

Generalizations Of Bernoulli's Inequality With Utility-Based Approach*

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Abstract

Utility function is one of the most useful tools in mathematical finance, decision analysis and economics. The concept of expected utility is used in economics to describe the behavior of a decision-maker choosing between a certain number of random gains. In this paper, some generalizations of the Bernoulli's inequality are derived by using methods from the utility theory. In particular, we obtain Harmonic-Geometric-Arithmetic mean inequality and give a generalization of it.

1 Introduction

The notion of utility goes back to Daniel Bernoulli [2]. Because the expected pay off in the game of the St. Petersburg Paradox is infinite, Bernoulli was the first who suggested a utility function as a solution to the St. Petersburg Paradox. The first axiomatic development of utility theory was achieved by Von Neumann and Morgenstern [18] who argued that the existence of a utility function could be derived from a set of axioms governing a preference ordering. Other developments are due to Marschak [13], Herstein and Milnor [9] and especially Fishburn [6, 7]. Borch [3] showed how utility theory could be used to formulate and solve some problems that are relevant to insurance. Most of the original papers of Borch have been reprinted in his books [3, 4]. Abbas [1] introduced a relation between probability and utility based on the concept of a utility density function and showed the application of this relation via the maximum entropy principle.

Friedman *et al.* [8] introduced utility-based generalizations of the Shannon entropy and Kullback-Leibler information measure and called them U-entropy and U-relative entropy and found various properties for these generalized quantities similar to the classical concepts of information theory. For a parametric family of utility functions Hoseinzadeh *et al.* [10] derived some links between U-relative entropy and other divergence measures. Hoseinzadeh *et al.* [11] extended Markov's inequality for probabilities

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from the U-entropy viewpoint and derived a link between conditional U-entropy and conditional Renyi entropy.

In this paper, some generalizations of Bernoulli's inequality are obtained by the concept of the certainty equivalent that was introduced in utility theory. Also, we derived an important and useful inequality in statistics, i.e. the Harmonic-Geometric-Arithmetic mean inequality and give a generalization for it.

2 Utility Functions

An investor's subjective probabilities numerically represent his beliefs and information, and his utilities represent his tastes and preferences. Utility functions provide us with a method to measure an investor's preferences for wealth and the amount of risk he is willing to undertake in the hope of gaining greater wealth.

A utility function is a twice-differentiable function of wealth $u(x)$ which has the following properties:

- (i) $u(x)$ is a strictly increasing function of x ,
- (ii) $u(x)$ is a strictly concave function of x .

The first property states that utility increases with wealth, i.e., that more wealth is preferred to less wealth. The second property states that the utility function is concave or, in other words, that the marginal utility of wealth, $u'(x)$, decreases as wealth increases, or equivalently, that the gain of utility resulting from the monetary gain of g , $u(x + g) - u(x)$, be a decreasing function of wealth x . The utility functions of most people tend to be concave, at least for large gains or large losses. It is easy to see that if $u(x)$ is a utility function, then for constant $a > 0$ and b , the function $w(x) = au(x) + b$ is also a utility function. For any utility function $u(x)$, we consider the following function:

$$r(x) = -\frac{u''(x)}{u'(x)}.$$

This is called absolute risk aversion function which is a positive function. It is also generally agreed in finance theory that for a utility function to be realistic with regard to economic behavior, its absolute risk aversion should be a decreasing function of wealth.

The most common utility functions are as follows:

- **Exponential** utility function with parameter $\alpha > 0$,

$$u(x) = \frac{1 - e^{-\alpha x}}{\alpha} \text{ for } -\infty < x < \infty.$$

Note that this utility tends to the finite value $\frac{1}{\alpha}$ as $x \rightarrow \infty$. For the exponential utility function, $r(x) = \alpha$, $-\infty < x < \infty$. Thus, the exponential utility function yields a constant risk aversion.

- **Logarithmic** utility function with parameters $\alpha > 0$, β , γ ,

$$u(x) = \alpha \ln(x - \beta) + \gamma \text{ for } x > \beta.$$

For the logarithmic utility function, $r(x) = \frac{1}{x-\beta}$, $x > \beta$. Hence the risk aversion is a decreasing function of wealth, which may be typical for some investors.

- **Iso-Elastic** utility function is a class of utility functions as follows:

$$u(x) = \frac{x^\alpha - 1}{\alpha} \text{ for } \alpha < 1, \alpha \neq 0 \text{ and } x > 0.$$

This class has the property of $u(kx) = f(k)u(x) + g(k)$, for all $k > 0$ and for some function $f(k) > 0$ and $g(k)$ which are independent of x . We note that this utility tends to $u(x) = \ln x$ as $\alpha \rightarrow 0$. For the Iso-Elastic utility function, $r(x) = \frac{1-\alpha}{x}$, that is a decreasing function of wealth, which may be typical for some decision-maker.

3 Stochastic Orders

Stochastic orders and inequalities play an important role in many diverse areas of probability and statistics and have been used during the last 40 years. Such areas include reliability theory, queuing theory, survival analysis, economics, insurance, actuarial sciences, operations research and management science. Several orders of two random variables (or their distribution functions) have been studied by Shaked and Shanthikumar [15]. The simplest way of comparing two random variables is by the comparison of the associated means. However, such a comparison is based on only two single numbers (the means) and therefore it is often not very informative. In addition to this, the means sometimes do not exist. In many instances in applications one has more detailed information, for the purpose of comparison of two distribution functions, than just the two means. Let X and Y be two random variables defined on a probability space. Several orders of univariate distribution functions are as follows:

- The random variable X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if and only if,

$$P\{X > x\} > P\{Y > x\} \text{ for all } x \in (-\infty, \infty).$$

- The random variable X is said to be smaller than Y in the convex order (denoted by $X \leq_{cx} Y$) if and only if,

$$E[\phi(X)] \leq E[\phi(Y)],$$

for all convex functions $\phi : R \rightarrow R$, provided the expectations exist.

- The random variable X is said to be smaller than Y in the increasing convex [concave] order (denoted by $X \leq_{icx} Y$ [$X \leq_{icv} Y$]) if and only if,

$$E[\phi(X)] \leq E[\phi(Y)],$$

for all increasing convex [concave] functions $\phi : R \rightarrow R$, provided the expectations exist.

Other orders such as hazard rate order, mean residual life order, likelihood ratio order have been studied by Shaked and Shanthikumar [15]. We will concentrate on a special case of this order which will be defined as expected utility order.

4 Main Results

In this section, via the monotone concave order, we order two random gains by means of their expected utilities. In particular, a random gain will be replaced by a fixed amount that is named the certainty equivalent. This notion can be used by the consumer who wants to determine the maximal premium he or she is willing to pay to obtain full coverage. Also, by means of this concept, we obtain some useful mathematical inequalities.

4.1 Preference Ordering of Random Gains

Now we consider a decision-maker with initial wealth w who has the choice between a certain number of random gains. By using the monotone concave order and a utility function, as an increasing concave function of wealth, we can compare two random gains as follows:

The random gain G_1 is preferred over the random gain G_2 in the expected utility order (denoted by $G_1 \leq_{Eu} G_2$) if and only if,

$$E[u(w + G_1)] > E[u(w + G_2)],$$

provided the expectations exist. This defines a preference ordering on the set of random gains. On the other hand since $u(x)$ is a concave function, by Jensen's inequality for any random variable G , we can write,

$$u(w + E[G]) > E[u(w + G)].$$

Therefore, if a decision-maker can choose between a random gain G and a fixed amount equal to its expectation, he will prefer the latter. This leads us to the following definition.

DEFINITION. The certainty equivalent, π , associated with the random gain G is defined by the condition that the decision-maker is indifferent between receiving G or the fixed amount π , i.e.,

$$u(w + \pi) = E[u(w + G)]. \quad (1)$$

Since $u(x)$ is a strictly increasing function, the last inequality leads us to the following important inequality:

$$\pi < E[G]. \quad (2)$$

In the next section, by using the inequality (2) and by means of a random gain with known distribution, we derive some useful mathematical inequalities.

4.2 Generalizations of Bernoulli's Inequality

Suppose that n is a positive integer and $x > -1$. Then the inequality

$$(1+x)^n \geq 1+nx,$$

is known as the Bernoulli's inequality which plays an important role in mathematical analysis and its applications. Recently, various generalizations of Bernoulli's inequality are obtained by many researches. For example, the following generalizations of this inequality were introduced by Kuang [12].

THEOREM 1. Let $x > -1$. If $\alpha > 1$ or $\alpha < 0$, then

$$(1+x)^\alpha \geq 1+\alpha x,$$

and if $0 < \alpha < 1$, then

$$(1+x)^\alpha \leq 1+\alpha x,$$

in which, equalities hold if and only if, $x = 0$.

THEOREM 2. Let $a_i \geq 0$, $x_i > -1$, $i = 1, \dots, n$ and $\sum_{i=1}^n a_i \leq 1$. If $a_i \geq 1$ or $a_i \leq 0$, then

$$\prod_{i=1}^n (1+x_i)^{a_i} \leq 1 + \sum_{i=1}^n a_i x_i,$$

and if $x_i > 0$, or $-1 < x_i < 0$, $i = 1, \dots, n$, then

$$\prod_{i=1}^n (1+x_i)^{a_i} \geq 1 + \sum_{i=1}^n a_i x_i.$$

By using methods on the theory of majorization, some generalizations of Bernoulli's inequality were established by Shi [16].

In the next theorems, by applying methods on the utility theory, some generalizations of Bernoulli's inequality are established.

THEOREM 3. For $w > 0$, and $0 \leq p \leq 1$, we have that

$$\left(1 + \frac{1}{w}\right)^p \leq 1 + \frac{p}{w}, \quad (3)$$

equality holds if and only if, $w \rightarrow \infty$.

PROOF. Suppose that the random gain G has the Bernoulli distribution with parameter $0 < p < 1$. Consider the logarithmic utility function and using equation (1) we can write

$$\ln(w + \pi) = (1-p)\ln(w) + p\ln(w+1).$$

Solving this equation with respect to π leads to $\pi = w[(1 + \frac{1}{w})^p - 1]$. Using $E(G) = p$ and applying inequality (2) leads to the inequality (3).

THEOREM 4. Let $0 \leq p_i \leq 1$, $w > 0$ and a_i , are positive real numbers such that $w + a_i > 0$, $i = 1, \dots, n$. Then we have

$$\prod_{i=1}^n \left(1 + \frac{a_i}{w}\right)^{p_i} \leq \sum_{i=1}^n \frac{a_i p_i}{w} + 1. \quad (4)$$

PROOF. Let G be a discrete random variable with probability distribution $p = (p_1, p_2, \dots, p_n)$ which is defined over a finite support $\{a_1, a_2, \dots, a_n\}$. By means of the logarithmic utility function and by using (1), we obtain:

$$\ln(w + \pi) = \sum_{i=1}^n p_i \ln(w + a_i),$$

solving this equation with respect to π yields

$$\pi = w \left[\prod_{i=1}^n \left(1 + \frac{a_i}{w}\right)^{p_i} - 1 \right].$$

Note that $E(G) = \sum_{i=1}^n a_i p_i$. Then by (2), we may then complete the proof.

COROLLARY 1. If $p = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ and $x_i = 1 + \frac{a_i}{w}$, $i = 1, \dots, n$, (4) is reduced to

$$\sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{\sum_{i=1}^n x_i}{n},$$

which is the Geometric-Arithmetic mean inequality.

Various proofs of the Geometric-Arithmetic mean inequality are known in the literature, (see, for example, Bullen [5]). A simple proof of the Geometric-Arithmetic mean inequality was given by Uchida [17].

THEOREM 5. Let $0 \leq p_i \leq 1$ and $x_i > 0$, $i = 1, \dots, n$ be positive real value numbers. Then we have

$$\sum_{j=1}^n p_j \prod_{i \neq j} x_i \geq \prod_{i=1}^n x_i^{1-p_i}. \quad (5)$$

PROOF. Define

$$y_i = \prod_{j \neq i} x_j \text{ for } i = 1, \dots, n.$$

Suppose that G is a discrete random variable on the support $\{y_1, y_2, \dots, y_n\}$ such that, $p = (p_1, p_2, \dots, p_n)$ is the probability distribution of G , i.e., $p_i = P(G = y_i)$, $i = 1, \dots, n$. Consider a decision-maker with initial wealth $w = 0$ and logarithmic utility function. Hence, from (1) we obtain

$$\ln \pi = E[\ln(G)] = \sum_{i=1}^n p_i \ln \left(\prod_{j \neq i} x_j \right) = \ln \left(\prod_{i=1}^n x_i^{1-p_i} \right).$$

Solving this equation with respect to π we conclude that $\pi = \prod_{i=1}^n x_i^{1-p_i}$. On the other hand, $E(G) = \sum_{j=1}^n p_j \prod_{i \neq j}^n x_i$ and inequality (2) imply the proof.

COROLLARY 2. Inequality (5) can be written as follows:

$$\frac{\prod_{i=1}^n x_i}{\sum_{j=1}^n p_j \prod_{i \neq j}^n x_i} \leq \prod_{i=1}^n x_i^{p_i}.$$

For $p = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, we derive:

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 \dots x_n},$$

that is the Harmonic-Geometric mean inequality.

COROLLARY 3. By setting $x_i = 1 + \frac{a_i}{w}$, $i = 1, \dots, n$ in Theorem 4 and using Theorem 5 we derive

$$\frac{\prod_{i=1}^n x_i}{\sum_{j=1}^n p_j \prod_{i \neq j}^n x_i} \leq \prod_{i=1}^n x_i^{p_i} \leq \sum_{i=1}^n p_i x_i.$$

So, these inequalities give a generalization for the Harmonic-Geometric-Arithmetic mean inequality, (See, for example, Napoca [14]).

THEOREM 6. Let m, n, k are nonnegative integer such that for $i = 0, 1, \dots, n$ inequalities $i \leq k \leq n \leq m$ hold. Then for all $w > 0$, we have

$$\prod_{i=0}^k \left[1 + \frac{i}{w} \right]^{C_n^i C_{m-n}^{k-i}} \leq \left[1 + \frac{nk}{mw} \right]^{C_m^k},$$

in which,

$$C_n^i = \frac{n!}{i!(n-i)!}.$$

PROOF. Let G be a random gain with the Hypergeometric probability distribution as follows:

$$p(i) = \frac{C_n^i C_{m-n}^{k-i}}{C_m^k} \text{ for } i = 0, 1, \dots, k, \quad k = 1, 2, \dots, n \text{ and } m \geq n.$$

By considering equation (1) and $u(x) = \ln x$, via some calculation we have

$$E[\ln(w + G)] = \ln \left[\prod_{i=0}^k (w + i)^{C_n^i C_{m-n}^{k-i}} \right]^{\frac{1}{C_m^k}}.$$

Now by solving this equation with respect to π we obtain

$$\pi = \left[\prod_{i=0}^k (w + i)^{C_n^i C_{m-n}^{k-i}} \right]^{\frac{1}{C_m^k}} - w.$$

By applying inequality (2) and $E(G) = k \frac{n}{m}$, we have the conclusion.

5 Conclusions

In this work, motivated by monotone concave order we compared two random gains by means of their expected utilities. Also, by using the concept of the certainty equivalent in utility theory, we obtained some generalization of the Bernoulli's inequality. As a special case, we derived the Harmonic-Geometric-Arithmetic mean inequality and obtained a generalization of it.

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