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Three Solutions For A Quasi-Linear Elliptic Problem*

Mostafa Allaoui†, Abdelrachid El Amrouss‡, Anass Ourraoui§

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Abstract

This paper shows the existence of at least three solutions for Navier problem involving the $p(x)$-biharmonic operator. Our technical approach is based on a theorem obtained by B. Ricceri.

1 Introduction

Analysis of solutions of specific boundary value problems is of considerable importance in the theory of partial differential equations, especially for equations of fourth order. Its interest is widely justified with many physical examples and arises from a variety of nonlinear phenomena. It is used in non-Newtonian fluids, in some reaction-diffusion problems, as well as in flow through porous media. It also appears in nonlinear elasticity petroleum extraction and in the theory of quasi-regular and quasi-conformal mappings. For more detailed references on physical and mathematical background, we refer to [1, 2, 3, 8].

The present work is concerned with the following $p(x)$-biharmonic problem with Navier boundary condition,

$$(\mathcal{P}) \quad \begin{cases} \Delta^2_{p(x)} u = \lambda f(x,u) + \mu g(x,u) \quad in \quad \Omega, \\ u = \Delta u = 0 \quad on \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta^2_{p(x)} u = \Delta(|\Delta|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic with $p \in C(\overline{\Omega})$, $p(x) > 1$ for every $x \in \overline{\Omega}$ and λ, $\mu \in \mathbb{R}_+$. We define $F(x,t) = \int_0^t f(x,s)ds$, $G(x,t) = \int_0^t g(x,s)ds$ and we denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$.

Throughout this paper, we suppose the following assumptions.

There exist two positive constants C, δ and $\alpha \in C(\overline{\Omega})$ with

$$\alpha^- := \inf_{x\in\overline{\Omega}} \alpha(x), \ \alpha^+ := \sup_{x\in\overline{\Omega}} \alpha(x) \text{ and } 1 < \alpha^- \le \alpha^+ < p^-,$$

such that

(F_1) $F(x,t) \ge 0$ for a.e. $x \in \Omega$ and $x \in [0,\delta]$.

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†University Mohamed I, Faculty of Science, Department of Mathematics, Oujda, Morocco
‡University Mohamed I, Faculty of Science, Department of Mathematics, Oujda, Morocco
§University Mohamed I, Faculty of Science, Department of Mathematics, Oujda, Morocco

(F_2) There exists $q_1(x) \in C(\overline{\Omega})$ with $p^+ < q_1^- \leq q_1(x) < p_2^*(x)$ such that

$$\limsup_{t \to 0} \frac{F(x,t)}{|t|^{q_1(x)}} < +\infty,$$

uniformly for a.e. $x \in \Omega$ with

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & if \quad p(x) < \frac{N}{2}, \\ +\infty & if \quad p(x) \geq \frac{N}{2}. \end{cases}$$

(F_3) $|F(x,t)| \leq C(1 + |t|^{\alpha(x)})$ for $x \in \Omega$ and for all $t \in \mathbb{R}$.
(F_4) $F(x,0) = 0$ for a.e. $x \in \Omega$.
(G) $\sup_{(x,t) \in \Omega \times \mathbb{R}} \frac{G(x,t)}{1+t^{q_2(x)}} < +\infty$, where $q_2(x) \in C(\overline{\Omega})$ and $q_2(x) < p_2^*(x)$ for $x \in \overline{\Omega}$.

The goal of this paper is to prove the following result.

THEOREM 1. Assume that (F_1) to (F_4) and (G) are satisfied. Then there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number e such that for every $\lambda \in \Lambda$, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, problem $(\mathcal{P})$ has at least three weak solutions whose norms in $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ are less than e.

Many authors consider the existence of nontrivial solutions for some fourth order problems such as [2, 3]. This is a generalization of the classical p-biharmonic operator $\Delta(|\Delta u|^{p-2})$ obtained in the case when p is a positive constant. Here we point out that the $p(x)$-biharmonic operator possesses more complicated nonlinearities than p-biharmonic, for example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its principle eigenvalue is zero. This study is inspired by the results of [6] and [7], we are to prove the existence of three solutions of problem $(\mathcal{P})$, and the technical approach is based on the three-critical-points theorem of Ricceri [11, 12].

This paper is divided into three sections organized as follows: in section 2 we start with some preliminary basic results on the theory of Lesbegue-Sobolev spaces with variables exponent (we refer to the book of Musielak [10], Mihăilescu and Rădulescu [9]), we recall the three-critical-points theorem of Ricceri with some required results. In section 3, we give the proof of the main result.

2 Preliminaries

In order to deal with the problem $(\mathcal{P})$, we need some theory of variable exponent Sobolev space. For convenience, we only recall some basic facts which will be used later. Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Let $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega})$ and $ess\inf_{x \in \overline{\Omega}} p(x) > 1\}$ for any $p(x) \in C_+(\overline{\Omega})$. Set $p^- = \min_{x \in \overline{\Omega}} p(x)$, $p^+ = \max_{x \in \overline{\Omega}} p(x)$ and

$$p_k^*(x) = \frac{Np(x)}{N - kp(x)} \text{ if } kp(x) < N \text{ and } p_k^*(x) = +\infty \text{ if } kp(x) \geq N.$$

Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^{p(x)}\, dx < \infty \right\}.$$

Then $L^{p(x)}(\Omega)$ endowed with the norm

$$\|u\|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1 \right\},$$

becomes a separable and reflexive Banach space.

PROPOSITION 1 (cf. [5]). Set $\rho(u) = \int_\Omega |u|^{p(x)}\, dx$. If $u \in L^{p(x)}(\Omega)$, we have

(1) $\|u\|_{p(x)} \ge 1 \Rightarrow \|u\|_{p(x)}^{p^-} \le \rho(u) \le \|u\|_{p(x)}^{p^+}$.

(2) $\|u\|_{p(x)} \le 1 \Rightarrow \|u\|_{p(x)}^{p^+} \le \rho(u) \le \|u\|_{p(x)}^{p^-}$.

Define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega) \text{ and } |\alpha| \le k\}$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_N} x_N}$$

with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ with the norm $\|u\| = \Sigma_{|\alpha \le k|} \|D^\alpha u\|_{p(x)}$ is a separable and reflexive Banach space.

PROPOSITION 2 (cf. [5]). For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \le p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

We denote $W_0^{k,p(x)}(\Omega)$ by the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

REMARK 1 (cf. [3]). $(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \| \,.\, \|)$ is a separable and reflexive Banach space. By the above remark and proposition 2.2 there is a continuous and compact embedding of $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ into $L^{r(x)}(\Omega)$ where $r(x) < p_2^*$ for all $x \in \overline{\Omega}$.

PROPOSITION 3 (cf. [5]). Set $\varrho(u) = \int_\Omega |\Delta u|^{p(x)}\, dx$. For $u, u_n \in W^{2,p(x)}(\Omega)$, we have

(1) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \varrho(u) \le \|u\|^{p^-}$.

(2) $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le \varrho(u) \le \|u\|^{p^+}$.

(3) $\| u_n \| \to 0 \Leftrightarrow \varrho(u_n) \to 0$.

(4) $\| u_n \| \to +\infty \Rightarrow \varrho(u_n) \to +\infty$.

The proof is similar to proof in ([5], Theorem 3.1).

PROPOSITION 4 (cf. [5]). For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left| \int_\Omega uvdx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(x)} \|v\|_{q(x)}$$

where

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1.$$

DEFINITION 1. We say that $u \in X$ is a **weak solution** of problem $(\mathcal{P})$ if

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx = \lambda \int_\Omega f(x,u)vdx + \mu \int_\Omega g(x,u)vdx$$

for all $v \in X$.

We define

$$I(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx, J(u) = -\int_\Omega F(x,u)dx$$

and

$$\psi(u) = -\int_\Omega G(x,u)dx$$

where

$$F(x,t) = \int_0^t f(x,s)ds, G(x,t) = \int_0^t g(x,s)ds.$$

Set

$$\langle L(u), v \rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx \text{ for } u, v \in X.$$

PROPOSITION 5 (cf. [2, 3]).

(i) $L : X \to X^*$ is a continuous, bounded and strictly monotone operator.

(ii) L is a mapping of type (S_+), i.e. if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0$, then $u_n \to u$ in X.

(iii) $L : X \to X^*$ is a homeomorphism.

PROPOSITION 6 (cf. Theorem 1 in [11]). Let X be a real reflexive Banach space, $K \subset \mathbb{R}$ an interval, $I : X \to \mathbb{R}$ be a sequentially weakly lower semi-continuous C^1 function whose derivative admits a continuous inverse on X^* and $J : X \to \mathbb{R}$ be a C^1

functional with compact derivative. In addition, I is bounded on each bounded subset of X. Assume that

$$\lim_{\|x\|\to\infty} I(x) + \lambda J(x) = +\infty \tag{1}$$

for $\lambda \in K$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in K}\inf_{x\in X}(I(x) + \lambda(J(x) + \rho)) < \inf_{x\in X}\sup_{\lambda\in K}(I(x) + \lambda(J(x) + \rho)).$$

Then there exist a nonempty set $A \subseteq K$ and a positive number e with the following property: for every $\lambda \in A$ and every C^1 functional $\psi : X \to \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, the equation

$$I'(u) + \lambda J'(u) + \mu\psi'(u) = 0$$

has at least three solutions in X whose norms are less than e.

PROPOSITION 7 (cf.[12]). Let X be a nonempty set, and I and J are two real functions on X. Suppose there are $\gamma > 0$ and $u_0, u_1 \in X$ such that

$$I(u_0) = J(u_0) = 0,\ I(u_1) > \gamma \text{ and } \sup_{u\in I^{-1}(]-\infty,\gamma])} J(u) < \gamma\frac{J(u_1)}{I(u_1)}.$$

Then for each ρ satisfying

$$I(u_0) = J(u_0) = 0,\ I(u_1) > \gamma \text{ and } \sup_{u\in I^{-1}(]-\infty,\gamma])} J(u) < \rho < \gamma\frac{J(u_1)}{I(u_1)},$$

we have

$$\sup_{\lambda\geq 0}\inf_{u\in X}(I(u) + \lambda(\rho - J(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(I(u) + \lambda(\rho - J(u))).$$

3 Proof of the Main Result

We now turn to the proof of Theorem 1. First, we check the conditions of proposition 6.

According to proposition 5, it is clear that I is continuously Gâteaux differentiable, whose Gâteaux derivative admits a continuous inverse on X^*. Notice that I is a convex and continuous functional, and then it is a weakly lower semi-continuous function. Moreover, ψ and J are continuously Gâteaux differentiable functions and its Gâteaux derivatives are compact. By a similar analysis to that in Fan and Zhang (cf. [4]), by (F3) and (G), we know that $J, \psi \in C^1(X, \mathbb{R})$ such that

$$J'(u)v = \int_\Omega f(x, u(x))vdx \text{ and } \psi'(u)v = \int_\Omega g(x, u(x))vdx$$

for $u, v \in X$. Since the identity operator from X to $L^{\alpha(x)}$ is compact, so the operators J' and ψ'^* are compact. Obviously, I is bounded on each bounded subset of X.

For $\|u\| < 1$,

$$\frac{1}{p^+} \|u\|^{p^+} \leq I(u) \leq \frac{1}{p^-} \|u\|^{p^-} .$$

Let $C_0 > 0$ such that $C_0 \geq \frac{1}{p^+} \|u\|^{p^-} - \frac{1}{p^-} \|u\|^{p^+}$. Then

$$I(u) \geq \frac{1}{p^+} \|u\|^{p^-} - C_0.$$

Since $\|u\| \geq 1$, we have $I(u) \geq \frac{1}{p^+} \|u\|^{p^-}$, and thus for any $u \in X$ we infer that

$$\begin{aligned} \lambda J(u) &= -\lambda \int_\Omega F(x,u)dx \\ &\geq -\lambda \int_\Omega C(1 + |u|^{\alpha(x)}\, dx) \\ &\geq -\lambda C(|\Omega| + \|u\|^{\alpha^+}_{\alpha(x)} + \|u\|^{\alpha^-}_{\alpha(x)}) \\ &\geq -C_1(1 + \|u\|^{\alpha^+}_{\alpha(x)}) \\ &\geq -C_2(1 + \|u\|^{\alpha^+}), \end{aligned}$$

with $C_1 \geq 0$ and $C_2 \geq 0$. Consequently, we obtain

$$I(u) + \lambda J(u) \geq \frac{1}{p^+} \|u\|^{p^-} - C_2(1 + \|u\|^{\alpha^+}) - C_0.$$

Therefore, for $u \in X$ and $\lambda \geq 0$, since $\alpha^+ < p^-$, we get

$$\lim_{\|u\| \to +\infty} (I(u) + \lambda J(u)) = +\infty.$$

Then the assumption (1) is satisfied. In order to prove the assumption (2), we need to verify the conditions of proposition 7.

Let $u_0 = 0$. Then $I(u_0) = -J(u_0) = 0$. We show that the assumption (3) of proposition 7 holds. Let $x^0 \in \Omega$ (because Ω is a nonempty bounded open set) and $r_2 > r_1 > 0$. Take $\omega(x) \in C_0^\infty(\overline{\Omega})$ with $\omega(x) = 0$ for $x \in \overline{\Omega} \setminus B(x^0, r_2)$, $\omega(x) = \frac{\delta}{r_2 - r_1}(r_2 - \| x_i - x_i^0 \|_2)$ when $x \in B(x^0, r_2) \setminus B(x^0, r_1)$ and $\omega(x) = \delta$ when $x \in B(x^0, r_1)$ with $\| x \|_2 = (\Sigma_{i=1}^N (x_i)^2)^{\frac{1}{2}}$.

Here $u_1(x) = \omega(x)$. Then we can get

$$-J(u_1) = -J(\omega) = \int_\Omega F(x,\omega)dx > 0.$$

By (F_2), there exist $\eta \in [0,1]$ and $C_1 > 0$ such that

$$F(x,t) \leq C_1 |t|^{q_1(x)} \text{ for } |t| < \eta \text{ and a.e. } x \in \Omega.$$

Putting

$$K_1 = \sup_{|t|<\eta} \frac{C[1 + |t|^{\alpha^+}]}{|t|^{q_1^-}},\ K_2 = \sup_{|t|>\eta} \frac{C[1 + |t|^{\alpha^+}]}{|t|^{q_1^-}},\ K_3 = \sup_{|t|<1} \frac{C[1 + |t|^{\alpha^+}]}{|t|^{q_1^-}},$$

$$K_4 = \sup_{|t|>1} \frac{C[1+|t|^{\alpha^+}]}{|t|^{q_1^-}} \text{ and } M^* = \max\{C_1, K_i, i = 1, ..., 4\}.$$

Thus

$$F(x,t) < M^* |t|^{q_1^-} \text{ for } t \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Now, fix γ such that $0 < \gamma < 1$. If we have $\frac{1}{p^+} \|u\|^{p^+} \leq \gamma < 1$. By the Sobolev embedding Theorem, for suitable positive constants C_2 and C_3, we entail that

$$-J(u) = \int_\Omega F(x,u)dx < M^* \int_\Omega |u|^{q_1^-} \leq C_2 \|u\|^{q_1^-} \leq C_3 \gamma^{\frac{q_1^-}{p^+}}.$$

It follows from $q_1^- > p^+$ that

$$\lim_{\gamma \to 0^+} \frac{\sup_{\frac{1}{p^+}\|u\|^{p^+} \leq \gamma} \{-J(u)\}}{\gamma} = 0. \tag{2}$$

Let $\omega \in X$ as previously mentioned with the fact $-J(\omega) > 0$. Fix γ_0 where $\gamma < \gamma_0 < \frac{1}{p^+} \min\{\|\omega\|^{p^+}, \|\omega\|^{p^-}, 1\} \leq 1$. Now, there are two cases to be considered.

If $\|\omega\| < 1$, we have

$$\begin{aligned} I(u_1) &= I(\omega) = \int_\Omega \frac{1}{p(x)} |\Delta\omega|^{p(x)} dx \geq \frac{1}{p^+} \int_\Omega |\Delta\omega|^{p(x)} dx \\ &\geq \frac{1}{p^+} \|\omega\|^{p^+} \geq \gamma_0 > \gamma. \end{aligned}$$

By (2), it yields

$$\sup_{\frac{1}{p^+}\||u\|^{p^+} \leq \gamma} -J(u) \leq \frac{\gamma}{2} \frac{-J(u_1}{\frac{1}{p^-}\|\omega\|^{p^-}} \leq \frac{\gamma}{2} \frac{-J(u_1)}{I(u_1)} < \gamma \frac{-J(u_1)}{I(u_1)}.$$

Else if $\|\omega\| \geq 1$ we obtain

$$\begin{aligned} I(u_1) &= I(\omega) = \int_\Omega \frac{1}{p(x)} |\Delta\omega|^{p(x)} dx \geq \frac{1}{p^+} \int_\Omega |\Delta\omega|^{p(x)} dx \\ &\geq \frac{1}{p^+} \|\omega\|^{p^-} \geq \gamma_0 > \gamma. \end{aligned}$$

From (2), since $\gamma > 0$, we get

$$\sup_{\frac{1}{p^+}\|u\|^{p^+} \leq \gamma} -J(u) \leq \frac{\gamma}{2} \frac{-J(u_1)}{\frac{1}{p^-}\|\omega\|^{p^+}} \leq \frac{\gamma}{2} \frac{-J(u_1)}{I(u_1)} < \gamma \frac{-J(u_1)}{I(u_1)}.$$

Thereby,

$$\sup_{\frac{1}{p^+}\|u\|^{p^+} \leq \gamma} -J(u) < \gamma \frac{-J(u_1)}{I(u_1)}. \tag{3}$$

For any $u \in I^{-1}(]-\infty, \gamma])$, we have

$$\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx \leq \gamma.$$

Then

$$\int_\Omega \frac{1}{p^+} |\Delta u|^{p(x)}\, dx \leq \gamma.$$

Hence,

$$\int_\Omega |\Delta u|^{p(x)}\, dx \leq \gamma \frac{1}{\frac{1}{p^+}} < \gamma_0 \frac{1}{\frac{1}{p^+}} < 1.$$

The last inequality implies that $\|u\| < 1$ and

$$\frac{1}{p^+} \|u\|^{p^+} < \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx \leq \gamma,$$

so we conclude

$$I^{-1}(]-\infty, \gamma]) \subset \{u \in X : \frac{1}{p^+} \|u\|^{p^+} < \gamma\}.$$

We deduce from the relation (3) that

$$\sup_{u \in I^{-1}(]-\infty, \gamma])} -J(u) < \gamma \frac{-J(u_1)}{I(u_1)}.$$

We can find ρ such that

$$\sup_{u \in I^{-1}(]-\infty, \gamma])} -J(u) < \rho < \gamma \frac{-J(u_1)}{I(u_1)}.$$

Taking $K = [0, +\infty[$, the assumptions of proposition 7 are satisfied. Then, we may easily obtain the condition (2) of proposition 6. Consequently, I, J and ψ verify the conditions of proposition 6. So the proof is complete.

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References

[1] J. I. Diaz, Nonlinear Partial Differential Equations and Free Boundaries Vol 1. Elliptic Equations, In: Pitman Res. Notes Ser 106, Boston, MA, Pitman, 1985.

[2] A. R. El Amrouss, F. Moradi and M. Moussaoui, Existence of solutions for fourth-order PDES with variable exponent, Electronic Journal of Differential Equations, 13(153)(2009), 1–13.

[3] A. R. El Amrouss and A. Ourraoui, Existence of solutions for a boundary problem involving $p(x)$-biharmonic operator, Bol. Soc. Paran. Mat.(3s.), 1(31)(2013), 179–192.

[4] X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems, Nonlinear Anal., 52(2003), 1843–1852.

[5] X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263(2001), 424–446.

[6] C. Li and C. L. Tang, Three solutions for a Navier boundary value problem involving the p-biharmonic, Nonlinear Anal., 72(2010), 1339–1347.

[7] J. Liu and X. Shi, Existence of three solutions for a class of quasi-linear elliptic systems involving the $(p(x), q(x))$-Laplacian, Nonlinear Anal., 71(2009), 550–557.

[8] L. K. Martinson and K. B. Pavlov, Unsteady shear flows of a conducting fluid with a rheological power flow, Magnit Gidrodinamika, (1970), 5869–5875.

[9] M. Mihăilescu and V. Rădulescu, A multiplicity result for nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. Lond. Ser. A, 462(2006), 2625–2641.

[10] J. Musielak, Orlicz Spaces and Modular Spaces, in Lecture Notes in Mathematics, Springer, Berlin, Vol. 1043, 1983.

[11] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal., 70(2009), 3084–3089.

[12] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, Math. Comp. Model., 32(2000), 1485–1494.