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On Explicit Representations Of Solutions Of Linear Delay Systems*

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Abstract

Let A and B be real square matrices of dimension $d \geq 2$ and $r > 0$. We consider the system

$$\acute{X}(t) = AX(t) + BX(t-r),\ t \geq 0,$$

where X is specified on the interval $[-r, 0]$ and give explicit solutions for the system when the matrix A has a single eigenvalue, generalizing results of [7]. By decoupling, we obtain explicit representations for solutions of a certain class of these systems in which A has several distinct eigenvalues.

1 Introduction

Delay equations play an important role in mathematical modeling since the effects of delays are inherent in the dynamics of the evolution of many systems. A number of applications of delay equations are discussed in Hauptmann et. al [3], Morelli et. al [6], Decoa et. al [1] etc. and also in the monographs of Hale [2] and Smith [9].

A close look at the literature shows that explicit representations of solutions of delay equations are known only in a few simple cases e.g. Küchler and Mensch [4], where a formula appears for the one dimensional linear case with a single delay. Similar closed form representations for the one dimensional equation with a single delay are also found in a more recent paper by Khusainov et. al [5].

In [7] we gave a formula for the general solution of the two dimensional linear system $\acute{X}(t) = AX(t) + BX(t-r)$, $t \geq 0$, where the matrix A has a single eigenvalue. The results were used in [8] to give easily verifiable sufficient conditions for the stability of the system.

Our aim in the present paper is to generalize the results of [7] to systems in $\mathbb{R}^d$ for arbitrary $d \geq 2$ and also to extend them to a certain class of these systems in which A has several eigenvalues. The fundamental matrix solution which we present here generalizes that known in the case of systems of ordinary differential equations.

The rest of the paper is organized as follows: In Section 2 we introduce Definitions. We also prove a number of technical Lemmas which we use in Section 3. Our main results appear in Section 3 and Section 4.

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2 Prerequisites

Let $d \geq 2,\ r > 0,\ A \in \mathbb{M}(d,d,\mathbb{R}),\ B \in \mathbb{M}(d,d,\mathbb{R})$ where $\mathbb{M}(d,d,\mathbb{R})$ denotes the set of $d \times d$ real matrices, $g : [-r,0] \to \mathbb{R}^d,\ f : [0,\infty) \to \mathbb{R}^d$ and consider the system

$$\acute{X}(t) = AX(t) + BX(t-r) + f(t),\ t \geq 0 \tag{1}$$

$$X(t) = g(t),\ t \in [-r,0]. \tag{2}$$

DEFINITION 1.

(i) We say that (1)(2) is homogeneous if $f \equiv 0$ otherwise we say that it is inhomogeneous.

(ii) We say that the system is of Type I if A has a single eigenvalue otherwise we say that it is of Type II.

REMARK 1. If the system is homogeneous then (1)(2) becomes

$$\acute{X}(t) = AX(t) + BX(t-r),\ t \geq 0 \tag{3}$$

$$X(t) = g(t),\ t \in [-r,0]. \tag{4}$$

By a solution of (1)(2) we understand a function X which satisfies the following:

DEFINITION 2. A function $X : [-r,\infty) \to \mathbb{R}^d$ is called a solution of (1) with initial condition (2), if it is continuous, satisfies (1) Lebesgue almost everywhere on $[0,\infty)$ and (2).

Let $1_{\{0\}}$ denote the indicator function of the singleton set $\{0\}$ defined for $t \in \mathbb{R}$ by

$$1_{\{0\}}(t) = \begin{cases} 1 & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases}$$

If f and g are integrable then the solution of (1) (2) is given by

$$X(t) := G(t)g(0) + \int_{-r}^{0} G(t-s-r)Bg(s)ds + \int_0^t G(t-s)f(s)ds \tag{5}$$

(see Smith [9] for the derivation of (5)), where the $\mathbb{M}(d,d,\mathbb{R})$-valued function G is the fundamental matrix solution for the homogeneous system (3) in the sense of the following Definition:

DEFINITION 3. We call the function $G : [-r,\infty) \to \mathbb{M}(d,d,\mathbb{R})$ the fundamental matrix solution for the homogeneous system (3) if for any $\eta \in \mathbb{R}^d$

$$X(t) := G(t)\eta \text{ for } t \in [-r,\infty)$$

is a solution of (3) with the initial condition $X(t) = \eta 1_{\{0\}}(t)$ for $t \in [-r, 0]$.

Explicit solutions of the system (1)(2) can therefore be obtained by determining G explicitly. We shall first consider Type I systems which are homogeneous.

Consider the homogeneous system (3)(4) and assume that it is of Type I. Let $\xi \in \mathbb{C}$ be the eigenvalue of A, then A admits a decomposition $A = QJQ^{-1}$, where J is the Jordan canonical form of A i.e., if E denotes the $d \times d$ identity matrix, then

$$J = \xi \mathrm{E} + M, \tag{6}$$

where $M := (m_{ij})$, $\tau_i \in \{0, 1\}$ and

$$m_{ij} := \begin{cases} 0, & i \geq j \text{ or } j \geq i+2, \\ \tau_i, & \text{otherwise.} \end{cases}$$

Let $Z := Q^{-1}X$ and $H := Q^{-1}BQ$, then in case of existence, solutions of (3)(4) can be obtained by solving the system

$$\begin{aligned} \acute{Z}(t) &= JZ(t) + HZ(t-r), \ t \geq 0 & (7) \\ Z(t) &= Q^{-1}g(t), \ t \in [-r, 0]. & (8) \end{aligned}$$

For the matrix M in (6), we define $M^0 = \mathrm{E}$. If $z \in \mathbb{R}^d$ and $m \in \{0, 1, 2, \ldots\}$, then we define $z_m := M^m z$. Note that $z_m = 0$ for $m \geq d$.

If $x,\ y \in \mathbb{M}(d, d, \mathbb{R})$, then we define $T_x(y) := xy$ and if $A \subseteq \mathbb{M}(d, d, \mathbb{R})$, then we write $T_x A$ for $\{T_x(y) : y \in A\}$. We define $I_0 := I_0^j := \{\mathrm{E}\}\,,\ j = 0, \ldots, d-1$. For $k \geq 1$ and $j \in \{0, 1, \ldots, d-1\}$, we set $I_k^j := T_{(M^j H)} I_{k-1}$ and $I_k := \cup\{I_k^j : j = 0, \ldots, d-1\}$. We shall define $p(E) := 0,\ p(H) := p(M) := 1$ and if $n \geq 1$, $x_i \in \{M,\ H,\ E\}$, $i = 1, \ldots, n$, then we define $p(x_1 \cdots x_n) := \sum_{i=1}^{n} p(x_i)$. If $x := x_1 \cdots x_m = 0$ for some m where $x_i \in \{M, H, E\},\ i = 1, \ldots, m$ then we call x a zero. If a set contains more than one zero, then, where it is convenient, we shall jointly denote all the zeroes by a single symbol 0. As an example $\{H, M^d H, M^d\}$ shall be represented as $\{H, 0\}$ where it is convenient. Note however, that $M^d H$ and M^d are distinct with $p(M^d H) = d+1$ and $p(M^d) = d$. We shall write x^m for the product $\overbrace{x \cdots x}^{m\times}$.

The following observations will be useful:

LEMMA 1.

$$\min\{p(x) : x \in I_k\} = \min\{p(x) : x \in I_k^0\} \tag{9}$$

and

$$\max\{p(x) : x \in I_k\} = \max\{p(x) : x \in I_k^{d-1}\}. \tag{10}$$

PROOF.

$$I_k = \cup\{T_{(M^j H)} I_{k-1} : j = 0, \ldots, d-1\} = \cup\{(M^j Hx) : x \in I_{k-1}, j = 0, \ldots, d-1\}.$$

Therefore,

$$\min\{p(x) : x \in I_k\} = \min\{p(M^0 Hx) : x \in I_{k-1}\} = \min\{p(x) : x \in I_k^0\}.$$

This proves (9). The argument proving (10) is similar.

LEMMA 2. Let $j \in \{0, \ldots, d-1\}$, then for $k \geq 1$

$$\min\{p(x) : x \in I_k^j\} = j + k \tag{11}$$

and

$$\max\{p(x) : x \in I_k^j\} = j + 1 + (k-1)d. \tag{12}$$

PROOF. Let $j \in \{0, \ldots, d-1\}$. If $k = 1$, then $I_1^j = \{(M^j H)\}$. Therefore $\max\{p(x) : x \in I_1^j\} = j + 1 = j + 1 + (k-1)d$. Assume that the assertion is true for $k = m$. Then for $k = m + 1$,

$$\begin{aligned} \max\{p(x) : x \in I_{m+1}^j\} &= \max\{p(x) : x \in T_{(M^j H)} I_m\} \\ &= j + 1 + \max\{p(x) : x \in I_m\} \\ &= j + 1 + \max\{p(x) : x \in I_m^{d-1}\}, \end{aligned}$$

where the last equality follows from (10). By the assumption of the induction,

$$\max\{p(x) : x \in I_m^{d-1}\} = (d-1) + 1 + (m-1)d = md = ((m+1)-1)d.$$

This proves (12). (11) is proven similarly.

LEMMA 3. For $k \geq 0$ and $l \in \{k, \ldots, kd\}$, the set $\{x \in I_k : p(x) = l\}$ is non-empty.

PROOF. We shall also prove this statement by induction. If $k = 0$ then $I_k = \{E\}$, $dk = 0$ and hence $l \in \{0\}$. Also, $\{x \in I_0 : p(x) = 0\} = \{E\} \neq \emptyset$. Assume that the statement is true for some $k \geq 0$ then $\{x \in I_k : p(x) = l\} \neq \emptyset$ where $l = k, \ldots, dk$. We now show that $\{x \in I_{k+1} : p(x) = l\} \neq \emptyset$ where $l = k+1, \ldots, d(k+1)$. Since

$$I_{k+1} = \cup\{(M^j H)x : x \in I_k, j = 0, \ldots, d-1\},$$

$\{p(x) : x \in I_{k+1}\}$ consists of the distinct elements of the following collection:

$$\begin{aligned} & \{j + 1 + p(x) : x \in I_k \text{ and } j \in \{0, \ldots, d-1\}\} \\ = \; & \{j + 1 + m : m \in \{k, \ldots, dk\} \text{ and } j \in \{0, \ldots, d-1\}\}. \end{aligned}$$

However, the distinct elements of this collection are the numbers in the set $\{(k+1), \ldots, d(k+1)\}$.

For typographical convenience, let us use the following notation:

$$\Lambda_{k,l} := \{x \in \cup_{j=1}^{d-1} I_k^j : p(x) = l\}$$

and

$$\Delta_{k,l} := \{x \in I_k^0 : p(x) = l\}.$$

We note that $\Lambda_{k,l} \cup \Delta_{k,l} = \{x \in I_k : p(x) = l\}$. We have the following Remark:

REMARK 2.

(i) By Lemma 2, for $j = 0, \dots, d-1$ and $k \geq 1$,

$$\{x \in I_k^j : p(x) < j + k\} = \emptyset \text{ and } \{x \in I_k^j : p(x) > j + 1 + (k-1)d\} = \emptyset.$$

So

(a) $\Lambda_{k,k} = \emptyset$ and
(b) $\left\{x \in I_k^0 : p(x) > (k-1)d + 1\right\} = \emptyset$.

Also $\{x \in I_k : p(x) < k\} = \emptyset$ and $\{x \in I_k : p(x) > kd\} = \emptyset$ and hence

(ii) In view of Lemma 3, I_k is the disjoint union of the non-empty sets $\{x \in I_k : p(x) = l\}$ where $l = k, \dots, kd$.

(iii) For $l \in \{k, \dots, kd\}$,

$$\{x \in I_k : p(x) = l\} = \{x \in \cup_{j=0}^{d-2} I_k^j : p(x) = l\} \cup \{x \in I_k^{d-1} : p(x) = l\}.$$

Since $T_M I_k^j = I_k^{j+1}$, $\min\{p(x) : x \in I_k^{d-1}\} = k + d - 1$ and $T_M I_k^{d-1} = \{0\}$ for all $j = 0, \dots, d-2$, we observe that

$$T_M\{x \in I_k : p(x) = l\} = \begin{cases} \Lambda_{k,(l+1)} \cup F, & l \in \{k, \dots, kd-1\}, \\ \{0\}, & l = kd, \\ \emptyset, & l \leq k-1 \text{ or } l \geq kd+1, \end{cases}$$

where

$$F := \begin{cases} \emptyset, & l < k + d - 1 \\ \{0\}, & l \in \{k + d - 1, \dots, kd - 1\}. \end{cases}$$

(iv) From (ii) and the definition of I_{k+1}^0,

$$T_H\{x \in I_k : p(x) = l\} = \begin{cases} \Delta_{(k+1),(l+1)}, & k \leq l \leq kd, \\ \emptyset, & \text{otherwise.} \end{cases}$$

3 Systems of Type I

In this Section, we give an explicit solution of the system (3)(4) when it is of Type I.

LEMMA 4. Let the matrices J and M be as in (6), $d \geq 2$ and $H \in \mathbb{M}(d, d, \mathbb{R})$ be arbitrary, then the system

$$\acute{Z}(t) = JZ(t) + HZ(t-r), \quad t \geq 0 \tag{13}$$

$$Z(t) = z1_{\{0\}}(t), \quad t \in [-r, 0], \quad z \in \mathbb{R}^d \tag{14}$$

admits a unique solution given by

$$Z(t) := \begin{cases} \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in I_k : p(x)=l\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} z_m, & t \geq 0, \\ z1_{\{0\}}(t), & t \in [-r, 0], \end{cases}$$

where $z_m = M^m z$.

PROOF. The uniqueness of the solution follows from the step method for solving the equation. Also, it is easy to see from its definition, that Z is continuous. To complete the proof, we will show that it satisfies (13) on the intervals $(nr, (n+1)r)$, $n = 0, 1, 2, \ldots$. Let $n = 0$, then for $t \in (0, r)$, $[\frac{t}{r}] = 0$, hence

$$Z(t) = e^{\xi t} \sum_{m=0}^{d-1} \frac{t^m}{m!} z_m.$$

Therefore,

$$\acute{Z}(t) = \xi e^{\xi t} \sum_{m=0}^{d-1} \frac{t^m}{m!} z_m + e^{\xi t} \sum_{m=1}^{d-1} \frac{t^{m-1}}{(m-1)!} z_m \tag{15}$$

$$= \xi E Z(t) + e^{\xi t} \sum_{m=1}^{d-1} \frac{t^{m-1}}{(m-1)!} z_m. \tag{16}$$

Since

$$JZ(t) + HZ(t-r) = MZ(t) + \xi EZ(t) + HZ(t-r) \tag{17}$$

and $Z(t-r) = 0$ for $t \in [0, r)$, we have to show that $MZ(t) = e^{\xi t} \sum_{m=1}^{d-1} \frac{t^{m-1}}{(m-1)!} z_m$. But then,

$$MZ(t) = e^{\xi t} \sum_{m=0}^{d-2} \frac{t^m}{m!} Z_{m+1} = e^{\xi t} \sum_{m=1}^{d-1} \frac{t^{m-1}}{(m-1)!} z_m.$$

The assertion is thus true for $n = 0$. Let now $n \geq 2$ and assume that the formula holds on $((n-1)r, nr)$. We will show that it holds on $(nr, (n+1)r)$. On $(nr, (n+1)r)$, we have

$$Z(t) = e^{\xi t} \sum_{j=0}^{d-1} \frac{t^j}{j!} z_j + \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in I_k : p(x)=l\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} z_m.$$

Therefore,

$$\acute{Z}(t) = \xi EZ(t)$$

$$+ e^{\xi t} \sum_{j=1}^{d-1} \frac{t^{j-1}}{(j-1)!} z_j \tag{18}$$

$$+ \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in I_k : p(x)=l\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m. \tag{19}$$

By (17), it suffices to show that

$$MZ(t) + HZ(t-r) = (18) + (19).$$

For $t \in (nr, (n+1)r)$, $t - r \in ((n-1)r, nr)$, hence

$$\begin{aligned}
& MZ(t) + HZ(t-r) \\
= \quad & e^{\xi t} \sum_{j=0}^{d-1} \frac{t^j}{j!} M z_j && (20) \\
& + \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x \in I_k : p(x)=l\}} Mx \sum_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} z_m && (21) \\
& + e^{\xi(t-r)} \sum_{j=0}^{d-1} \frac{(t-r)^j}{j!} H z_j && (22) \\
& + \sum_{k=1}^{n-1} e^{\xi(t-(k+1)r)} \sum_{l=k}^{dk} \sum_{\{x \in I_k : p(x)=l\}} Hx \sum_{m=0}^{d-1} \frac{(t-(k+1)r)^{l+m}}{(l+m)!} z_m. && (23)
\end{aligned}$$

Now (20) $= e^{\xi t} \sum\limits_{j=0}^{d-2} \frac{t^j}{j!} z_{j+1} = e^{\xi t} \sum\limits_{j=1}^{d-1} \frac{t^{j-1}}{(j-1)!} z_j$, which is (18). We will now show that $(21) + (22) + (23) = (19)$. We shall use the convention that if g is a map taking values in a set in which addition is defined and contains a zero, then $\sum\limits_{\{x \in \emptyset\}} g(x) = 0$.

$$\begin{aligned}
(21) \quad = \quad & \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk-1} \sum_{\{x \in \Lambda_{k,(l+1)}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} z_m && (24) \\
= \quad & \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k+1}^{dk} \sum_{\{x \in \Lambda_{k,l}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m \\
= \quad & \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x \in \Lambda_{k,l}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m, && (25)
\end{aligned}$$

where (24) follows from Remark 2 (iii), while (25) follows from Remark 2 (i)(a). Since $I_1^0 = \{H\} = \{x \in I_1^0 : p(x) = 1\}$, it follows that

$$(22) = e^{\xi(t-r)} \sum_{l=1}^{d} \sum_{\{x \in \triangle_{1,l}\}} x \sum_{m=0}^{d-1} \frac{(t-r)^{(l-1)+m}}{((l-1)+m)!} z_m. \qquad (26)$$

By Remark 2 (iv),

$$
\begin{aligned}
(23) \quad &= \sum_{k=1}^{n-1} e^{\xi(t-(k+1)r)} \sum_{l=k}^{dk} \sum_{\{x\in\Delta_{(k+1),(l+1)}\}} x \sum_{m=0}^{d-1} \frac{(t-(k+1)r)^{l+m}}{(l+m)!} z_m \\
&= \sum_{k=2}^{n} e^{\xi(t-kr)} \sum_{l=k-1}^{dk-d} \sum_{\{x\in\Delta_{k,(l+1)}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} z_m \\
&= \sum_{k=2}^{n} e^{\xi(t-kr)} \sum_{l=k}^{(k-1)d+1} \sum_{\{x\in\Delta_{k,l}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m \\
&= \sum_{k=2}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in\Delta_{k,l}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m, \qquad (27)
\end{aligned}
$$

where (27) follows from Remark 2 (i)(b). Therefore,

$$
\begin{aligned}
& (21)+(22)+(23) \\
= \;& (25)+(26)+(27) \\
= \;& \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in\Lambda_{k,l}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m \\
& + \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in\Delta_{k,l}\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m \\
= \;& \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in I_k : p(x)=l\}} x \sum_{m=0}^{d-1} \frac{(t-kr)^{(l-1)+m}}{((l-1)+m)!} z_m \\
= \;& (19).
\end{aligned}
$$

In the notation of Section 2 and Lemma 4 we have the following

THEOREM 1. For $t \geq 0$, define

$$
K(t) := \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x\in I_k : p(x)=l\}} QxQ^{-1} \sum_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} QM^m Q^{-1},
$$

then the fundamental matrix solution for the homogeneous system (3) is given by

$$
G(t) = \begin{cases} K(t) & \text{for } t \geq 0 \\ E1_{\{0\}}(t) & \text{for } t \in [-r, 0]. \end{cases}
$$

PROOF. Let $\eta \in \mathbb{R}^d$ and define $X(t) := G(t)\eta, \; t \in [-r\ \infty)$. We have to show that $X(t)$ satisfies (3) Lebesgue a.e. on $[0, \infty)$ with $X(t) = \eta 1_{\{0\}}(t), \; t \in [-r, 0]$. For

$t \in [-r, 0]$, $X(t) = G(t)\eta = E1_{\{0\}}(t)\eta = \eta 1_{\{0\}}(t)$. By Lemma 4,

$$Z(t) := \begin{cases} \sum\limits_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum\limits_{l=k}^{dk} \sum\limits_{\{x \in I_k : p(x)=l\}} x \sum\limits_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} z_m, & \text{for } t \geq 0, \\ z1_{\{0\}}(t), & \text{for } t \in [-r, 0] \end{cases}$$

satisfies (7) Lebesgue almost everywhere on $[0, \infty)$ with the initial condition $Z(t) = z1_{\{0\}}(t),\ t \in [-r, 0]$. In particular, if $z = Q^{-1}\eta$, then $X(t) = QZ(t)$ satisfies (3) Lebesgue a.e. on $[0, \infty)$ with $X(t) = \eta 1_{\{0\}}(t),\ t \in [-r, 0]$. Now for $t \geq 0$,

$$\begin{aligned} QZ(t) &= \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{dk} \sum_{\{x \in I_k : p(x)=l\}} QxQ^{-1} \sum_{m=0}^{d-1} \frac{(t-kr)^{l+m}}{(l+m)!} QM^m Q^{-1}\eta \quad (28) \\ &= K(t)\eta = G(t)\eta. \end{aligned}$$

Therefore $X(t) = G(t)\eta$ for $t \in [-r\ \infty)$ satisfies (3) Lebesgue a.e. on $[0, \infty)$ with $X(t) = \eta 1_{\{0\}}(t)$ for $t \in [-r, 0]$.

From Theorem 1 we obtain the following Corollary which generalizes formula 2.3 in [4]:

COROLLARY 1. If the matrix A is diagonal then the fundamental matrix solution for the homogeneous system (3) is given by

$$G(t) = \begin{cases} \sum\limits_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \frac{(t-kr)^k}{k!} B^k, & \text{for } t \geq 0, \\ E1_{\{0\}}(t), & \text{for } t \in [-r, 0]. \end{cases}$$

PROOF. If A is diagonal i.e. $\tau_i = 0$ for all i, then $M^m = 0$ for all $m \geq 1$ and hence (28) implies

$$X(t) = \begin{cases} \sum\limits_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum\limits_{l=k}^{dk} \sum\limits_{\{x \in I_k : p(x)=l\}} QxQ^{-1} \frac{(t-kr)^l}{l!} E\eta, & \text{for } t \geq 0, \\ \eta 1_{\{0\}}(t), & \text{for } t \in [-r, 0]. \end{cases}$$

Since

$$\{x \in I_k : p(x) = l\} = \{0\}$$

for $l = k+1, \ldots, dk$, it follows that

$$G(t) = \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} QH^k Q^{-1} \frac{(t-kr)^k}{k!},\ t \geq 0.$$

Now note that $B = QHQ^{-1}$.

We also have the following generalization of formula 2.4 in [4]

COROLLARY 2. If the matrix A is diagonalizable then the solution of (1) (2) with integrable g and f is given for $t \geq 0$ by

$$X(t) := G(t)g(0) + B\int_{-r}^{0} G(t-s-r)g(s)ds + \int_{0}^{t} G(t-s)f(s)ds.$$

PROOF. The assertion follows from (5) and the fact that if A is diagonalizable then G and B commute.

We will now give an example illustrating the considerations above: Consider the Type I system

$$\begin{aligned} \acute{x}(t) &= Ax(t) + Bx(t-r), \; t \geq 0, \\ x(t) &= g(t), \; t \in [-r, 0], \end{aligned}$$

where $r > 0$,

$$A = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } g(t) = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}.$$

The only eigenvalue of A is $\xi = 1$, $Q^{-1}AQ = J$ where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix} \text{ and } Q^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \\ 2 & -\frac{3}{2} & -1 \end{pmatrix}.$$

For $t \in [0, r)$, $K(t) = e^t \sum\limits_{m=0}^{1} \frac{t^m}{m!} QM^mQ^{-1} = e^t[E + tQMQ^{-1}]$ since $M^2 = 0$.

Therefore the fundamental matrix solution is given on $[-r, r)$ by

$$G(t) := \begin{cases} e^t \left[E + tQMQ^{-1}\right], & \text{for } t \in [0, r), \\ E1_{\{0\}}(t), & \text{for } t \in [-r, 0]. \end{cases}$$

Since

$$QMQ^{-1} = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \text{ and } g(t) = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

we see that for $t \in [0, r)$,

$$\begin{aligned}
X(t) &= G(t)g(0) + \int_{-r}^{0} G(t-s-r)Bg(s)ds \\
&= \int_{-r}^{t-r} K(t-s-r)Bg(s)ds \\
&= \int_{-r}^{t-r} e^{t-s-r}\left[E + (t-s-r)\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ds \\
&= \int_{-r}^{t-r} se^{t-s-r}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (t-s-r)\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix}\right] ds \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\int_{-r}^{t-r} se^{t-s-r}ds + \begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix}\int_{-r}^{t-r} se^{t-s-r}(t-s-r)ds \\
&= [e^t(1-r) - (t-r+1)]\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + [(2+t-r) - \{2+tr-t-r\}e^t]\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix}
\end{aligned}$$

and for $t \in [r, 2r)$,

$$\begin{aligned}
K(t) &= \sum_{k=0}^{1} e^{(t-kr)} \sum_{l=k}^{3k} \sum_{\{x \in I_k : p(x)=l\}} QxQ^{-1} \sum_{m=0}^{1} \frac{(t-kr)^{(l+m)}}{(l+m)!} QM^mQ^{-1} \\
&= e^t[E + tQMQ^{-1}] + e^{t-r} \sum_{l=1}^{3} \sum_{\{x \in I_1 : p(x)=l\}} QxQ^{-1} \sum_{m=0}^{1} \frac{(t-r)^{m+l}}{(m+l)!} QM^mQ^{-1}
\end{aligned}$$

since $I_0 = \{E\}$. Now $I_1^0 = \{H\}$, $I_1^1 = \{(MH)\}$, $I_1^2 = \{M^2H\}$ and so $I_1 = \{H, (MH), (M^2H)\}$, $p(E) = 0$, $p(H) = 1$, $p(MH) = 2$ and $p(M^2H) = 3$. However $M^2H = 0$. Therefore

$$\begin{aligned}
K(t) &= e^t[E + tQMQ^{-1}] + e^{t-r}QHQ^{-1} \sum_{m=0}^{1} \frac{(t-r)^{m+1}}{(m+1)!} QM^mQ^{-1} \\
&\quad + e^{t-r}QMHQ^{-1} \sum_{m=0}^{1} \frac{(t-r)^{m+2}}{(m+2)!} QM^mQ^{-1} \\
&= e^t[E + tQMQ^{-1}] + \sum_{m=0}^{1} \frac{(t-r)^{m+1}}{(m+1)!} e^{t-r}QHM^mQ^{-1} \\
&\quad + \sum_{m=0}^{1} \frac{(t-r)^{m+2}}{(m+2)!} e^{t-r}QMHM^mQ^{-1} \\
&= e^t[E + tQMQ^{-1}] + (t-r)e^{t-r}QHQ^{-1} + \frac{(t-r)^2}{2} e^{t-r}Q(HM + MH)Q^{-1}
\end{aligned}$$

where the last equality is explained by the fact that $QMHMQ^{-1} = 0$. QMQ^{-1} was evaluated above, $QHQ^{-1} = B$ and one can check that

$$Q(HM + MH)Q^{-1} = \begin{pmatrix} 8 & -5 & -2 \\ 8 & -4 & 0 \\ 4 & -4 & -4 \end{pmatrix}.$$

Therefore,

$$K(t) = e^t[E + tQMQ^{-1}] + (t-r)e^{t-r}QHQ^{-1} + \frac{(t-r)^2}{2}e^{t-r}Q(HM+MH)Q^{-1}$$

and

$$\begin{aligned}
X(t) &= \int_{-r}^{0} G(t-s-r)Bg(s)ds = \int_{-r}^{0} K(t-s-r)Bg(s)ds \\
&= \int_{-r}^{0} s\Big[e^{t-s-r}\{E + (t-s-r)QMQ^{-1}\} + (t-s-2r)e^{t-s-2r}QHQ^{-1} \\
&\quad + \frac{(t-s-2r)^2}{2}e^{t-s-2r}Q(HM+MH)Q^{-1}\Big]\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ds \\
&= \int_{-r}^{0} se^{t-s-r}ds \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_{-r}^{0} e^{t-s-r}s(t-s-r)ds \begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix} \\
&\quad + \int_{-r}^{0} s(t-s-2r)e^{t-s-2r}ds \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_{-r}^{0} s\frac{(t-s-2r)^2}{2}e^{t-s-2r}ds \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix} \\
&= \Big(e^t\{t - tr + r - 2\} - e^{t-r}\{t-r-2\}\Big)\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix} \\
&\quad + \Big(e^t - re^t + e^{t-r}\{-r(t-r) + t - 3\} - e^{t-2r}\{t-2r-2\}\Big)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&\quad + \Big\{e^{t-r}\left[-r\frac{(t-r)^2}{2} + \frac{(t-r)^2}{2} - (t-r) + 1\right] \\
&\quad - e^{t-2r}\left[\frac{(t-2r)^2}{2} - (t-2r) + 1\right] \\
&\quad - \Big[e^{t-r}\{-r(t-r) + t - 2\} - e^{t-2r}\{t-2r-2\}\Big]\Big\}\begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix}.
\end{aligned}$$

The process can then be continued on the interval $[kr, (k+1)r)$, $k \geq 2$.

4 Systems of Type II

In this Section, we will use the results of Section 3 to give an explicit representation for the fundamental matrix solution for the homogeneous system (3) and hence a solution of (1)(2) for a certain class of Type II systems.

In what follows all vectors will be understood to be column vectors and we will write x^* for the transpose of the vector x. Let $d \geq 2$ and consider the system (3)(4). There exists an invertible matrix $Q \in \mathbb{M}(d, d, \mathbb{R})$ such that $A = QJQ^{-1}$, where for some $n \geq 1$,

$$J = Diag(J_{11}, \ldots, J_{nn}),$$

a block-diagonal matrix such that for each $i = 1, \ldots, n$, J_{ii} is a square matrix having the same structure as the matrix J in (6). For $i = 1, \ldots, n$, let d_i be the dimension of J_{ii} and define

$$\nu_i := \begin{cases} 1, & i = 1 \\ d_1 + \cdots + d_{i-1} + 1, & i = 2, \ldots, n \end{cases} \quad \text{and } \mu_i := d_1 + \cdots + d_i.$$

Let $H = (h_{ij})_{ij=1,\ldots,d}$. The matrix J induces a partition of H into sub-matrices- $H = (H_{ij})_{ij=1,\ldots,n}$ where $H_{ij} := (h_{kl})_{\substack{k=\nu_i,\ldots,\mu_i \\ l=\nu_j,\ldots,\mu_j}}$. With this notation we have the following:

LEMMA 5. Assume that $H_{ij} = 0$ for all $i < j$ and for $i = 1, \ldots, n$ let G_{ii} denote the fundamental matrix solution for the system

$$\acute{W}_i(t) = J_{ii} W_i(t) + H_{ii} W_i(t - r), \ t \geq 0.$$

For $i = 1, 2, \ldots, n$ successively, $t \geq 0$ and each $j \in \{1, \ldots, n\}$ define the sub-matrices $F_{ij}(t)$ by

$$F_{ij}(t) := \begin{cases} 0 & \text{for } j > i, \\ G_{ii}(t) & \text{for } i = j, \\ \sum\limits_{l=j}^{i-1} \int_0^t F_{ii}(t-s) H_{il} F_{lj}(s-r) ds & \text{for } i > j, \end{cases}$$

and let $F(t) := (F_{ij}(t))_{ij=1,\ldots,n}$. For $z \in \mathbb{R}^d$, let

$$Z(t) := \begin{cases} F(t)z & \text{for } t \geq 0, \\ z 1_{\{0\}}(t) & \text{for } t \in [-r, 0]. \end{cases}$$

Then Z solves (13)(14).

PROOF. For $i = 1, \ldots, n$ let S_i be the system

$$\acute{W}_i(t) = J_{ii} W_i(t) + H_{ii} W_i(t-r) + \sum_{j=1}^{i-1} H_{ij} W_j(t-r), \ t \geq 0, \tag{29}$$

$$W_i(t) = \eta_i 1_{\{0\}}(t), \ t \in [-r, 0], \tag{30}$$

where $\eta_i := (z_{\nu_i}, \ldots, z_{\mu_i})^*$. Each system S_i is a system of Type I. Under the assumptions above, solving (13)(14) reduces to solving the n Type I systems, $S_i, \ i = 1, \ldots, n$.

In particular $((W_1(t))^*, \ldots, (W_n(t))^*)^* = Z(t)$ where Z is the solution of (13)(14). If we solve the systems $\{S_i\}$ successively in the order $S_1, S_2, \ldots, S_n$ then S_1 is a homogeneous system of Type I while the systems $S_2, \ldots, S_n$ are inhomogeneous Type I systems.

Let $\xi_1, \ldots, \xi_n$ be the distinct eigenvalues of A such that J_{ii} corresponds to ξ_i. Let E_i be the identity matrix of dimension d_i and $J_{ii} = \xi_i E_i + M_i$, $i = 1, \ldots, n$. Further let $I^j_{i0} = \{E_i\}$ for all $j = 0, \ldots, d_i - 1$, $I^j_{ik} = T_{(M_i^j H_{ii})} I_{i(k-1)}$ and $I_{ik} = \cup\{I^j_{ik} : j = 0, \ldots, d_i - 1\}$.

By Lemma 4,

$$G_{ii}(t) = \begin{cases} K_i(t), & t \geq 0, \\ E_i 1_{\{0\}}(t), & t \in [-r, 0], \end{cases}$$

where

$$K_i(t) := \sum_{k=0}^{[\frac{t}{r}]} e^{\xi_i(t-kr)} \sum_{l=k}^{d_i k} \sum_{\{x \in I_{ik} : p(x) = l\}} x \sum_{m=0}^{d_i - 1} \frac{(t-kr)^{l+m}}{(l+m)!} M_i^m.$$

By (5) the solution of (29), (30) is given by

$$W_i(t) = G_{ii}(t)\eta_i + \sum_{j=1}^{i-1} \int_0^t G_{ii}(t-s) H_{ij} W_j(s-r) ds \tag{31}$$

and hence $W(t) = ((W_1(t))^*, \ldots, (W_n(t))^*)^*$ solves

$$\begin{aligned} \acute{W}(t) &= JW(t) + HW(t-r), \ t \geq 0, \\ W(t) &= z 1_{\{0\}}(t), \ t \in [-r, 0]. \end{aligned}$$

We will now show that for $t \geq -r$,

$$W(t) = Z(t). \tag{32}$$

It is obvious that $W(t) = Z(t)$ for all $t \in [-r, 0]$. We will now show that $W(t) = Z(t)$ for $t > 0$.

To do this, we show by induction that the i-th vector $W_i(t)$ on the left hand side of (32) equals the i-th vector on the right hand side, $i = 1, \ldots, n$. For $i = 1$, using (31), $W_1(t) = G_{11}(t)\eta_1 = F_{11}(t)\eta_1$. Assume that the assertion is true for some i, $1 \leq i \leq n-1$. We now show that it is true for $i+1$. Again by (31) and defining $\varphi(i, t, s, l) := F_{ii}(t-s)H_{il}$,

$$\begin{aligned} W_{i+1}(t) &= G_{(i+1)(i+1)}(t)\eta_{i+1} + \sum_{l=1}^{i} \int_0^t G_{(i+1)(i+1)}(t-s) H_{(i+1)l} W_l(s-r) ds \\ &= F_{(i+1)(i+1)}(t)\eta_{i+1} + \sum_{l=1}^{i} \int_0^t \varphi(i+1, t, s, l) W_l(s-r) ds. \end{aligned}$$

By the assumption of the induction,

$$\begin{aligned}
\sum_{l=1}^{i}\int_0^t \varphi(i+1,t,s,l)W_l(s-r)ds &= \sum_{l=1}^{i}\int_0^t \varphi(i+1,t,s,l)\sum_{j=1}^{l}F_{lj}(s-r)\eta_j ds \\
&= \sum_{l=1}^{i}\sum_{j=1}^{l}\int_0^t \varphi(i+1,t,s,l)F_{lj}(s-r)\eta_j ds \\
&= \sum_{j=1}^{i}\sum_{l=j}^{i}\int_0^t \varphi(i+1,t,s,l)F_{lj}(s-r)ds\eta_j \\
&= \sum_{j=1}^{i}F_{(i+1)j}(t)\eta_j.
\end{aligned}$$

Hence $W_{i+1}(t) = \sum_{j=1}^{i+1} F_{(i+1)j}(t)\eta_j$.

THEOREM 2. Under the assumptions of Lemma 5 the fundamental matrix solution for the homogeneous system (3) is given by

$$G(t) := \begin{cases} QF(t)Q^{-1} & t \geq 0 \\ E1_{\{0\}}(t) & t \in [-r, 0]. \end{cases}$$

PROOF. The proof is similar to the proof of Theorem 1 if we use Lemma 5.

REMARK 3. Similar to Theorem 2 an explicit representation of the fundamental matrix solution for the homogeneous system (3) can be obtained if $H_{ij} = 0$ for all $i > j$.

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