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Two Differential Equations For The Linear Generating Function Of The Charlier Polynomials*

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Abstract

We provide an interesting way to obtain the linear generating function for the classical discrete Charlier orthogonal polynomials by implementing what we entitle the 'Inverse Method'. This method transforms a given three-term recurrence relation into a differential equation, the solution of which is a linear generating function. To demonstrate the details of the procedure, we first apply the Inverse Method to the three-term recurrence relation that defines the Charlier polynomials. We then apply it to a new three-term recurrence relation, which is established via a certain connection between the Charlier polynomials and a variation of the Laguerre polynomials. The solution to each of these differential equations is the intended generating function.

1 Introduction

In this paper, we ultimately construct and solve two different differential equations that both admit the linear generating function (see (4)) for the classical discrete Charlier orthogonal polynomial sequences $\{C_n(x;a)\}_{n=0}^{\infty}$ (refer to (1)) as their solution. The analysis leading to the second of these differential equations does not appear in the current literature. For other papers related to developing characterizations for linear generating functions via differential equations consider [1, 8, 10, 15].

In order to efficiently discuss the details of this paper, we first address some preliminary definitions, nomenclature and concepts. We begin with the definition of the Charlier polynomials:

$$C_n(x;a) := {}_2F_0\left(\begin{array}{c} -n,-x \\ - \end{array} \middle| -\frac{1}{a} \right) = \sum_{k=0}^{n} \frac{(-n)_k(-x)_k}{k!} \left(-\frac{1}{a}\right)^k, \tag{1}$$

which we have written in hypergeometric form, adhering to the following definition.

DEFINITION 1 ([16]). A generalized classical ***Hypergeometric Function*** (${}_rF_s$) has the form

$${}_rF_s\left(\begin{array}{c} a_1,...,a_r \\ b_1,...,b_s \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1,...,a_r)_k}{(b_1,...,b_s)_k} \frac{x^k}{k!}, \tag{2}$$

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where the ***Pochhammer Symbol*** $(a)_k$ is defined as

$$(a)_k := a\,(a+1)\,(a+2)\cdots(a+k-1)\,, \quad (a)_0 := 1,$$

and

$$(a_1,\ldots,a_j)_k := (a_1)_k \cdots (a_j)_k.$$

The sum (2) terminates if one of the numerator parameters is a negative integer, e.g., consider $-n$ as in (1).

It is worth mentioning that the Charlier polynomials are classified as a classical Sheffer *B-Type 0* orthogonal polynomial sequence (cf. [6, 17]) and satisfy the discrete orthogonality relation

$$\sum_{n=0}^{\infty} \frac{a^x}{x!} C_m(x;a) C_n(x;a) = a^{-n} e^a n! \delta_{m,n}, \quad a > 0.$$

Moreover, the Charlier polynomials are very important in various applications, including; statistical planning and inference [7], quantum mechanics [5, 14, 18], difference equations [4], combinatorics [12] and lattices [9] (e.g., the semi-infinite Toda lattice).

An indispensable structure in the theory and applications of orthogonal polynomials is the linear generating function, which we define next.

DEFINITION 2 ([16]). A ***Linear Generating Function*** for a polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ has the form

$$\sum_{\Lambda} \zeta_n P_n(x) t^n = F(x,t), \tag{3}$$

with $\Lambda \subseteq \{0,1,2,3,\ldots\}$ and $\{\zeta_n\}_{n=0}^{\infty}$ a sequence in n that is independent of x and t. Moreover, we say that the function $F(x,t)$ ***generates*** the set $\{P_n(x)\}_{n=0}^{\infty}$.

The Charlier polynomials (1) have the linear generating function:

$$\sum_{n=0}^{\infty} \frac{1}{n!} C_n(x;a) t^n = e^t \left(1 - \frac{t}{a}\right)^x. \tag{4}$$

The relation (4) can be derived from first principles. To demonstrate this, we take the left-hand side of (4) and substitute (1), which leads to

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} C_n(x;a) t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (-n)_k (-x)_k}{k! n! a^k} t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-x)_k}{k!(n-k)! a^k} t^n \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-x)_k}{k! a^k} t^k \\ &= e^t \left(1 - \frac{t}{a}\right)^x,\end{aligned}$$

where we first used $(-n)_k = (-1)^k n!/(n-k)!$, then the relation (cf. p.57 of [16]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k)$$

and binomial expansion.

Interestingly, (4) is the only known linear generating function for the Charlier polynomials (1) of the form (3). In fact, various papers have been published on obtaining other types of generating functions for the Charlier polynomials, e.g., [3, 13], and even more recently, [2, 7].

Lastly, we utilize the following necessary and sufficient condition for orthogonality throughout this work.

DEFINITION 3 ([16]). Every orthogonal polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a ***Three-Term Recurrence Relation*** of the form

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad A_n A_{n-1} C_n > 0$$

where

$$P_{-1}(x) = 0 \text{ and } P_0(x) = 1. \tag{5}$$

The three-term recurrence relation for the Charlier polynomials is

$$-xC_n(x;a) = aC_{n+1}(x;a) - (n+a)C_n(x;a) + nC_{n-1}(x;a). \tag{6}$$

We can now discuss the details of this paper. Namely, we develop and implement an appealing way to obtain the linear generating function (4) for the Charlier polynomials (1) other than the first principles approach. To accomplish this, we utilize what we have deemed the 'Inverse Method', which transforms a three-term recurrence relation of the form (5) into a differential equation that has a generating function of the form (3) as its unique solution. In Section 2, we explain the details of the Inverse Method and then apply it to (6) in order to obtain (4) as a solution. Our main result is developed in Section 3, wherein we construct a certain relationship between the Charlier polynomials and a variation of the Laguerre polynomials, which results in a new three-term recurrence relation. The Inverse Method is then applied to this relation yielding (4) as a solution.

2 The Inverse Method

In this section, we demonstrate the details of the Inverse Method by applying it to (6) in order to obtain (4) as a solution. We use the term 'Inverse Method' due to the relation our approach has to the Inverse Problems as defined in Chapter 5 of [11], for example. This method is again utilized in Section 3.

We begin by describing the general idea behind the Inverse Method. First, we multiply both sides of the given three-term recurrence relation of the form (5) by $c_n t^n$, where c_n is a function in n that is independent of x and t, and sum the result for $n = 0, 1, 2, \ldots$. We then define $F(t) := \sum_{n=0}^{\infty} c_n P_n(x) t^n$, whence it follows that $\frac{\partial}{\partial t} F(t) = \sum_{n=1}^{\infty} n c_n P_n(x) t^{n-1}$. From there, we use algebraic manipulations to obtain a differential equation[1], with the initial condition $F(0) = P_0(x) = 1$. The unique solution to this differential equation will have the form $\sum_{n=0}^{\infty} c_n P_n(x) t^n = f(x, t)$. We now apply this method to (6).

To begin, we note that from examining (4), c_n, as described above, must be $1/n!$. Thus, we multiply (6) by $t^n/n!$ and sum the result for $n = 0, 1, 2, \ldots$, which yields

$$\begin{aligned} -x \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n = a \sum_{n=0}^{\infty} \frac{C_{n+1}(x;a)}{n!} t^n - \sum_{n=1}^{\infty} \frac{C_n(x;a)}{(n-1)!} t^n \\ - a \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n + \sum_{n=1}^{\infty} \frac{C_{n-1}(x;a)}{(n-1)!} t^n. \end{aligned}$$

We then define $F := F(t; x, a) = \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n$ and it therefore immediately follows that $\dot{F} := \frac{\partial}{\partial t} F(t; x, a) = \sum_{n=1}^{\infty} \frac{C_n(x;a)}{(n-1)!} t^{n-1}$. Then, we see that our relation directly above leads to the first-order differential equation below

$$\dot{F} - \left(1 + \frac{x}{t-a}\right) F = 0; \quad F(0; x, a) = 1. \tag{7}$$

The integrating factor for (7) is $\mu = \exp\left(-\int 1 + \frac{x}{t-a} dt\right) = e^{-t}(t-a)^{-x}$ and therefore, a general solution is

$$F(t; x, a) = c(x; a) e^t (t-a)^x,$$

where $c(x; a)$ is an arbitrary function of x. From the initial condition, it is immediate that $c(x; a) = (-a)^{-x}$ and thus, the unique solution to (7) turns out to be (4):

$$F(t; x, a) = \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n = e^t \left(1 - \frac{t}{a}\right)^x.$$

REMARK. It is worth discussing the ramifications of attempting to find a linear generating function for the Charlier polynomials of the form $\sum_{n=0}^{\infty} C_n(x; a) t^n = g(x, t)$ via the Inverse Method. Following suit to the analysis above leads to the non-homogeneous differential equation

$$\dot{G} + \left(\frac{a + (x - a + t)t}{(t-1)t^2}\right) G = \frac{a}{(t-1)t^2}; \quad G(0; x; a) = 1. \tag{8}$$

The integrating factor for this equation is $\mu = e^{a/t}(t-1)^{x+1} t^{-x}$ and therefore the general solution is

$$G(t; x; a) = d(x; a) \frac{a e^{-a/t} t^x}{(t-1)^{x+1}} \int \frac{e^{a/t}}{t^2} \left(1 - \frac{1}{t}\right)^x dt.$$

[1] For some orthogonal polynomial sequences, e.g., the Chebyshev polynomials, the resulting equation is algebraic.

This solution can be manipulated in various ways. For example, we can equivalently write

$$G(t;x;a) = d(x;a)\frac{e^{-a/t}t^x}{(t-1)^{x+1}}\sum_{k=0}^{x}\frac{(-1)^{k+1}(-x)_k}{a^k k!}\Gamma(k+1,-a/t),$$

where $\Gamma(\alpha,z) := \int_z^\infty \tau^{\alpha-1}e^{-\tau}d\tau$ is the Incomplete Gamma Function. However, due to the discontinuity at $t=0$ in the coefficients of (8), no solution exists that satisfies the initial value problem.

3 The Main Result

We next derive a new three-term recurrence relation for a variation of the Laguerre polynomials - these polynomials are in fact are equivalent to a multiple of the Charlier polynomials. We then apply the Inverse Method to this relation in order to yield (4) as a solution. The Laguerre polynomials are defined as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{array}{c}-n\\ \alpha+1\end{array}\middle|\, x\right), \quad \alpha > -1. \tag{9}$$

The restriction on α is essential, as $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ is undefined for all $\alpha \le -1$. However, taking into account that $(\alpha+1)_n/(\alpha+1)_k = (\alpha+k+1)_{n-k}$ and expanding (9) as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k k!}x^k$$

leads to the equivalent form

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}(\alpha+k+1)_{n-k}x^k. \tag{10}$$

Moreover, using relationship (10), we are able to define the Laguerre polynomials for all $\alpha \in \mathbb{R}$. We can now establish the following statement.

LEMMA 1. The Charlier polynomials (1) and the Laguerre polynomials (10) are related in the following way:

$$\frac{(-a)^n}{n!}C_n(x;a) = L_n^{(x-n)}(a). \tag{11}$$

PROOF. From (10), we see that

$$L_n^{(x-n)}(a) = \frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}(x-n+k+1)_{n-k}a^k. \tag{12}$$

Making use of the following relations

$$(x-(n-k)+1)_{n-k} = (-1)^{n+k}(-x)_{n-k}, \quad (-n)_k = \frac{(-1)^k n!}{(n-k)!}$$

and

$$(-1)^{n+k}\frac{n!}{k!} = (-n)_{n-k},$$

we see that (12) becomes

$$\begin{aligned} L_n^{(x-n)}(a) &= \frac{(-a)^n}{n!}\sum_{k=0}^{n}\frac{(-n)_{n-k}(-x)_{n-k}}{(n-k)!}\left(-\frac{1}{a}\right)^{n-k} \\ &= \frac{(-a)^n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(-x)_k}{k!}\left(-\frac{1}{a}\right)^{k} \\ &= \frac{(-a)^n}{n!}C_n(x;a). \end{aligned}$$

LEMMA 2. A three-term recurrence relation for $\{L_n^{(x-n)}(a)\}_{n=0}^{\infty}$ as in (11) is

$$(n+1)L_{n+1}^{(x-n-1)}(a) = (x-a-n)L_n^{(x-n)}(a) - aL_{n-1}^{(x-n+1)}(a). \tag{13}$$

PROOF. From LEMMA 1 and (4), we obtain the linear generating function

$$\sum_{n=0}^{\infty} L_n^{(x-n)}(a)t^n = e^{-at}(1+t)^x. \tag{14}$$

From writing $L_n^{(x-n)}(a) = c_{n,0}x^n + c_{n,1}x^{n-1} + c_{n,2}x^{n-2} + O(x^{n-3})$, it follows from comparing the coefficients of $x^n t^n$, $x^{n-1}t^n$ and $x^{n-2}t^n$ in (14) that

$$\begin{aligned} L_n^{(x-n)}(a) = &\frac{1}{n!}x^n - \left[\frac{a}{(n-1)!} + \frac{1}{2(n-2)!}\right]x^{n-1} \\ &+ \frac{1}{8}\left[\frac{4a^2}{(n-2)!} + \frac{12a+8}{3(n-3)!} + \frac{1}{(n-4)!}\right]x^{n-2} + O(x^{n-3}). \end{aligned}$$

Putting this into the general three-term recurrence relation (5) and comparing coefficients of x^{n+1}, x^n and x^{n-1} yields the recurrence coefficients for $\{L_n^{(x-n)}(a)\}_{n=0}^{\infty}$:

$$A_n = \frac{1}{n+1}, \quad B_n = -\frac{a+n}{n+1} \text{ and } C_n = \frac{a}{n+1},$$

from which (13) follows.

This leads us to our main result.

THEOREM 1. For $H := H(t;x,a)$, the linear generating function (14) satisfies the first-order differential equation

$$\dot{H} + \left(a - \frac{x}{1+t}\right)H = 0; \quad H(0;x,a) = 1. \tag{15}$$

PROOF. We establish this result by applying the Inverse Method to (13), analogous to deriving (4) via (6). We first multiply (13) by t^n and sum for $n = 0, 1, 2, \ldots$. From defining $H := H(t; x, a) = \sum_{n=0}^{\infty} L_n^{(x-n)}(a)t^n$ we obtain (15).

The integrating factor turns out to be $\mu = \exp\left(\int a - \frac{x}{1+t}dt\right) = e^{at}(1+t)^{-x}$ and it follows that the solution to (15) is (14):

$$H(t; x, a) = \sum_{n=0}^{\infty} L_n^{(x-n)}(a)t^n = e^{-at}(1+t)^x.$$

Clearly, we see that $H(-t/a; x, a)$ above gives us (4).

In conclusion, we have established our two distinct differential equations (7) and (15), which characterize the only linear generating function for the Charlier polynomials. As a byproduct of our analysis, we also derived the new three-term recurrence relation (13).

Lastly, we state that an interesting future consideration is to develop the q-analogue of this paper. That is, construct q-difference equations for the quantized Charlier (q-Charlier) polynomials (see [19]):

$$C_n(q^{-x}; a, q) := {}_2\phi_1\left(\begin{array}{c} q^{-n}, q^{-x} \\ 0 \end{array} \middle| \, q; -\frac{q^{n+1}}{a}\right),$$

with

$${}_r\phi_s\left(\begin{array}{c} a_1, ..., a_r \\ b_1, ..., b_s \end{array} \middle| \, q; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ..., a_r; q)_k}{(q, b_1, ..., b_s; q)_k} z^k \left(-q^{(k-1)/2}\right)^{k(s+1-r)}$$

and

$$(a; q)_k := (1-a)(1-aq)\left(1-aq^2\right)\cdots\left(1-aq^{k-1}\right), \quad (a; q)_0 := 1,$$

that have the corresponding linear generating function(s) as solutions. Directly above, ${}_r\phi_s$ represents the Generalized q-Hypergeometric Series and $(a; q)_k$ the q-Pochhammer Symbol.

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