Applied Mathematics E-Notes, 13(2013), 36-50 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On A Generalization Of The Laurent Expansion Theorem*

Branko Sarić†

Received 19 March 2012

Abstract

Based on the properties of the non-univalent conformal mapping $e^z = s$ a causal connection has been established between the *Laurent* expansion theorem and the *Fourier* trigonometric series expansion of functions. This connection combined with two highly significant results proved in the form of lemmas is a foundation stone of the theory. The main result is in the form of a theorem that is a natural generalization of the *Laurent* expansion theorem. The paper ends with a few examples that illustrate the theory.

1 Introduction

A trigonometric series is a series of the form

$$\frac{a_0}{2} + \sum_{k=1}^{+\infty} \left[a_k \cos(k\zeta) + b_k \sin(k\zeta)\right] \tag{1}$$

where the real coefficients $a_0, a_1, ..., b_1, b_2, ...$ are independent of the real variable ζ. Applying *Euler's* formulae to $\cos(k\zeta)$ and $\sin(k\zeta)$, we may write (1) in the complex form

$$\sum_{k=-\infty}^{+\infty} c_k e^{ik\zeta} \tag{2}$$

where $2c_k = a_k - ib_k$ and for $k \in \aleph$ ($\aleph$ is the set of natural numbers) and $2c_0 = a_0$. Here c_{-k} is conjugate to c_k [8].

Let $z = \xi + i\zeta$ be a complex variable. Suppose that series (2) converges at all points ζ of the interval $(-\pi, \pi)$ to a real valued point function $f(\zeta)$. If $f(-iz)$ is integrable on the interval $\Im_{-\pi}^{\pi} = \{z \mid \xi = 0, \zeta \in [-\pi, \pi]\}$, then the *Fourier* coefficients c_k of $f(\zeta)$ are determined uniquely as

$$c_k = \frac{1}{2\pi i} \int_{\Im_{-\pi}^{\pi}} f(-iz)\, e^{-kz} dz \ (k = 0, \pm1, \pm2, ...). \tag{3}$$

*Mathematics Subject Classifications: 15A15, 42A168,; 30B10, 30B50.

†Faculty of Sciences, University of Novi Sad. Trg Dositeja Obradovica 2, 21000 Novi Sad, Serbia; College of Technical Engineering Professional Studies, Svetog Save 65, 32 000 Čačak, Serbia

Similarly, if for $\xi = 0$ a trigonometric series $\sum_{k=-\infty}^{+\infty} \hat{c}_k \left(e^z\right)^k$ converges in $(-\pi, \pi)$ to a complex valued point function $f\left(e^z\right)$, which is integrable on $\Im_{-\pi}^{\pi}$, then

$$\hat{c}_k = \frac{1}{2\pi i} \int_{\Im_{-\pi}^{\pi}} f\left(e^z\right) e^{-kz} dz \ (k = 0, \pm 1, \pm 2, ...). \tag{4}$$

As is well-known, progress in *Fourier* analysis has gone hand in hand with progress in theories of integration. Perhaps this can be best exemplified by using the so-called total value of the generalized *Riemann* integrals [5, 6]. This brand new theory of integration, which takes the notion of residues of real valued functions into account, gives us the opportunity to integrate real valued functions that was not integrable in any of the known integration methods until now. All of this leads to the following very important problem. Let $\mathcal{C}$ be a simple closed contour in the z-plane, which consists of $\Im_{-\pi}^{\pi}$ and any regular curve $Q_{-\pi}^{\pi}$ connecting the endpoints of $\Im_{-\pi}^{\pi}$, inside and on which a complex valued point function $f(z)$ is analytic except for a finite number of isolated singular points $z_1, z_2, ..., z_n$ belonging to the interior (int) of $\Im_{-\pi}^{\pi}$. Chose $\rho > 0$ to be small enough so that $Q_{-\pi}^{\pi}$ and the semi-circumferences $\gamma_\nu^+ = \{z \mid |z - z_\nu| = \rho, \xi > 0\}$ and $\gamma_\nu^- = \{z \mid |z - z_\nu| = \rho, \xi < 0\}$ $(\nu = 1, 2, ..., n)$ are disjoint. Having formed the numbers $\hat{c}_k$ by means of

$$\begin{aligned} \hat{c}_k &= \frac{1}{2\pi i} vt \int_{\Im_{-\pi}^{\pi}} f(z) e^{-kz} dz \\ &= \frac{1}{2\pi i} \lim_{\rho \to 0^+} \left[\sum_{\nu=0}^{n} \int_{l_\nu} f(z) e^{-kz} dz + \sum_{\nu=1}^{n} \left\{ \begin{array}{l} \int_{\gamma_\nu^-}^{\frown} f(z) e^{-kz} dz \\ \int_{\gamma_\nu^+}^{\frown} f(z) e^{-kz} dz \end{array} \right. \right] \end{aligned} \tag{5}$$

for $k = 0, \pm 1, \pm 2, ...$ where vt denotes the *total* value of an improper integral, $l_0 = [-i\pi, z_1 - i\rho]$, $l_\nu = [z_\nu + i\rho, z_{\nu+1} - i\rho]$ $(\nu = 1, 2, ..., n-1)$ and $l_n = [z_n + i\rho, i\pi]$, we may call the series $\sum_{k=-\infty}^{+\infty} \hat{c}_k e^{ik\zeta}$ the *Fourier* series of $f(z)$ on $\Im_{-\pi}^{\pi}$. This means that the numbers $\hat{c}_k$ are connected with $f(z)$ by the formula (5). Clearly, in this case, the numbers $\hat{c}_k$, which are the *Fourier* coefficients of $f(z)$, are not determined uniquely since on account of *Cauchy's* residue theorem

$$vt \int_{\Im_{-\pi}^{\pi}} f(z) e^{-kz} dz = \int_{Q_{-\pi}^{\pi}}^{\frown} f(z) e^{-kz} dz + \left\{ \begin{array}{c} 0 \\ 2\pi i \sum_{\nu=1}^{n} Res_{z=z_\nu} [f(z) e^{-kz}] \end{array} \right. \tag{6}$$

whenever $Q_{-\pi}^{\pi}$ is in the left half plane of the z-plane. A question that arises is whether this *Fourier* series of $f(z)$ does in fact converge and if it does not whether it is summable in any other sense and under what conditions is its sum equal to $f(z)$ in $int\Im_{-\pi}^{\pi} \backslash \{z_1, z_2, ..., z_n\}$. As we shall see, in what follows, the answer to this question is closely related to the *Laurent* expansion theorem.

2 Main Results

By the *Laurent* expansion theorem, a single-valued complex function $f(s)$ of a complex variable s, analytic in an arbitrary punctured disc $\{s \mid 0 < |s| < R\}$ of the s-plane, has a *Laurent* series expansion $f(s) = \sum_{k=-\infty}^{+\infty} \hat{c}_k s^k$. The *Laurent* coefficients $\hat{c}_k$ are determined uniquely as

$$\hat{c}_k = \frac{1}{2\pi i} \int_{\mathcal{C}_r}^{\circlearrowleft} \frac{f(s)}{s^{k+1}} ds \ (k = 0, \pm 1, \pm 2, ...) \tag{7}$$

where $\mathcal{C}_r = \{s : |s| = r, 0 < r < R\}$ [3]. On the other hand, it is well-known that the horizontal strip $\mathcal{D} = \{z = \xi + i\zeta : \xi \in (-\infty, \rho), \zeta \in [-\pi, \pi]\}$ of the z-plane is mapped onto the punctured disc $\mathcal{D}^\star = \{s : 0 < |s| < e^\rho\}$ in the s-plane by the non-univalent conformal mapping $s = e^z$. Accordingly, if a single-valued complex function $f(s)$ is analytic in $\mathcal{D}^\star$ and $\rho > 0$, then, on account of the *Laurent* expansion theorem, the *Fourier* series $\sum_{k=-\infty}^{+\infty} \hat{c}_k (e^z)^k$ with the *Fourier* coefficients (4) converges in $\mathcal{D}$ to $f(e^z)$. Furthermore, it follows from (7) that for any real number $\sigma < \rho$

$$\hat{c}_k = \frac{1}{2\pi i} \int_{\sigma - i\pi}^{\sigma + i\pi} f(e^z) e^{-kz} dz \ (k = 0, \pm 1, \pm 2, ...). \tag{8}$$

Here a single-valued complex function $f(s)$ is taken to be the function $f(e^z)$ of period 2π with respect to ζ in the z-plane.

Generally, a complex valued point function $f(z)$ is said to be integrable, of bounded variation and the like, with respect to either ξ or ζ, in the strip $\mathcal{D}$, if its real and imaginary parts satisfy separately the aforementioned properties for all fixed values of the second variable in $\mathcal{D}$.

As is well known, the first theorem, chronologically, in the theory of *Fourier* trigonometric series, is as follows [8].

THEOREM 1. If a real valued point function f is of bounded variation, then the *Fourier* series of f converges at every point ζ to the value

$$\frac{1}{2} \lim_{\tau \to 0^+} [f(\zeta + \tau) + f(\zeta - \tau)]. \tag{9}$$

If f is in addition continuous at every point on an interval $I = [a, b]$, then the *Fourier* series of f is uniformly convergent in I.

The result of the preceding theorem implies the following result.

THEOREM 2. If a single valued complex function $f(s)$ is of bounded variation with respect to $\arg s$ in a punctured disc $\mathcal{D} = \{s : 0 < |s| < e^\rho\}$ of the s-plane, then the *Laurent* series $\sum_{k=-\infty}^{+\infty} \hat{c}_k s^k$, with the coefficients

$$\hat{c}_k = \frac{1}{2\pi i} \int_{\mathcal{C}_\xi}^{\circlearrowleft} \frac{f(s)}{s^{k+1}} ds \ (k = 0, \pm 1, \pm 2, ...), \tag{10}$$

where $\mathcal{C}_\xi = \{s : |s| = e^\xi, \xi < \rho\}$, converges at every point $s \in \mathcal{C}_\xi$ to the value

$$\frac{1}{2} \lim_{\tau \to 0^+} \left[f\left(se^{i\tau}\right) + f\left(se^{-i\tau}\right)\right]. \tag{11}$$

If f is in addition continuous on $\mathcal{C}_\xi$, then the *Laurent* series of f is uniformly convergent on $\mathcal{C}_\xi$.

This result strongly suggests that single valued complex functions having isolated singularities inside $\Im_{-\pi}^{\pi}$ may be expanded in the *Fourier* trigonometric series, too. Let's show this works. Let w be a complex parameter independent of $z = \xi + i\zeta$. As is well known, if a real valued point function $f(\zeta)$ is of bounded variation in the interval $[-\pi, \pi]$, then for $\hat{\zeta} \in (-\pi, \pi)$ and $\mathrm{Re}(w) \geq 0$ there exist finite limits, [3],

$$\lim_{|w|\to+\infty} w \int_{\Im_{-\pi}^{\hat{\zeta}}} e^{iw(i\hat{\zeta}-z)} f(-iz)\, dz = i \lim_{\tau\to 0^+} f(\hat{\zeta} - \tau)$$

and

$$\lim_{|w|\to+\infty} w \int_{\Im_{\hat{\zeta}}^{\pi}} e^{-iw(i\hat{\zeta}-z)} f(-iz)\, dz = -i \lim_{\tau\to 0^+} f(\hat{\zeta} + \tau) \tag{12}$$

where $\Im_{-\pi}^{\hat{\zeta}} = \{z = \xi + i\zeta : \xi = 0, \zeta \in [-\pi, \hat{\zeta}]\}$ and $\Im_{\hat{\zeta}}^{\pi} = \{z = \xi + i\zeta : \xi = 0, \zeta \in [\hat{\zeta}, \pi]\}$. Since the above conditions play a key role in the proof of *Theorem 1* based on the *Cauchy* calculus of residues, see [3], it follows that all functions that satisfy these conditions have a *Fourier* series expansion

$$\lim_{\tau\to 0^+} \frac{f(\zeta+\tau) + f(\zeta-\tau)}{2} = \sum_{k=-\infty}^{+\infty} \hat{c}_k e^{ik\zeta},$$

with the *Fourier* coefficients (4). Furthermore, pursuing the matter in the proof of *Theorem 1*, one easily comes to a conclusion that conditions (12) cannot be satisfied for $\mathrm{Re}(w) \geq 0$ but only for $\mathrm{Re}(w) > 0$. This is an immediate consequence of the result presented in the following form.

LEMMA 1. For $w \in \mathbb{C}$, let a single valued complex function $f(w)$ be analytic in the w-plane with the exception of an infinite but countable number of isolated singularities w_k $(k = 1, 2, ...)$ on the half-straight lines $\{w = |w|\, e^{i\theta} : |w| > 0, \theta = \hat{\theta}\}$ and $\{w = |w|\, e^{i\theta} : |w| > 0, \theta = \pi + \hat{\theta}\}$ by which the w-plane is divided into two open adjacent regions $\mathcal{R} = \{w = |w|\, e^{i\theta} : |w| > 0, -\pi + \hat{\theta} < \theta < \hat{\theta}\}$ and $\mathcal{L} = \{w = |w|\, e^{i\theta} : |w| > 0, \hat{\theta} < \theta < \pi + \hat{\theta}\}$. If $\lim_{|w|\to+\infty} wf(w) = C_{\mathcal{R}}$, whenever $w \in \mathcal{R}$ and $\lim_{|w|\to+\infty} wf(w) = C_{\mathcal{L}}$, whenever $w \in \mathcal{L}$, then

$$\sum_{k=1}^{+\infty} Res_{w=w_k} f(w) = \frac{C_{\mathcal{R}} + C_{\mathcal{L}}}{2}. \tag{13}$$

PROOF. By definition, the residue of $f(w)$ at the point at infinity is equal to the residue of the function $-f(1/w)/w^2$ at the point $w = 0$. In symbols,

$$Res_{|w|=+\infty} f(w) = -Res_{w=0} f(1/w)/w^2.$$

Since $-f(1/w)/w^2$ has no singularity at infinity, it follows from

$$\sum_{k=1}^{+\infty} Res_{w=w_k} f(1/w)/w^2 = -Res_{w=0} f(1/w)/w^2$$

that

$$\sum_{k=1}^{+\infty} Res_{w=w_k} f(w) = -Res_{|w|=+\infty} f(w). \tag{14}$$

For $\gamma_{\mathcal{R}}(w) = wf(w) - C_{\mathcal{R}}$ and $\gamma_{\mathcal{L}}(w) = wf(w) - C_{\mathcal{L}}$ let $M_{\mathcal{R}} = \max\{|\gamma_{\mathcal{R}}(w)| \mid w \in \mathcal{R}\}$ and $M_{\mathcal{L}} = \max\{|\gamma_{\mathcal{L}}(w)| \mid w \in \mathcal{L}\}$, respectively. By means of an arc length parametrization for the circular arcs

$$\mathcal{C}_\rho^{\mathcal{R}} = \{w = |w| e^{i\theta} : |w| = \rho, -\pi + \hat{\theta} + \alpha(\rho) \le \theta \le \hat{\theta} - \alpha(\rho)\}$$

and

$$\mathcal{C}_\rho^{\mathcal{L}} = \{w = |w| e^{i\theta} : |w| = \rho, \hat{\theta} + \alpha(\rho) \le \theta \le \pi + \hat{\theta} - \alpha(\rho)\}$$

where $\alpha(\rho)$ is an angular function, which can be made as small as one desires, such that $\lim_{\rho\to+\infty} \alpha(\rho) = 0$, we obtain

$$\begin{aligned} & \int_{\mathcal{C}_\rho^{\mathcal{R}}}^{\frown} f(w)\, dw + \int_{\mathcal{C}_\rho^{\mathcal{L}}}^{\frown} f(w)\, dw \\ = \; & i\left[\int_{-\pi+\hat{\theta}+\alpha(\rho)}^{\hat{\theta}-\alpha(\rho)} \rho e^{i\theta} f\left(\rho e^{i\theta}\right) d\theta + \int_{\hat{\theta}+\alpha(\rho)}^{\pi+\hat{\theta}-\alpha(\rho)} \rho e^{i\theta} f\left(\rho e^{i\theta}\right) d\theta\right] \\ = \; & i\left\{\int_{-\pi+\hat{\theta}+\alpha(\rho)}^{\hat{\theta}-\alpha(\rho)} \left[C_{\mathcal{R}} + \gamma_{\mathcal{R}}\left(\rho e^{i\theta}\right)\right] d\theta + \int_{\hat{\theta}+\alpha(\rho)}^{\pi+\hat{\theta}-\alpha(\rho)} \left[C_{\mathcal{L}} + \gamma_{\mathcal{L}}\left(\rho e^{i\theta}\right)\right] d\theta\right\} \\ = \; & i[\pi - 2\alpha(\rho)](C_{\mathcal{R}} + C_{\mathcal{L}}) + G_{\mathcal{R}} + G_{\mathcal{L}} \end{aligned}$$

where $G_{\mathcal{R}} = i\int_{-\pi+\hat{\theta}+\alpha(\rho)}^{\hat{\theta}-\alpha(\rho)} \gamma_{\mathcal{R}}\left(\rho e^{i\theta}\right) d\theta$ and $G_{\mathcal{L}} = i\int_{\hat{\theta}+\alpha(\rho)}^{\pi+\hat{\theta}-\alpha(\rho)} \gamma_{\mathcal{L}}\left(\rho e^{i\theta}\right) d\theta$, that is, $|G_{\mathcal{R}}| \le [\pi - 2\alpha(\rho)]M_{\mathcal{R}}$ and $|G_{\mathcal{L}}| \le [\pi - 2\alpha(\rho)]M_{\mathcal{L}}$. Therefore, the lemma conditions: $\lim_{|w|\to+\infty} wf(w) = C_{\mathcal{R}}$ $(w \in \mathcal{R})$ and $\lim_{|w|\to+\infty} wf(w) = C_{\mathcal{L}}$ $(w \in \mathcal{L})$, lead us to $G_{\mathcal{R}} \to 0$ and $G_{\mathcal{L}} \to 0$, as $\rho \to +\infty$. This implies that $\lim_{\rho\to+\infty} \int_{\mathcal{C}_\rho^{\mathcal{R}}}^{\frown} f(w)\, dw = i\pi C_{\mathcal{R}}$ and $\lim_{\rho\to+\infty} \int_{\mathcal{C}_\rho^{\mathcal{L}}}^{\frown} f(w)\, dw = i\pi C_{\mathcal{L}}$ so that

$$\frac{1}{2\pi i}\int_{\mathcal{C}_\infty}^{\circlearrowleft} f(w)\, dw = \frac{1}{2\pi i}\lim_{\rho\to+\infty}\left[\int_{\mathcal{C}_\rho^{\mathcal{R}}}^{\frown} f(w)\, dw + \int_{\mathcal{C}_\rho^{\mathcal{L}}}^{\frown} f(w)\, dw\right] = \frac{C_{\mathcal{R}} + C_{\mathcal{L}}}{2}$$

where $\mathcal{C}_\infty = \{w \mid |w| = +\infty\}$. Since $\int_{\mathcal{C}_\infty}^{\circlearrowleft} f(w)\, dw = -2\pi i Res_{|w|=+\infty} f(w)$, it follows from (14) that

$$\sum_{k=1}^{+\infty} Res_{w=w_k} f(w) = \frac{C_{\mathcal{R}} + C_{\mathcal{L}}}{2}.$$

REMARK: Clearly, the circular arcs $\mathcal{C}_\rho^\mathcal{R}$ and $\mathcal{C}_\rho^\mathcal{L}$ form the circular path not until at infinity. Thus, if the limit $\lim_{\rho\to+\infty}\int_{\mathcal{C}_\rho}^{\circlearrowleft} f(w)\,dw$, where $\mathcal{C}_\rho = \{w : |w| = \rho\}$ is a circumference that encloses the n singularities of $f(w)$ in its interior, does not exist, then it is more convenient to use the partial limits $\lim_{\rho\to+\infty}\int_{\mathcal{C}_\rho^\mathcal{R}}^{\frown} f(w)\,dw$ and $\lim_{\rho\to+\infty}\int_{\mathcal{C}_\rho^\mathcal{L}}^{\frown} f(w)\,dw$, as was done above.

So, if the limits (12) do not exist on the straight-line $\mathrm{Re}(w) = 0$, but only strongly in the open right half-plane $\mathrm{Re}(w) > 0$, then the infinite sum of all residues of $g(w,\hat{\zeta}) = \int_{\Im_{-\pi}^{\pi}} e^{-iw(i\hat{\zeta}-z)} f(-iz)\,dz/(1-e^{-2\pi w})$, which is a meromorphic function with countably infinite set of isolated simple poles $w_k = ik$ $(k = 0, \pm1, \pm2, ...)$ in the w-plane, does exist in some sense more general than the *Cauchy* one. In symbols, for any $\hat{\zeta} \in (-\pi,\pi)$, for which the conditions (12) are satisfied in the half plane $\mathrm{Re}(w) > 0$,

$$\sum_{k=-\infty}^{+\infty} \frac{1}{2\pi i}\int_{\Im_{-\pi}^{\pi}} e^{k(i\hat{\zeta}-z)} f(-iz)\,dz = \lim_{\tau\to 0^+}\frac{f(\hat{\zeta}+\tau)+f(\hat{\zeta}-\tau)}{2}. \tag{15}$$

Clearly, if $f(\zeta)$ satisfies the *Riemann-Lebesgue* lemma condition, more precisely if the *Fourier* coefficients (3) for $k\to+\infty$ tend to 0, the region of convergence of (12) extends to $\mathrm{Re}(w) \geq 0$. Now, to answer affirmatively the question from the introductory part of the paper, it is sufficient to prove the result in the following form.

LEMMA2. For $z = \xi + i\zeta$ let a single valued complex function $f(z)$ be analytic in the z-plane with the exception of a finite number of isolated poles $z_1, z_2, ..., z_n$ inside $\Im_{-\pi}^{\pi}$. If $\hat{\zeta} \in int\Im_{-\pi}^{\pi}$ is not a singular point of $f(z)$, then for $\mathrm{Re}(w) > 0$ there holds

$$\lim_{|w|\to+\infty} w\left[vt\int_{\Im_{-\pi}^{\hat{\zeta}}} e^{iw(i\hat{\zeta}-z)} f(z)\,dz\right] = if(i\hat{\zeta})$$

where $\Im_{-\pi}^{\hat{\zeta}} = \{z : \xi = 0, \zeta \in [-\pi,\hat{\zeta}]\}$.

PROOF. Suppose, without loss of generality, that $\hat{\zeta} > 0$ and all poles $z_1, ..., z_n$ of $f(z)$ lie in the interval $(-i\pi, i\hat{\zeta})$. Chose $\rho > 0$ to be small enough so that the endpoints of $\Im_{-\pi}^{\hat{\zeta}}$ and the semi-circumferences $\gamma_\nu^+ = \{z : |z - z_\nu| = \rho, \xi > 0\}$ and $\gamma_\nu^- = \{z : |z - z_\nu| = \rho, \xi < 0\}$ $(\nu = 1, 2, ..., n)$ are disjoint. If we apply the rule for integration by parts to the integrals $\int_{l_\nu} e^{iw(i\hat{\zeta}-z)} df(z)$, where $l_0 = [-i\pi, z_1 - i\rho]$, $l_\nu = [z_\nu + i\rho, z_{\nu+1} - i\rho]$ $(\nu = 1, 2, ..., n-1)$ and $l_n = [z_n + i\rho, i\hat{\zeta}]$, as well as to the integrals $\int_{\gamma_\nu^-}^{\frown} e^{iw(i\hat{\zeta}-z)} df(z)$ and $\int_{\gamma_\nu^+}^{\frown} e^{iw(i\hat{\zeta}-z)} df(z)$, then (5) yields

$$\begin{aligned}
&\lim_{\rho\to 0^+}\left[\sum_{\nu=0}^{n}\int_{l_\nu}^{\frown} e^{iw(i\hat{\zeta}-z)} df(z) + \sum_{\nu=1}^{n}\left\{\begin{array}{l}\int_{\gamma_\nu^-}^{\frown} e^{iw(i\hat{\zeta}-z)} df(z)\\ \int_{\gamma_\nu^+}^{\frown} e^{iw(i\hat{\zeta}-z)} df(z)\end{array}\right.\right] \\
&= \left[f(i\hat{\zeta}) - e^{-w(\hat{\zeta}+\pi)} f(-i\pi)\right] + iw\left[vt\int_{\Im_{-\pi}^{\hat{\zeta}}} e^{iw(i\hat{\zeta}-z)} f(z)\,dz\right].
\end{aligned} \tag{16}$$

It is easy to prove, by the complex mean value theorem [1], that

$$\lim_{|w|\to+\infty}\sum_{\nu=0}^{n}\int_{l_\nu}^{\frown} e^{iw(i\hat{\zeta}-z)}df\,(z)=0$$

whenever $\mathrm{Re}(w)>0$, and

$$\lim_{|w|\to+\infty}\sum_{\nu=1}^{n}\left\{\begin{array}{l}\int_{\gamma_\nu^-}^{\frown} e^{iw(i\hat{\zeta}-z)}df\,(z)\\ \int_{\gamma_\nu^+}^{\frown} e^{iw(i\hat{\zeta}-z)}df\,(z)\end{array}\right.=0$$

in the region of w-plane in which $\mathrm{Re}(w)\neq 0$ and

$$Re[iw(i\hat{\zeta}-\rho e^{i\theta})]=-Re(w)\left[(\hat{\zeta}-\rho\sin\theta)-\rho\frac{Im(w)}{Re(w)}\cos\theta\right]<0.$$

In view of the fact that $\hat{\zeta}>\rho>0$ there exists a positive real number k such that $\hat{\zeta}=(1+k)\,\rho$. Hence, the previous condition reduces to the condition

$$Re(w)[(1-\sin\theta)+k-\tan\varphi\cos\theta]>0$$

where $\tan\varphi=Im\,(w)/Re\,(w)$. As $|\sin\theta|\leq 1$ and $|\cos\theta|\leq 1$, then $Re(w)[(1-\sin\theta)+k-\tan\varphi\cos\theta]>0$, whenever $\varphi\in(-\pi/2,\arctan k)$. Since k tends to $+\infty$ ($\arctan k$ tends to $\pi/2$) as ρ tends to 0^+, it follows finally that both sums of integrals converge to zero in the region $\mathrm{Re}(w)>0$ of the w-plane. This last result combined with (16) implies that for $\mathrm{Re}(w)>0$ there holds

$$\lim_{|w|\to+\infty} w\left[vt\int_{\Im_{-\pi}^{\hat{\zeta}}} e^{iw(i\hat{\zeta}-z)}f\,(z)\,dz\right]=if(i\hat{\zeta}).$$

By using the results of *Lemmas 1* and *2* we are now able to extend an existing class of functions which can be represented by a *Fourier* series, as follows.

THEOREM 3. For $z\in\mathbb{C}$, let a single valued complex function $f\,(z)$ be analytic in the strip $\mathcal{D}=\{z=\xi+i\zeta:\xi\in(-\infty,+\infty)\,,\zeta\in[-\pi,\pi]\}$ of the z-plane, with the exception of a finite number of isolated poles inside $\Im_{-\pi}^{\pi}$. If $\hat{\zeta}\in int\Im_{-\pi}^{\pi}$ is not a singular point of $f\,(z)$, then its *Fourier* series $\sum_{k=-\infty}^{+\infty}\hat{c}_k e^{ik\hat{\zeta}}$, with the coefficients

$$\hat{c}_k=\frac{1}{2\pi i}vt\int_{\Im_{-\pi}^{\pi}} f\,(z)\,e^{-kz}dz=\frac{1}{2\pi i}\left\{\begin{array}{l}\int_{\hat{\xi}-i\pi}^{\hat{\xi}+i\pi} f\,(z)\,e^{-kz}dz \text{ for any } \hat{\xi}<0\\ \int_{\tilde{\xi}-i\pi}^{\tilde{\xi}+i\pi} f\,(z)\,e^{-kz}dz \text{ for any } \tilde{\xi}>0\end{array}\right. \tag{17}$$

for $k=0,\pm1,\pm2,...$, is summable and has a sum equal to the function value at the point $\hat{\zeta}$. In symbols,

$$\sum_{k=-\infty}^{+\infty}\hat{c}_k e^{ik\hat{\zeta}}=f(\hat{\zeta}).$$

Furthermore, this still holds for all points $\hat{\zeta} \in int\mathcal{D}$ on either the right or the left of $\Im_{-\pi}^{\pi}$, in addition to the fact that if $\hat{\zeta}$ lies to the right of $\Im_{-\pi}^{\pi}$, then

$$\hat{c}_k = \int_{\tilde{\xi}-i\pi}^{\tilde{\xi}+i\pi} f(z)\, e^{-kz} dz \ (k = 0, \pm 1, \pm 2, ...) \text{ for any } \tilde{\xi} > 0, \tag{18}$$

and if $\hat{\zeta}$ lies to the left of $\Im_{-\pi}^{\pi}$, then

$$\hat{c}_k = \int_{\hat{\xi}-i\pi}^{\hat{\xi}+i\pi} f(z)\, e^{-kz} dz \ (k = 0, \pm 1, \pm 2, ...) \text{ for any } \hat{\xi} < 0. \tag{19}$$

The theorem proof, see [4], is omitted because it is an immediate consequence of *Lemmas 1* and *2* as well as a formula for the expansion of functions into an infinite series derived from the *Cauchy* calculus of residues, see [3]. In view of the non-univalent conformal mapping $e^z = s$ the result of the preceding theorem implies the result in the form of the following theorem representing the extended *Laurent* expansion theorem.

THEOREM 4. Suppose that a single valued complex function $f(s)$ is analytic in the s-plane with the exception of a finite number of isolated poles on the unit circumference $\mathcal{C}_1$. Then $f(s)$ has a *Laurent* series expansion

$$f(s) = \sum_{k=-\infty}^{+\infty} \hat{c}_k s^k,$$

at all regular points of $f(s)$ lying on the unit circumference with the coefficients

$$\hat{c}_k = \frac{1}{2\pi i} vt \int_{\mathcal{C}_1}^{\circlearrowleft} \frac{f(s)}{s^{k+1}} ds = \frac{1}{2\pi i} \begin{cases} \int_{\mathcal{C}_R}^{\circlearrowleft} \frac{f(s)}{s^{k+1}} ds \text{ for any } R > 1 \\ \int_{\mathcal{C}_r}^{\circlearrowleft} \frac{f(s)}{s^{k+1}} ds \text{ for any } r < 1 \end{cases} \quad (k = 0, \pm 1, \pm 2, ...) \tag{20}$$

where $\mathcal{C}_r$ and $\mathcal{C}_R$ are circumferences with center at the origin and radius r and R, respectively.

Furthermore, this still holds for all points s inside and outside the unit circle, in addition to the fact that if s lies inside $\mathcal{C}_1$, then

$$\hat{c}_k = \frac{1}{2\pi i} \int_{\mathcal{C}_r}^{\circlearrowleft} \frac{f(s)}{s^{k+1}} ds \ (k = 0, \pm 1, \pm 2, ...) \text{ for any } r < 1, \tag{21}$$

and if s lies outside $\mathcal{C}_1$, then

$$\hat{c}_k = \frac{1}{2\pi i} \int_{\mathcal{C}_R}^{\circlearrowleft} \frac{f(s)}{s^{k+1}} ds \ (k = 0, \pm 1, \pm 2, ...) \text{ for any } R > 1. \tag{22}$$

3 Examples

We consider *an expansion of* $\sin\zeta/(1-\cos\zeta)$ *in a Fourier trigonometric series on the interval* $[-\pi,\pi]$.

As $f(\zeta)=\sin\zeta/(1-\cos\zeta)$ is an odd function, the *Cauchy* principal value of $\int_{-\pi}^{\pi} f(\zeta)\cos(k\zeta)\,d\zeta$ is equal to 0, for each $k\in\aleph_0$, where $\aleph_0$ is the set of natural numbers plus the number zero.

In addition, since $\int_0^{\pi}[\sin[(k+1/2)\zeta]/\sin(\zeta/2)]d\zeta=\pi$ and $\int_{-\pi}^{\pi}\cos(k\zeta)\,d\zeta=0$, as well as $\lim_{z\to 0} zf(z)\cos(kz)=2$, for each $k\in\aleph_0$, it follows that

$$a_0=\frac{1}{2\pi}vt\int_{-\pi}^{\pi} f(\zeta)\,d\zeta=\frac{1}{2\pi}vp\int_{-\pi}^{\pi} f(\zeta)\,d\zeta+\frac{1}{2\pi}\lim_{\rho\to 0^+}\left\{\begin{array}{l}\int_{\gamma^+}^{\frown} f(z)\,dz\\ \int_{\gamma^-}^{\frown} f(z)\,dz\end{array}\right.=\left\{\begin{array}{l} i\\ -i\end{array}\right. \tag{23}$$

and

$$\begin{aligned}a_k &= \frac{1}{\pi}vt\int_{-\pi}^{\pi} f(\zeta)\cos(k\zeta)\,d\zeta\\ &= \frac{1}{\pi}vp\int_{-\pi}^{\pi} f(\zeta)\cos(k\zeta)\,d\zeta+\frac{1}{\pi}\lim_{\rho\to 0^+}\left\{\begin{array}{l}\int_{\gamma^+}^{\frown} f(z)\cos(kz)\,dz\\ \int_{\gamma^-}^{\frown} f(z)\cos(kz)\,dz\end{array}\right.=\left\{\begin{array}{l} 2i\\ -2i\end{array}\right.\end{aligned} \tag{24}$$

for $k\in\aleph$ where γ^+ and γ^- are semi-circumferences centered at the origin and of a small enough radius ρ in the right and the left half plane of the z-plane, respectively, as well as that

$$b_k=\frac{1}{\pi}vt\int_{-\pi}^{\pi}\frac{\sin\zeta\sin(k\zeta)}{1-\cos\zeta}d\zeta=\frac{1}{\pi}\int_{-\pi}^{\pi}\cot\left(\frac{\zeta}{2}\right)\sin(k\zeta)\,d\zeta=2\ (k\in\aleph). \tag{25}$$

Accordingly, a *Fourier* trigonometric series of $f(\zeta)$ can be expressed by the following functional form

$$\frac{\sin\zeta}{1-\cos\zeta}=2\sum_{k=1}^{+\infty}\sin(k\zeta)\pm i\left[1+2\sum_{k=1}^{+\infty}\cos(k\zeta)\right]$$

for every $\zeta\in[-\pi,\pi]$ and $\zeta\neq 0$. Separating the real and imaginary parts in the preceding equation, we finally obtain that for every $\zeta\in[-\pi,\pi]$ and $\zeta\neq 0$

$$\frac{\sin\zeta}{1-\cos\zeta}=2\sum_{k=1}^{+\infty}\sin(k\zeta)\ \text{ and }\ \sum_{k=-\infty}^{+\infty}\cos(k\zeta)=0. \tag{26}$$

In the complex form, for every $\zeta\in[-\pi,\pi]$ and $\zeta\neq 0$,

$$\left\{\begin{array}{l} 1+2\sum_{k=1}^{+\infty}e^{-ik\zeta}\\ -1-2\sum_{k=1}^{+\infty}e^{ik\zeta}\end{array}\right.=\frac{e^{i\zeta}+1}{e^{i\zeta}-1}. \tag{27}$$

At the endpoints of $[-\pi,\pi]$ there holds $\sum_{k=-\infty}^{+\infty}\cos(k\pi)=0$. This implies that

$$\sum_{k=1}^{+\infty}(-1)^k=-1/2.$$

The complex form (27) points out the fact that we may use the *Laurent* expansion of the single valued complex function $f(s) = (s+1)/(s-1)$ to get the same result. Namely, since

$$\frac{1}{2\pi i} vt \int_{\mathcal{C}_1}^{\circlearrowleft} \frac{s+1}{s-1} \frac{ds}{s^{k+1}} = \left\{ \begin{array}{l} \left\{ \begin{array}{c} 2 \\ 0 \end{array} \right. , \text{ if } k<0 \\ \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right. , \text{ if } k=0 \\ \left\{ \begin{array}{c} 0 \\ -2 \end{array} \right. , \text{ if } k>0 \end{array} \right. ,$$

where $\mathcal{C}_1$ is the unit circumference in the s-plane, it follows from *Theorem 4* that for every $|s| = 1$ and $s \neq 1$

$$\left\{ \begin{array}{c} 1 + 2\sum_{k=1}^{+\infty} s^{-k} \\ -1 - 2\sum_{k=1}^{+\infty} s^{k} \end{array} \right. = \frac{s+1}{s-1}.$$

Clearly, this is the same as (27). Furthermore, $\sum_{k=1}^{+\infty} s^{-k} = 1/(s-1)$ for $|s| > 1$ and $\sum_{k=1}^{+\infty} s^{k} = s/(1-s)$ for $|s| < 1$.

An expansion of $1/(1-\cos\zeta)$ *in a Fourier trigonometric series on the interval* $[-\pi, \pi]$.

On the one hand,

$$\begin{aligned} a_0 &= vt \int_{-\pi}^{\pi} \frac{d\zeta}{1-\cos\zeta} \\ &= \lim_{\rho \to 0^+} [\int_{-\pi}^{-\rho} \frac{d\zeta}{1-\cos\zeta} + \left\{ \begin{array}{c} \int_{\gamma_0^+}^{\frown} \frac{dz}{(1-\cos z)} \\ \int_{\gamma_0^-}^{\frown} \frac{dz}{(1-\cos z)} \end{array} \right. + \int_{\rho}^{\pi} \frac{d\zeta}{1-\cos\zeta}] \\ &= \lim_{\rho \to 0^+} [\frac{\sin\rho}{1-\cos\rho} - \frac{2\sin\rho}{1-\cos\rho} + \frac{\sin\rho}{1-\cos\rho}] = 0 \end{aligned} \tag{28}$$

where $\gamma_0^+ = \{z = |z| e^{i\theta} : |z| = \rho \text{ and } \theta \in [-\pi, 0]\}$ and $\gamma_0^- = \{z = |z| e^{i\theta} : |z| = \rho$ and $\theta \in [0, \pi]\}$ are circular arcs bypassing the second order pole at the point $z = 0$ as a singularity of $f(z) = 1/(1-\cos z)$.

On the other hand, for each $k \in \aleph$ the function $f(s) = s^k/(s-1)$ is a meromorphic function having at the point $s = 1$ a simple pole as singularity. Since $\lim_{s\to 1}(s-1) f(s) = 1$, it follows from the *Jordan* lemmas that for any $k \in \aleph$

$$\begin{aligned} \frac{2}{\pi i} vp \int_{\mathcal{C}_1}^{\circlearrowleft} \frac{s^k}{s-1} ds &= \frac{2}{\pi i} vp \int_{-\pi}^{\pi} \frac{i e^{ik\theta}}{1-e^{-i\theta}} d\theta \\ &= \frac{1}{\pi} vp [\int_{-\pi}^{\pi} \frac{\cos(k\theta)}{1-\cos\theta} d\theta - \int_{-\pi}^{\pi} \frac{\cos[(k+1)\theta]}{1-\cos\theta} d\theta] \\ &= 2 \end{aligned}$$

where $\mathcal{C}_1$ is a unit circumference in the s-plane. Similarly, as the functions $z\cos(kz)$ and $z\sin(kz)\sin z/(1-\cos z)$ tend to 0 as $z \to 0$, then for any $k \in \aleph$

$$\frac{1}{2\pi i} \lim_{\rho\to 0^+} \left\{ \begin{array}{c} \int_{\gamma_0^+}^{\frown} \frac{\cos(kz) - \cos[(k+1)z]}{1-\cos z} dz \\ \int_{\gamma_0^-}^{\frown} \frac{\cos(kz) - \cos[(k+1)z]}{1-\cos z} dz \end{array} \right. = \frac{1}{2\pi i} \lim_{\rho\to 0^+} \left\{ \begin{array}{c} \int_{\gamma_0^+}^{\frown} [\cos(kz) - \frac{\sin(kz)\sin z}{1-\cos z}] dz \\ \int_{\gamma_0^-}^{\frown} [\cos(kz) - \frac{\sin(kz)\sin z}{1-\cos z}] dz \end{array} \right. = 0$$

From the preceding two results it follows further that

$$\frac{1}{\pi}vt[\int_{-\pi}^{\pi}\frac{\cos\left[\left(k+1\right)\theta\right]}{1-\cos\theta}d\theta-\int_{-\pi}^{\pi}\frac{\cos\left(k\theta\right)}{1-\cos\theta}d\theta]=-2.$$

Since

$$\frac{1}{\pi}vt\int_{-\pi}^{\pi}\frac{\cos\theta}{1-\cos\theta}d\theta=-2+\frac{1}{\pi}vt\int_{-\pi}^{\pi}\frac{d\theta}{1-\cos\theta}=-2,$$

we obtain finally that

$$a_k=\frac{1}{\pi}vt\int_{-\pi}^{\pi}\frac{\cos\left(k\zeta\right)}{1-\cos\zeta}d\zeta=-2k\ \ (k\in\aleph)\,. \tag{29}$$

As $\sin\left(k\zeta\right)/\left(1-\cos\zeta\right)$ is an odd function, the *Cauchy* principal value (vp) of the improper integral $\int_{-\pi}^{\pi}[\sin\left(k\zeta\right)/\left(1-\cos\zeta\right)]dt$ vanishes, so that for each $k\in\aleph$,

$$vt\int_{-\pi}^{\pi}\frac{\sin\left(k\zeta\right)}{1-\cos\zeta}d\zeta=\lim_{\rho\to0^{+}}\left\{\begin{array}{l}\int_{\gamma_0^{+}}^{\frown}\frac{\sin(kz)}{1-\cos z}dz\\ \int_{\gamma_0^{-}}^{\frown}\frac{\sin(kz)}{1-\cos z}dz\end{array}\right..$$

Since $\lim_{z\to0}\sin\left(kz\right)/\left(1-\cos z\right)=k$ for each $k\in\aleph$, it follows finally that

$$b_k=\frac{1}{\pi}vt\int_{-\pi}^{\pi}\frac{\sin\left(k\zeta\right)}{1-\cos\zeta}d\zeta=\left\{\begin{array}{l}ik\\ -ik\end{array}\right.\quad(k\in\aleph)\,. \tag{30}$$

Therefore, a *Fourier* trigonometric series expansion of $1/\left(1-\cos\zeta\right)$ on the segment $[-\pi,\pi]$ $(\zeta\neq0)$ is as follows

$$\frac{1}{1-\cos\zeta}=-2\sum_{k=1}^{+\infty}k\cos\left(k\zeta\right)\pm i\sum_{k=1}^{+\infty}k\sin\left(k\zeta\right). \tag{31}$$

Separating the real and imaginary parts in the preceding equation, we finally obtain that for each $\zeta\in[-\pi,\pi]$ and $\zeta\neq0$

$$\frac{1}{1-\cos\zeta}=-2\sum_{k=1}^{+\infty}k\cos\left(k\zeta\right)\ \text{ and }\ \sum_{k=-\infty}^{+\infty}k\sin\left(k\zeta\right)=0. \tag{32}$$

In the complex form, for each $\zeta\in[-\pi,\pi]$ and $\zeta\neq0$,

$$\sum_{k=1}^{+\infty}ke^{\mp ik\zeta}=\frac{e^{i\zeta}}{(1-e^{i\zeta})^2}.$$

At the endpoints of $[-\pi,\pi]$ there holds $\sum_{k=1}^{+\infty}k\cos\left(k\pi\right)=-1/4$. Therefore,

$$\sum_{k=1}^{+\infty}k\left(-1\right)^{k+1}=1/4.$$

As in the previous examples, the *Laurent* expansion of the single valued complex function $s/(1-s)^2$ comes to the same result. Namely, since

$$\frac{1}{2\pi i}vt\int_{\mathcal{C}_1}^{\circlearrowleft}\frac{1}{(1-s)^2}\frac{ds}{s^k}=\begin{cases}\begin{cases}k\\0\end{cases}, \text{ if } k<0\\ 0, \text{ if } k=0\\ \begin{cases}0\\k\end{cases}, \text{ if } k>0\end{cases},$$

where $\mathcal{C}_1$ is the unit circumference in the s-plane, it follows from *Theorem 4* that for every $|s|=1$ and $s\neq 1$

$$\sum_{k=1}^{+\infty} ks^{\mp k}=\frac{s}{(1-s)^2}.$$

Furthermore, $\sum_{k=1}^{+\infty} ks^{-k} = s/(1-s)^2$ for $|s|>1$ and $\sum_{k=1}^{+\infty} ks^{k} = s/(1-s)^2$ for $|s|<1$.

4 Remark

Fourier expansions of real-valued functions

$$f(t)=\begin{cases}\frac{\sin\zeta}{1-\cos\zeta}, \mathit{if}\ \tau_0\leq|\zeta|\leq\pi\\ 0, \mathit{if}\ \ |\zeta|<\tau_0\end{cases} \quad\text{and } g(t)=\begin{cases}b, \mathit{if}\ \tau_0\leq\zeta\leq\pi\\ 0, \mathit{if}\ \ |\zeta|<\tau_0\\ a, \mathit{if}\ \ -\pi\leq\zeta\leq-\tau_0\end{cases},$$

where $\tau_0>0$, satisfying *Dirichlet's* conditions in the closed interval $[-\pi,\pi]$, are as follows. For every $|\zeta|\in(\tau_0,\pi)$,

$$f(\zeta)=2\sum_{k=1}^{+\infty}\frac{1}{\pi}[\int_{\tau_0}^{\pi}\frac{\sin\tau}{1-\cos\tau}\sin(k\tau)\,d\tau]\sin(k\zeta)$$

and

$$\begin{aligned} g(\zeta) &= \frac{1}{2\pi}\left(\int_{-\pi}^{-\tau_0} a d\tau+\int_{\tau_0}^{\pi} b d\tau\right)+\sum_{k=1}^{+\infty}\frac{1}{\pi}\left\{\left[\int_{-\pi}^{-\tau_0} a\sin(k\tau)\,d\tau\right.\right.\\ &\quad\left.+\int_{\tau_0}^{\pi} b\sin(k\tau)\,d\tau\right]\sin(k\zeta)+\left.\left[\int_{-\pi}^{-\tau_0} a\cos(k\tau)\,d\tau+\int_{\tau_0}^{\pi} b\cos(k\tau)\,d\tau\right]\cos(k\zeta)\right\}. \end{aligned}$$

Using of the fundamental trigonometric identities we obtain the following recurrent formula for the *Fourier* coefficients of $f(\zeta)$

$$\begin{aligned} \frac{1}{\pi}\int_{\tau_0}^{\pi}\frac{\sin\zeta\sin[(k+1)\zeta]}{1-\cos\zeta}d\zeta &= \frac{1}{\pi}\int_{\tau_0}^{\pi}\frac{\sin\zeta\sin[(k-1)\zeta]}{1-\cos\zeta}d\zeta-\frac{2\sin(k\tau_0)}{k\pi}\\ &\quad -\frac{\sin[(k+1)\tau_0]}{(k+1)\pi}-\frac{\sin[(k-1)\tau_0]}{(k-1)\pi}\quad(k\in\aleph). \end{aligned}$$

Since

$$\frac{1}{\pi}\int_{\tau_0}^{\pi}\frac{(\sin\zeta)^2}{1-\cos\zeta}d\zeta=\frac{1}{\pi}\int_{\tau_0}^{\pi}(1+\cos\zeta)\,d\zeta=1-\frac{\tau_0}{\pi}-\frac{\sin\tau_0}{\pi}$$

and

$$\frac{1}{\pi}\int_{\tau_0}^{\pi}\frac{\sin(2\zeta)\sin\zeta}{1-\cos\zeta}d\zeta=\frac{2}{\pi}\int_{\tau_0}^{\pi}(1+\cos\zeta)\cos\zeta d\zeta=1-\frac{\tau_0}{\pi}-2\frac{\sin\tau_0}{\pi}\;-\frac{\sin(2\tau_0)}{2\pi}$$

it follows finally that for every $|\zeta|\in(\tau_0,\pi)$

$$f(\zeta)=2\sum_{k=1}^{+\infty}\left[1+\frac{\tau_0}{\pi}-2\sum_{\kappa=0}^{k-1}\frac{\sin(\kappa\tau_0)}{\kappa\pi}-\frac{\sin(k\tau_0)}{k\pi}\right]\sin(k\zeta)$$

and

$$\begin{aligned} g(\zeta) \quad = \quad & \frac{a+b}{2}-\frac{a+b}{2\pi}\left[\frac{1}{2}+\sum_{k=1}^{+\infty}\frac{\sin(k\tau_0)}{k\tau_0}\cos(k\zeta)\right]\tau_0 \\ & +\frac{b-a}{\pi}\sum_{k=1}^{+\infty}\left[\cos(k\tau_0)-(-1)^k\right]\frac{\sin(k\zeta)}{k}. \end{aligned} \tag{33}$$

Based on the expansion of $g(\zeta)$ if $a=b$, then for every $|\zeta|\in(\tau_0,\pi)$ and $\tau_0>0$

$$\sum_{k=1}^{+\infty}\frac{\sin(k\tau_0)\cos(k\zeta)}{k\tau_0}=-\frac{1}{2}. \tag{34}$$

Thus, for every $\zeta\in(\tau_0,\pi)$ and $\tau_0>0$

$$\sum_{k=1}^{+\infty}[\cos(k\tau_0)-(-1)^k]\frac{\sin(k\zeta)}{k}=\frac{\pi}{2}. \tag{35}$$

This implies that for every $\zeta\in(\tau_0,\pi)$

$$\sum_{k=1}^{+\infty}\cos(k\tau_0)\,\frac{\sin(k\zeta)}{k}=\frac{\pi}{2}-\frac{\zeta}{2} \tag{36}$$

since $\sum_{k=1}^{+\infty}(-1)^k\sin(k\zeta)/k=-\zeta/2$ for every $\zeta\in(-\pi,\pi)$. In view of the fact that

$$\lim_{k\to+\infty}\left[1+\frac{\tau_0}{\pi}-2\sum_{\kappa=0}^{k-1}\frac{\sin(\kappa\tau_0)}{\kappa\pi}-\frac{\sin(k\tau_0)}{k\pi}\right]=0,$$

it follows finally that for every $\tau_0\in(0,\pi)$

$$\sum_{k=0}^{+\infty}\frac{\sin(k\tau_0)}{k}=\frac{\pi}{2}+\frac{\tau_0}{2}. \tag{37}$$

Since $\tau_0 \in (0,\pi)$, it would be natural to ask whether functional expressions (34), (35) and (36), hold in the limit as $\tau_0 \to 0^+$? In other words, is the limit of a sum equal to a sum of the limit, as $\tau_0 \to 0^+$. Based on the results of the previous examples, as well as on the well-known result of the series theory $\pi/4 = \sum_{k=1}^{+\infty} \sin\left[(2k-1)\zeta\right]/(2k-1)$ for $\zeta \in (0,\pi)$, we may give an affirmative answer to the former questions. However, the problem of generalization of the preceding conclusion stays open and can be a subject of a separate analysis.

Similarly, if we already know that $d\left[\sin\zeta/(1-\cos\zeta)\right]/d\zeta = -1/(1-\cos\zeta)$ and $d\ln(1-\cos\zeta)/d\zeta = \sin\zeta/(1-\cos\zeta)$ for $|\zeta| \in (0,\pi)$, then closely related to results (26) and (32) of the paper, as well as to the well-known result of the series theory $-\ln(1-\cos\zeta)/2 = \sum_{k=1}^{+\infty} \cos(k\zeta)/k$ for $|\zeta| \in (0,2\pi)$ [7], is the following question, is the derivative of a sum of *Fourier* series equal to a sum of the derivative of series members, separately. This question also stays open for a separate analysis.

Some of the paper results have been predictable. So, the alternative numerical series $\sum_{k=0}^{+\infty} (-1)^k$ has a sum, more precisely it is $(C,1)$ summable and its sum is equal to $1/2$ [2], just as it has been assumed yet by *Euler* and *Leibniz*. By using this assumption they obtained absolutely exact results. It is nothing other to be left than to prove the validity of this assumption. As for result (32) from *Example 2,* one can say that it is causality related to result (26) from *Example 1.* Namely, since $\sum_{k=1}^{+\infty} k\sin(k\zeta) = 0$ for $\zeta = \pi/2$, it follows from $\sum_{k=0}^{+\infty} (2k+1)(-1)^k = 0$ that $2\sum_{k=0}^{+\infty} k(-1)^k = -\sum_{k=0}^{+\infty} (-1)^k = -1/2$.

Acknowledgment. The author's research is supported by the Ministry of Science, Technology and Development, Republic of Serbia (Project 174024).

References

[1] J. Evard and F. Jafari, A complex Rolle's theorem, Amer. Math. Monthly, 99(9)(1992), 858–861.

[2] J. Marcinkiewicz and A. Zygmund, Two theorems on trigonometrical series, Rec. Math. [Mat. Sbornik] N.S., 2(44)No.4(1937), 733–737.

[3] D. Mitrinović and J. Kečkić, The Cauchy Method of Residues, Vol. 2: Theory and Applications, Dordrecht: Kluwer Academic Publishers, 1993.

[4] B. Sarić, The Fourier series of one class of functions with discontinuities, Doctoral dissertation defended on 20th of October 2009 at the University of Novi Sad, Faculty of Science, Department of Mathematics and Informathics.

[5] B. Sarić, Cauchy's residue theorem for a class of real valued functions, Czech. Math. J., 60(4)(2010), 1043–1048.

[6] B. Sarić, On totalization of the H_1-integral, Taiwanese J. Math., 15(4)(2011), 1691–1700.

[7] D. Slavić, On summation of series, Publication of school of electrical engineering of Belgrade University, Series: Mathematics and Physics, 302-319(1970), 53–59.

[8] A. Zygmund, Trigonometric Series, University Press, Cambridge, 2003.