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Ul'yanov Type Inequalities For Moduli Of Smoothness*

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Abstract

Let T denote the interval $[-\pi,\pi]$. In this work we investigate the inequality of Ul'yanov type for moduli of smoothness of an integer order in the $L_p(T)$, $p\geq 1$ spaces. In particular, we study (p,q) inequalities for moduli of smoothness of a derivative of a function via the modulus of smoothness of the function itself.

1 Introduction

Let f be 2π-periodic and let $f\in L_p[0,2\pi]=L_p$ for $p\geq 1$. Throughout this work, $\|\cdot\|_p$ will denote the L_p-norm and will be defined by

$$\|f\|_p=\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p\,dx\right\}^{1/p},\ f\in L_p,\ 1\leq p<\infty.$$

The modulus of smoothness $\omega_k(f,\delta)_p$ of a function $f\in L_p$, $1\leq p\leq\infty$, of fractional order $k>0$ are defined by

$$\omega_k(f,\delta)_p=\sup{}_{|h|\leq\delta}\left\|\Delta_h^k f(x)\right\|_p \tag{1}$$

where

$$\Delta_h^k f(x)=\sum_{\nu=0}^{\infty}(-1)^\nu\binom{k}{\nu}f(x+(k-\nu)h),\quad k>0.$$

Note that, the following (p,q) inequalities between moduli of smoothness, nowadays called Ul'yanov-type inequalities, are known:

$$\omega_k\left(f^{(r)},\delta\right)_q\leq C\left(\int_0^\delta\left(u^{-\theta}\omega_{k+r}(f,t)_p\right)^{q_1}\frac{du}{u}\right)^{1/q_1}, \tag{2}$$

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where

$$r \in N \cup \{0\}, \quad 0 < p < q \leq \infty, \;\; \theta = \frac{1}{p} - \frac{1}{q},$$

$$q_1 = \begin{cases} q & \text{if } q < \infty, \\ 1 & \text{if } q = \infty. \end{cases}$$

In the case $r = 0$, $p \geq 1$ the inequality (2) was proved by Ul'yanov [15]. In other cases, (p, q) estimates (the modulus of smoothness $\omega_k(f, \delta)_p$ of an integer order, the r-the derivative, $r \in N$ and the fractional derivative of order $r > 0$ of the function) were obtained in references [3], [4], [14].

Note that the inequality between moduli of smoothness of various orders in different metrics was investigated by [6].

We denote by $E_n(f)_p$ the best approximation of $f \in L_p(T)$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_p := \inf{}_{T_n \in \Pi_n} \|f - T_n\|_p, \quad n = 0, 1, 2, ...,$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let $W_p^r[0, 2\pi] = W_p^r$, $(r = 1, 2, ...)$ be the linear space of functions for which $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_p(T), p > 1$. It becomes a Banach space with the norm

$$\|f\|_{W_p^r} := \|f\|_p + \left\|f^{(r)}\right\|_p.$$

Let $f \in L_p$. For $\delta > 0$, the K-functional is defined by

$$K\left(\delta, f; L_p, W_p^r\right) := \inf\left\{\|f - \psi\|_p + \delta\left\|\psi^{(r)}\right\|_p : \psi \in W_p^r\right\}.$$

Let $1 < p < \infty$. We define an operator on $L_p(T)$ by

$$(\sigma_h g)(x) := \frac{1}{2h}\int_{-h}^{h} g(x + t)\,dt, \;\; 0 < h < \pi, \;\; x \in T.$$

The k-modulus of smoothness $\Omega_k(\cdot, g)_p$, $(k = 1, 2, ...)$, of $g \in L_p(T)$ is defined by

$$\Omega_k(\delta, g)_p := \sup{}_{\substack{0<h_i<\delta \\ 1\leq i\leq k}} \left\|\prod_{i=1}^{k}(I - \sigma_{h_i})\,g\right\|_{L_p(T)}, \quad \delta > 0, \tag{3}$$

where I is the identity operator [1], [5], [7].

In the case of $k = 0$ we set $\Omega_k(\delta, g)_p := \|g\|_{L_p(T)}$ and if $k = 1$ we write $\Omega(\delta, g)_p := \Omega_1(\delta, g)_p$.

It can be shown easily that the modulus of smoothness $\Omega_k(\cdot, g)_p$ is a nondecreasing, nonnegative, continuous function satisfying the conditions

$$\lim{}_{\delta \to 0}\Omega_k(\delta, g)_p = 0,$$

$$\Omega_k\left(\delta, f+g\right)_p \leq \Omega_k\left(\delta, f\right)_p + \Omega_k\left(\delta, g\right)_p$$

for $f, g \in L_p(T)$,

$$f \sim \sigma(f) := \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}\left(a_k(f)\cos kx + b_k(f)\sin kx\right) \tag{4}$$

is the Fourier series of the function $f \in L_1(T)$.

The n-th *partial sums* and *de La Vallée-Poussin sum* of the series (4) are defined, respectively, as

$$S_n\left(x, f\right) := \frac{a_0(f)}{2} + \sum_{k=1}^{n}\left(a_k(f)\cos kx + b_k(f)\sin kx\right)$$

and

$$V_n\left(f\right) := V_n\left(x, f\right) := \frac{1}{n}\sum_{\nu=n}^{2n-1} S_\nu\left(x, f\right).$$

The following Lemma holds.

LEMMA 1. For $f \in L_p$, $1 \leq p \leq \infty$; and $k = 1, 2, ...$ we have

$$\begin{aligned} c_1\left(p,k\right)\Omega_k\left(\frac{1}{n}, f\right)_p &\leq \left(n^{-2k}\left\|V_n^{(2k)}\left(f,x\right)\right\|_p + \left\|f\left(x\right) - V_n\left(f,x\right)\right\|_p\right) \\ &\leq c_2\left(p,k\right)\Omega_k\left(\frac{1}{n}, f\right)_p. \end{aligned}$$

PROOF. Considering reference [7], the inequality

$$\Omega_k\left(\frac{1}{n}, T_n\right)_p \leq c_3\left(p,k\right)n^{-2k}\left\|T_n^{(2k)}\right\|_p \tag{5}$$

holds, where T_n is a trigonometric polynomial of order n. Using the properties of smoothness $\Omega_k\left(\cdot, f\right)_p$ [5], [7] and (5), we have

$$\begin{aligned} \Omega_k\left(\frac{1}{n}, f\right)_p &\leq c_4\left(p,k\right)\left(\Omega_k\left(\frac{1}{n}, T_n\right)_p + \left\|f - T_n\right\|_p\right) \\ &\leq c_5\left(p,k\right)\left(n^{-2k}\left\|T_n^{(2k)}\right\|_p + \left\|f - T_n\right\|_p\right). \end{aligned}$$

By reference [7] the Jackson inequality

$$E_n\left(f\right)_p \leq c_6\Omega_k\left(\frac{1}{n+1}, f\right)_p, \qquad k = 1, 2, ..., \tag{6}$$

holds, with a constant $c_6 > 0$ independent of n.

Note that, to estimate $\Omega_k\left(\frac{1}{n},f\right)_p$ from below we shall use the following inequality in [7]

$$n^{-2k}\left\|T_n^{(2k)}\right\|_p \le c_7(p,k)\,\Omega_k\left(\frac{1}{n},T_n\right)_p. \tag{7}$$

Let $V_n(f)$ be de La Vallée-Poussin sum of the series (4).

We denote by $T_n^*(x,f)$ the best approximating polynomial of degree at most n to f in $L_p(T)$. In this case, from the boundedness of V_n in $L_p(T)$, we obtain

$$\begin{aligned}\|f-V_n(f)\|_p &\le \|f(x)-T_n^*(x,f)\|_p + \|T_n^*(x,f)-V_n(x,f)\|_p\\ &\le \le c_7(p)\,E_n(f)_p + \|V_n(x,T_n^*(x,f)-f(x))\|_p\\ &\le c_8(p,k)\,E_n(f)_p. \end{aligned}\tag{8}$$

Using (7) and (8) we reach

$$\begin{aligned}& n^{-2k}\left\|V_n^{(2k)}(x,f)\right\|_p + \|f(x)-V_n(x,f)\|_p\\ \le\ & c_9(p,k)\left(\Omega_k\left(\frac{1}{n},V_n\right)_p + E_n(f)_p\right)\\ \le\ & c_{10}(p,k)\left(\Omega_k\left(\frac{1}{n},f\right)_p + \Omega_k\left(\frac{1}{n},f-V_n\right)_p\right)\\ \le\ & c_{11}(p,k)\,\Omega_k\left(\frac{1}{n},f\right)_p. \end{aligned}$$

Thus the proof of Lemma 1 is completed.

In this work we study (p,q)-inequalities of Ul'yanov type for the modulus of smoothness $\Omega_k\left(f^{(r)},\delta\right)_p$, $k=1,2,...$, $r=1,2,...$ defined in the form (3). To prove we use the method of the proof given in the study [14].

Main result in the present work is the following theorem.

THEOREM 1. Let $f\in L_p$, $1<p<q<\infty$, $\theta=\frac{1}{p}-\frac{1}{q}$. Then for any $k=1,2,...$, $r=1,2,...$ the following estimate holds:

$$\Omega_k\left(\delta,f^{(r)}\right)_q \le C\left(\int_0^\delta \left(u^{-(\theta+r)}\Omega_{r+k}(u,f)_p\right)^q \frac{du}{u}\right)^{1/q}. \tag{9}$$

2 Proof of the Main Result

According to reference [7] for $1<q<\infty$ the following equivalence holds:

$$\begin{aligned}\Omega_k\left(\frac{1}{2^n},f^{(r)}\right)_q &\approx K\left(\frac{1}{2^n},f^{(r)},L_q(T),W_q^{2k}(T)\right)\\ &= \inf\left\{\left\|f^{(r)}-\psi\right\|_q + 2^{-2nk}\left\|\psi^{(2k)}\right\|_q : \psi\in W_q^{2k}(T)\right\}. \end{aligned}\tag{10}$$

If V_n is the de La Vallée-Poussin sum of the function f using Lemma 1 we get

$$K\left(\frac{1}{2^n}, f^{(r)}, L_q(T), W_q^{2k}\right) \approx \left\| f^{(r)} - V_{2^n}\left(f^{(r)}\right)\right\|_q + 2^{-2nk}\left\| V_{2^n}^{(2k)}\right\|_q := I_1 + I_2. \quad (11)$$

Taking account of (8) we have

$$\|f - V_n(f)\|_p \le c_{12} E_n(f)_p . \quad (12)$$

Considering [16] and [4], the following (p,q)-inequality holds:

$$\left\| (V_{2^l})^{(r)} - (V_{2^n})^{(r)}\right\|_q \le c_{13}\left(\sum_{m=n}^{l-1} 2^{m\theta q}\left\| (V_{2^{m+1}})^{(r)} - (V_{2^m})^{(r)}\right\|_p^q\right)^q . \quad (13)$$

Using the Bernstein-type inequality [7], [9], [14] we obtain

$$\left\| V_{2^{m+1}}^{(r)} - V_{2^m}^{(r)}\right\|_p \le c_{14} 2^{mr}\|V_{2^{m+1}} - V_{2^m}\|_p . \quad (14)$$

Taking into account the relations (13), (14) and Jackson inequality [6] we have

$$\begin{aligned} I_1 &= \left\| f^{(r)} - V_{2^n}\left(f^{(r)}\right)\right\|_q \\ &\le c_{15}\sum_{m=n}^{\infty} 2^{m\theta q} 2^{mqr} E_{2^m}^q(f) \\ &\le c_{16}\left(\sum_{m=n}^{\infty} 2^{m\theta q} 2^{mqr}\Omega_{k+r}\left(\frac{1}{2^m}, f\right)_p^q\right)^{1/q} \\ &\le c_{17}\left(\int_0^{2^{-n}}\left(u^{-(\theta+r)}\Omega_{k+r}(u,f)_p\right)^q \frac{du}{u}\right)^{1/q} . \end{aligned} \quad (15)$$

On the order hand, for $\delta_1 \approx \delta_2$ the following equivalence holds:

$$\Omega_k(\delta_1, f)_p \approx \Omega_k(\delta_2, f)_p . \quad (16)$$

It is known that for trigonometric polynomials of degree n the following Nikol'skii inequality holds [4], [8], [10] :

$$\|T_n\|_q \le c_{18} n^{\frac{1}{p}-\frac{1}{q}}\|T_n\|_p , \qquad 0 < p \le q \le \infty. \quad (17)$$

Use of inequality (17) gives us

$$\begin{aligned} I_2 &= 2^{-nk} \left\| \left(V_{2^n}^{(k)} \right)^{(r)} \right\|_q \\ &\leq c_{19} 2^{-nk} 2^{n\theta} \left\| V_{2^n}^{(k+r)} \right\|_p \\ &\leq c_{20} 2^{n\theta} 2^{nr} \Omega_{k+r} \left(\frac{1}{2^n}, f \right)_p \\ &\leq c_{21} \left(\int_0^{2^{-n}} \left(u^{-\theta} u^{-r} \Omega_{k+r} (u, f)_p \right)^q \frac{du}{u} \right)^{1/q} \\ &= c_{21} \left(\int_0^{2^{-n}} \left(u^{-(\theta+r)} \Omega_{k+r} (f, u)_p \right)^q \frac{du}{u} \right)^{1/q} . \end{aligned} \tag{18}$$

Using (10), (11), (15) and (18), we have (9).

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