Planar Motions and Lorentzian Euler-Savary Formula

Let P_0 and P_1 be fixed and moving planes in Lorentzian space respectively. The perpendicular coordinate system of the planes P_0 and P_1 are $\{O_0; \vec{p}_{01}, \vec{p}_{02}\}$ and $\{O_1; \vec{p}_{11}, \vec{p}_{12}\}$, respectively. If we suppose that M, M' and M'' are nonnull (timelike or spacelike) points linked to moving plane P_1, then the conjugate points γ, γ' and γ'' of these nonnull points are curvature centers of the trajectory drawn by M, M' and M'' in the fixed plane P_0.

The normals of this trajectory pass from an instantaneous center of rotation denoted by I and called as pole point. At each t moment there is a rotation pole and the geometric locus of the pole points is called fixed pole curve C_0 in the plane P_0 and moving pole curve C_1 in the plane P_1 during the one-parameter Lorentzian motion $P_1 \backslash P_0$(see Figures 2.1 and 2.2).

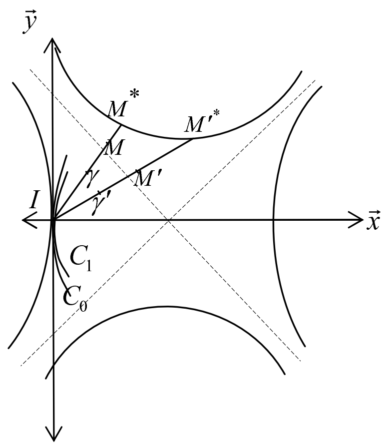

Figure 2.1 Timelike $\overrightarrow{IM}, \overrightarrow{IM'}$ vectors

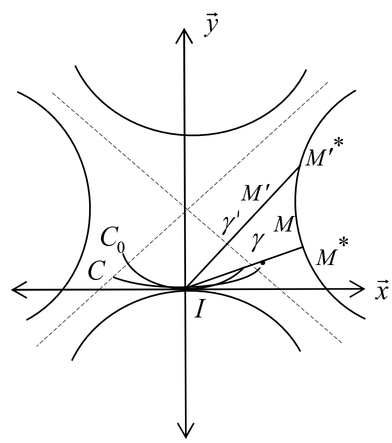

Figure 2.2 Spacelike $\overrightarrow{IM}, \overrightarrow{IM'}$ vectors

If θ is a hyperbolic angle of Lorentzian motion of P_1 with respect to P_0 at each t moment, then each nonnull point M linked to P_1 makes a rotation motion with $\dot{\theta}$ angular velocity at the center I. The pole curves C_0 and C_1 roll upon each other without sliding during the one parameter Lorentzian planar motion, namely, C_0 and C_1 pole curves are always tangent to each other and have the same velocity at each t moment.

Since the causal character of a curve is determined with respect to the causal character of the tangent of this curve in Lorentzian plane, C_0 and C_1 are timelike if the common tangent of these curves is timelike (see Figure 2.1) or C_0 and C_1 are spacelike if the common tangent of these curves is spacelike (see Figure 2.2).

It is seen from Figure 2.1 that γ, γ' and γ'' are timelike curvature centers of a trajectory drawn by the timelike points M, M' and M'' linked to the moving plane P_1 in the fixed plane P_0. Also, $\vec{x}$ and $\vec{y}$ are, respectively, common normal and common tangent of timelike pole curves C_0 and C_1. In Figure 2.2, it is indicated that γ, γ' and γ'' are spacelike curvature centers of a trajectory in the fixed plane and this trajectory is drawn by spacelike points M, M' and M'' linked to moving plane P_1. The common tangent and the common normal of the spacelike pole curves C_0 and C_1 are $\vec{x}$ and $\vec{y}$, respectively.

From now on, we will investigate these two different situations together.

Let $\overrightarrow{X}$, $\overrightarrow{X'}$ and $\overrightarrow{X''}$ be timelike (spacelike) unit vectors, then these unit vectors can be given as follows:

$$\overrightarrow{X} = \frac{\overrightarrow{IM}}{\left\|\overrightarrow{IM}\right\|},\ \overrightarrow{X'} = \frac{\overrightarrow{IM'}}{\left\|\overrightarrow{IM'}\right\|},\ \overrightarrow{X''} = \frac{\overrightarrow{IM''}}{\left\|\overrightarrow{IM''}\right\|}, \tag{1}$$

see Figure 2.3 (resp. see Figure 2.4).

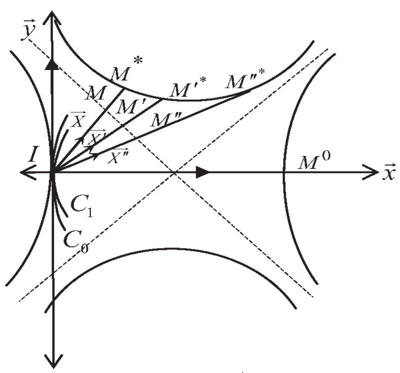

Figure 2.3 Timelike $\overrightarrow{X}, \overrightarrow{X'}$ and $\overrightarrow{X''}$ vectors

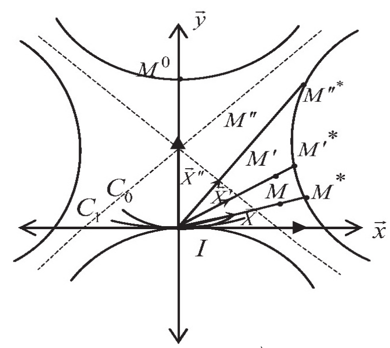

Figure 2.4 Spacelike $\overrightarrow{X}, \overrightarrow{X'}$ and $\overrightarrow{X''}$ vectors

If the abscissa of γ and M timelike (spacelike) points on the axis (I, X) are ρ_1 and ρ_0, respectively, then there are the relationships

$$\left\langle \overrightarrow{I\gamma}, \overrightarrow{X} \right\rangle = \varepsilon\rho_0, \left\langle \overrightarrow{IM}, \overrightarrow{X} \right\rangle = \varepsilon\rho_1 \tag{2}$$

where $\varepsilon = -1$ if the pole curves are timelike or $\varepsilon = +1$ if the pole curves are spacelike. Similarly, we can give

$$\left\langle \overrightarrow{I\gamma'}, \overrightarrow{X}' \right\rangle = \varepsilon\rho_0', \left\langle \overrightarrow{IM'}, \overrightarrow{X}' \right\rangle = \varepsilon\rho_1'$$

and

$$\left\langle \overrightarrow{I\gamma''}, \overrightarrow{X}'' \right\rangle = \varepsilon\rho_0'' , \left\langle \overrightarrow{IM''}, \overrightarrow{X}'' \right\rangle = \varepsilon\rho_1''.$$

3 Inflection Points and Inflection Circle

An inflection point may be defined to be a point whose trajectory momentarily has an infinite radius of curvature [14]. Such points also have zero acceleration normal to their trajectory. Let the inflection points, by referring to Figure 2.3 and Figure 2.4 be

M^*, M'^* and M''^*. The locus of such points is a circle in the Lorentzian plane called as an inflection circle. The abscissae of the inflection points can be written:

$$\left\langle \overrightarrow{IM^*}, \overrightarrow{X} \right\rangle = \varepsilon\rho, \left\langle \overrightarrow{IM'^*}, \overrightarrow{X}' \right\rangle = \varepsilon\rho', \left\langle \overrightarrow{IM''^*}, \overrightarrow{X}'' \right\rangle = \varepsilon\rho'' \tag{3}$$

Let h be a distance from a timelike (spacelike) point M^0 on the hyperbolic (respectively Lorentzian) inflection circle at the direction of the common normal to the instantaneous rotation center I, see Figure 2.3 (Figure 2.4). Then there is a relationship between h and ρ as follows

$$h \sinh\theta = \rho$$

where θ is a hyperbolic angle of the motion $P_1 \backslash P_0$. If the canonical relative systems of a plane with respect to other planes are taken into consideration then the Lorentzian Euler-Savary Formula can be constructed for timelike and spacelike pole curves, separately. In [9] it is proved that this formula remains unchanged whether the pole curves are spacelike or timelike.

The Lorentzian Euler-Savary Formula, which gives the relation between the curvatures of the trajectory curves drawn by the points of the moving plane in fixed plane, is

$$\left(\tfrac{1}{\rho_1} - \tfrac{1}{\rho_0}\right) \sinh\theta = \tfrac{1}{R_1} - \tfrac{1}{R_0} \tag{4}$$

where R_0 and R_1 are the abscissa (ordinates) on $(O, \vec{x})$ (on $(O, \vec{y})$) of the curvature centers of the timelike (spacelike) pole curves C_0 and C_1, respectively. Also, ρ_0 and ρ_1 are the distance from the timelike (spacelike) points γ and M to the center I, respectively, see Figure 3.1 (see Figure 3.2).

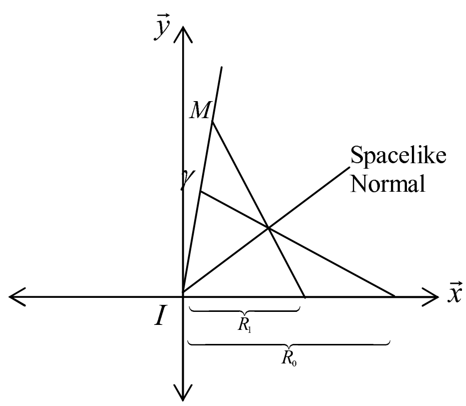

Figure 3.1 R_0 and R_1 lengths

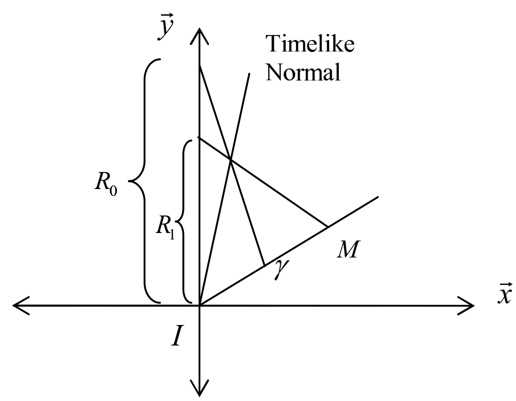

Figure 3.2 R_0 and R_1 lengths

Since there is the relation $\frac{1}{\rho} = \frac{1}{\rho_1} - \frac{1}{\rho_0}$ the formula given by the equation (4) is

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right) \sinh\theta = \frac{1}{R_1} - \frac{1}{R_0} = \frac{1}{h}$$

in which $\frac{1}{h} = \frac{1}{R_1} - \frac{1}{R_0}$ (first form) or $\frac{1}{h} = \pm\frac{\omega}{V}$ (second form) where ω is the angular velocity of the motion of the plane P_1 with respect to P_0 and V is the common velocity of I on the pole curves C_0 and C_1.

4 Lorentzian Bobillier Formula from the Lorentzian Euler-Savary Formula

If we consider the timelike (spacelike) points Q, Q', Q'' and Q^0 defined by

$$\overrightarrow{IQ} = \varepsilon\frac{1}{\rho}\overrightarrow{X}, \quad \overrightarrow{IQ'} = \varepsilon\frac{1}{\rho'}\overrightarrow{X'}, \quad \overrightarrow{IQ''} = \varepsilon\frac{1}{\rho''}\overrightarrow{X''}, \quad \overrightarrow{IQ^0} = \varepsilon\frac{1}{h}\overrightarrow{x} \tag{5}$$

where $\varepsilon = -1$ if C_0 and C_1 are timelike and $\varepsilon = +1$ if C_0 and C_1 are spacelike pole curves, the timelike (spacelike) points Q, Q' Q'' and Q^0 are images of the timelike (spacelike) points M^*, M'^*, M''^* and M^0 of hyperbolic (Lorentzian) inflection circle which respectively belong to $\left(I, \overrightarrow{X}\right)$, $\left(I, \overrightarrow{X'}\right)$, $\left(I, \overrightarrow{X''}\right)$ and $(I, \overrightarrow{x})$. Hence, the following equations are obtained as follows,

$$\left\langle \overrightarrow{IQ}, \overrightarrow{X} \right\rangle = \frac{1}{\rho}, \quad \left\langle \overrightarrow{IQ'}, \overrightarrow{X}' \right\rangle = \frac{1}{\rho'}, \quad \left\langle \overrightarrow{IQ''}, \overrightarrow{X}'' \right\rangle = \frac{1}{\rho''}, \quad \left\langle \overrightarrow{IQ^0}, \overrightarrow{x} \right\rangle = \frac{1}{h},$$

see Figure 4.1 (see Figure 4.2).

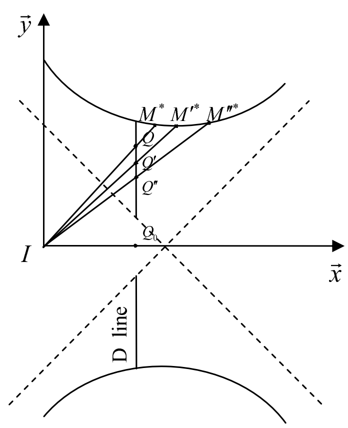

Figure 4.1 Timelike Q points

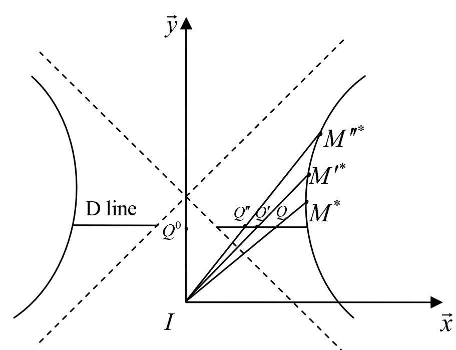

Figure 4.2 Spacelike Q points

From the equations (5) and $h sinh\theta = \rho$, the relationship

$$\overrightarrow{IQ}\sinh\theta = \frac{1}{\rho}\overrightarrow{X}\sinh\theta = \frac{1}{h}\overrightarrow{X}$$

is obtained. Similarly,

$$\overrightarrow{IQ'}\sinh\theta' = \frac{1}{\rho'}\overrightarrow{X'}\sinh\theta' = \frac{1}{h}\overrightarrow{X'}$$

and

$$\overrightarrow{IQ''}\sinh\theta'' = \frac{1}{\rho''}\overrightarrow{X''}\sinh\theta'' = \frac{1}{h}\overrightarrow{X''}$$

are given. If the last three equations are taken into consideration it is easily seen that

$$\left\langle \overrightarrow{IQ}, \overrightarrow{X} \right\rangle \sinh\theta = \left\langle \overrightarrow{IQ'}, \overrightarrow{X'} \right\rangle \sinh\theta' = \left\langle \overrightarrow{IQ''}, \overrightarrow{X''} \right\rangle \sinh\theta'' = \frac{1}{h}.$$

This means that the set of the timelike (spacelike) points Q is a straight line D parallel to axis $\vec{y}$ (axis $\vec{x}$). Thus the line D is an image of the hyperbolic (Lorentzian) inflection circle by this inversion at the rotation center I, see Figure 3.1 (Figure 3.2).

Since the timelike (spacelike) vectors $\left(\overrightarrow{IQ} - \overrightarrow{IQ'}\right)$ and $\left(\overrightarrow{IQ'} - \overrightarrow{IQ''}\right)$ are linearly dependent, the Lorentzian cross product of these vectors is

$$(\overrightarrow{IQ} \times \overrightarrow{IQ'}) - (\overrightarrow{IQ'} \times \overrightarrow{IQ'}) - (\overrightarrow{IQ} \times \overrightarrow{IQ''}) + (\overrightarrow{IQ'} \times \overrightarrow{IQ''}) = \overrightarrow{0}.$$

See Figure 4.3 (Figure 4.4).

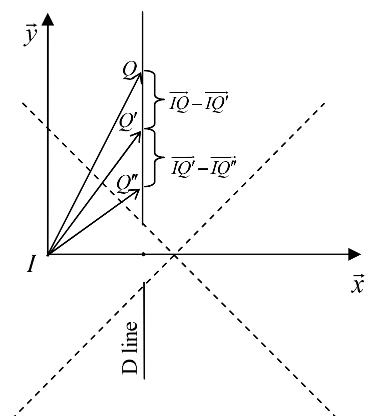

Figure 4.3 $\overrightarrow{IQ} - \overrightarrow{IQ'}$ and $\overrightarrow{IQ'} - \overrightarrow{IQ''}$ vectors

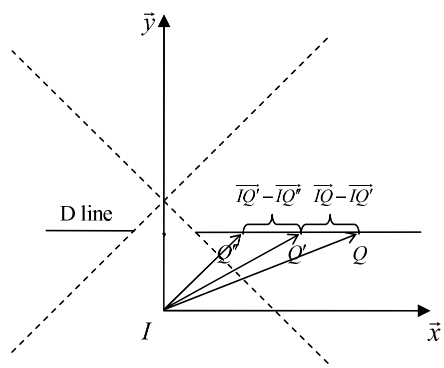

Figure 4.4 $\overrightarrow{IQ} - \overrightarrow{IQ'}$ and $\overrightarrow{IQ'} - \overrightarrow{IQ''}$ vectors

Then from the last equation, we obtain

$$\varepsilon^2\left(\frac{1}{\rho}\overrightarrow{X} \times \frac{1}{\rho'}\overrightarrow{X'}\right) + \varepsilon^2\left(\frac{1}{\rho''}\overrightarrow{X''} \times \frac{1}{\rho}\overrightarrow{X}\right) + \varepsilon^2\left(\frac{1}{\rho'}\overrightarrow{X'} \times \frac{1}{\rho''}\overrightarrow{X''}\right) = \overrightarrow{0}.$$

Since $\rho\rho'\rho'' \neq 0$ and $\varepsilon^2 = 1$, we write

$$\rho''\left(\overrightarrow{X} \times \overrightarrow{X'}\right) + \rho'\left(\overrightarrow{X} \times \overrightarrow{X''}\right) + \rho\left(\overrightarrow{X'} \times \overrightarrow{X''}\right) = \vec{0}.$$

If the definition of Lorentzian cross product is taken into consideration, then the last equation becomes

$$\rho \sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right) + \rho' \sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right) + \rho'' \sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right) = 0$$

where $\frac{1}{\rho} = \frac{1}{\rho_1} - \frac{1}{\rho_0}$, $\frac{1}{\rho'} = \frac{1}{\rho'_1} - \frac{1}{\rho'_0}$ and $\frac{1}{\rho''} = \frac{1}{\rho''_1} - \frac{1}{\rho''_0}$.

This last equation is called Lorentzian Bobillier Formula which is totally based on Lorentzian Euler-Savary Formula.

5 Direct Approach Towards the Lorentzian Bobillier Formula

The following approach allows us to obtain the Lorentzian Bobillier Formula, directly. Then the Lorentzian Euler-Savary Formula appears as a particular case of this formula.

Let $\overrightarrow{V}^0(M)$ and $\overrightarrow{J}^0(M)$ be absolute velocity vector and absolute acceleration vector of the timelike (spacelike) point M, respectively. If ω is the angular velocity of the motion $P_1 \backslash P_0$ then $\omega = \frac{\Delta\theta}{\Delta t}$ where θ is the rotation angle. By taking a unit vector $\vec{z}$ which is orthogonal to the planes P_0 and P_1 the angular velocity vector can be defined by $\vec{\omega} = \omega \vec{z}$. On the other hand the sliding velocity vector of the point M is

$$\overrightarrow{V}_1(M) = \overrightarrow{\omega} \times \overrightarrow{IM}.$$

During one parameter Lorentzian planar motion the relationship

$$\overrightarrow{V}^0(M) = \overrightarrow{V}_1^0(I) + \overrightarrow{V}_1(M) \tag{6}$$

holds where $\overrightarrow{V}^0(M), \overrightarrow{V}_1^0(I)$ and $\overrightarrow{V}_1(M)$ denote the absolute, relative and sliding velocity vectors of the motion, $P_1 \backslash P_0$ respectively [14]. If we substitute the equation (6) into the last equation we obtain

$$\overrightarrow{V}^0(M) = \overrightarrow{V}_1^0(I) + (\overrightarrow{\omega} \times \overrightarrow{IM}). \tag{7}$$

The differentiation of the equation (7) with respect to time t is

$$\overrightarrow{J}^0(M) = \overrightarrow{J}_1^0(I) + \left(\dot{\omega}\overrightarrow{z} \times \overrightarrow{IM}\right) + \left(\omega\overrightarrow{z} \times \left(\omega\overrightarrow{z} \times \overrightarrow{IM}\right)\right) \tag{8}$$

where $\overrightarrow{J}_1^0(I)$ is the acceleration vector of the point M on P_1 that coincides instantaneously with I. Here the first term is the trajectorywise invariant acceleration component, the second term is tangential acceleration component, and the third term is centripetal component.

If the Lagrange identity in the sense of Lorentz is taken into consideration, then the equation (8) becomes

$$\overrightarrow{J}^0(M) = \overrightarrow{J}_1^0(I) + \left(\dot{\omega}\overrightarrow{z} \times \overrightarrow{IM}\right) + \langle \omega\vec{z}, \omega\overrightarrow{z}\rangle \overrightarrow{IM} - \left\langle \overrightarrow{IM}, \omega\overrightarrow{z}\right\rangle \omega\overrightarrow{z}.$$

Since $\overrightarrow{IM}$ is orthogonal to the angular velocity vector, $\left\langle \overrightarrow{IM}, \omega\overrightarrow{z} \right\rangle = 0$. On the other hand $\langle \omega\overrightarrow{z}, \omega\overrightarrow{z} \rangle = -\varepsilon\omega^2$ where $\varepsilon = -1$ if $\overrightarrow{z}$ is spacelike or $\varepsilon = 1$ if $\vec{z}$ is timelike. Therefore we get

$$\overrightarrow{J}^{0}(M) = \overrightarrow{J}_{1}^{0}(I) + \left(\dot{\omega}\overrightarrow{z} \times \overrightarrow{IM}\right) - \varepsilon\omega^2\overrightarrow{IM}. \tag{9}$$

By considering the analysis of the equation (9) for the inflection points whose acceleration normal is zero then the absolute velocity and acceleration vector of the point M^* on the hyperbolic (Lorentzian) circle become linearly dependent, that is

$$\overrightarrow{V}^{0}(M^*) \times \overrightarrow{J}^{0}(M^*) = \overrightarrow{0}.$$

If we substitute the equations (7) and (9) into the last equation, we find the following equation

$$\left(\overrightarrow{V}_{1}^{0}(I) + \left(\omega\overrightarrow{z} \times \overrightarrow{IM^*}\right)\right) \times \left(\overrightarrow{J}_{1}^{0}(I) + \left(\dot{\omega}\overrightarrow{z} \times \overrightarrow{IM^*}\right) - \varepsilon\omega^2\overrightarrow{IM^*}\right) = \vec{0}.$$

By applying the Lorentzian cross product and considering $\overrightarrow{V_1^0}(I) = \vec{0}$, we obtain

$$\begin{array}{l} -\omega\left\langle \overrightarrow{z}, \overrightarrow{J}_{1}^{0}(I)\right\rangle \overrightarrow{IM^*} + \omega\left\langle \overrightarrow{IM^*}, \overrightarrow{J}_{1}^{0}(I)\right\rangle \overrightarrow{z} + \omega\dot{\omega}\left(\left(\overrightarrow{z} \times \overrightarrow{IM^*}\right) \times \left(\overrightarrow{z} \times \overrightarrow{IM^*}\right)\right) \\ +\omega^3\varepsilon\left\langle \overrightarrow{z}, \overrightarrow{IM^*}\right\rangle \overrightarrow{IM^*} - \varepsilon\omega^3\left\langle \overrightarrow{IM^*}, \overrightarrow{IM^*}\right\rangle \overrightarrow{z} = \vec{0}. \end{array}.$$

It is known that the relationships

$$\left\langle \overrightarrow{z}, \overrightarrow{J}_{1}^{0}(I)\right\rangle = 0, \left\langle \overrightarrow{z}, \overrightarrow{IM^*}\right\rangle = 0, \left\|\overrightarrow{IM^*}\right\|^2 = \varepsilon\left\langle \overrightarrow{IM^*}, \overrightarrow{IM^*}\right\rangle$$

and

$$\left(\overrightarrow{z} \times \overrightarrow{IM^*}\right) \times \left(\overrightarrow{z} \times \overrightarrow{IM^*}\right) = \vec{0}$$

hold. Then we find that

$$\left\langle \overrightarrow{IM^*}, \overrightarrow{J}_{1}^{0}(I)\right\rangle \overrightarrow{z} - \varepsilon^2\omega^2\left\|\overrightarrow{IM^*}\right\|^2 \overrightarrow{z} = \vec{0}.$$

There is always a constant hyperbolic angle between the timelike (spacelike)vector $\overrightarrow{J}_{1}^{0}(I)$ and the timelike (spacelike) normal vector $\overrightarrow{IM^*}$. These vectors are on the same branch of the Lorentzian (hyperbolic) circle. Let us denote this angle by α. So, the last equation becomes

$$\varepsilon\left\|\overrightarrow{IM^*}\right\| J_1^0(I)\cosh\alpha - \omega^2\left\|\overrightarrow{IM^*}\right\|^2 = 0.$$

where $\varepsilon^2 = 1$. From equation (3),

$$\varepsilon\rho J_1^0(I)\cosh\alpha - \omega^2\rho^2 = 0,$$

is obtained. After some rearrangements it becomes

$$\rho = \varepsilon \frac{J_1^0(I)\cosh\alpha}{\omega^2}.$$

Since the hyperbolic angle α is also an angle between the timelike (spacelike) vectors $\overrightarrow{J}_1^0(I)$ and $\overrightarrow{X}$ on the same branch of the Lorentzian (hyperbolic) circle, it is found that

$$\rho = \frac{\left\langle \overrightarrow{J}_1^0(I), \overrightarrow{X} \right\rangle}{\omega^2}. \tag{10}$$

The analogous equations can be written for points M' and M'' as

$$\rho' = \frac{\left\langle \overrightarrow{J}_1^0(I), \overrightarrow{X'} \right\rangle}{\omega^2}, \tag{11}$$

and

$$\rho'' = \frac{\left\langle \overrightarrow{J}_1^0(I), \overrightarrow{X''} \right\rangle}{\omega^2}. \tag{12}$$

So, from the equations (10), (11), (12), ρ, ρ' and ρ'' may be seen as the Lorentzian orthogonal projections of the same timelike (spacelike) vector $\frac{\overrightarrow{J}_1^0(I)}{\omega^2}$ on the timelike (spacelike) unit vectors X, X' and X'' which are linearly dependent. The dependence between X, X' and X'' may be written as follows;

$$\lambda\overrightarrow{X} + \mu\overrightarrow{X'} + \vartheta\overrightarrow{X''} = \vec{0}. \tag{13}$$

By successive Lorentzian cross products with X and X', the quantities λ, μ and ϑ are obtained as follows

$$\lambda = \sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right), \mu = \sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right), \vartheta = \sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right). \tag{14}$$

Substituting the equation (14) into the (13) the linear combination becomes

$$\sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right)\overrightarrow{X} + \sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right)\overrightarrow{X'} + \sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right)\overrightarrow{X''} = 0.$$

The Lorentzian scalar product of the previous equation with the vector $\frac{\overrightarrow{J}_1^0(I)}{\omega^2}$ is

$$\sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right)\frac{\left\langle \overrightarrow{X}, \overrightarrow{J}_1^0(X) \right\rangle}{\omega^2} + \sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right)\frac{\left\langle \overrightarrow{X'}, \overrightarrow{J}_1^0(X) \right\rangle}{\omega^2} + \sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right)\frac{\left\langle \overrightarrow{X''}, \overrightarrow{J}_1^0(X) \right\rangle}{\omega^2} = 0. \tag{15}$$

Finally, taking into account (10), (11) and (12) Lorentzian Bobillier Formula is obtained again, but using a direct way without the use of Lorentzian Euler-Savary Formula,

$$\rho\sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right) + \rho'\sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right) + \rho''\sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right) = 0. \tag{16}$$

Therefore, the following theorem can be given.

THEOREM 1. In one parameter Lorentzian planar motion of moving plane P_1 with respect to fixed plane P_0, the relationship between the centers of curvatures concerning second order instantaneous properties is given by the Lorentzian Bobillier Formula given in the equation (16).

From this point of view, we obtained the Lorentzian form of Bobillier Formula given in [4] and [8].

Let us investigate a particular case of Theorem 1. If a timelike (spacelike) point K linked to moving plane P_1 be coincident with instantaneous pole center I, then $\overrightarrow{V}^0(K) = 0$ and similarly $\overrightarrow{J}^0(K) = 0$. Under this condition the length ρ' is equal to zero. For timelike pole curves and spacelike pole curves Lorentzian Bobillier Formula becomes

$$\rho \sinh(\vec{y}, \vec{x}) + \rho'' \sinh\left(\overrightarrow{X}, \overrightarrow{y}\right) = 0$$

and

$$\rho \sinh(\vec{x}, \vec{y}) + \rho'' \sinh\left(\overrightarrow{X}, \overrightarrow{x}\right) = 0,$$

respectively. In addition to this by taking a hyperbolic angle θ between X and the axis y (axis x) for timelike (spacelike) pole curves, x and y are orthogonal in the sense of Lorentzian. So we can give the following corollary.

COROLLARY 1. Let a point K linked to moving plane P_1 be coincident with instantaneous pole center I. In that case Bobillier Formula becomes

$$\rho - \rho'' \sinh\theta = 0.$$

As announced, it is simply a particular case of Bobillier formula in sense of Lorentz.

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