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Combination Labelings Of Graphs*

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Abstract

Suppose $G = (V, E)$ is a simple, connected, undirected graph with n vertices and m edges. Let f be a bijection from V onto $\{1, 2, ..., n\}$ which labels the vertices of G. The vertex-labeling f induces an edge-labeling f^C of G as follows: an edge $uv \in E$ with $f(u) > f(v)$ is assigned the label $f^C(uv) = \binom{f(u)}{f(v)}$. If the edge labels of G are pairwise distinct, then we say G is a combination graph. In this paper, we will show that complete k-ary trees, wheel graphs, Petersen graphs $GP(n, 1), GP(n, 2)$, grid graphs and certain caterpillar graphs are combination graphs. We will also show that, except for several special cases, complete bipartite graphs are not combination graphs.

1 Introduction

Suppose $G = (V, E)$ is a simple, connected, undirected graph with n vertices and m edges. Let f be a bijection from V onto $\{1, 2, ..., n\}$ which labels the vertices of G. The vertex-labeling f induces an edge-labeling f^C of G as follows: an edge $uv \in E$ with $f(u) > f(v)$ is assigned the label $f^C(uv) = \binom{f(u)}{f(v)}$. The labeling f^C is called the *combination labeling* of G induced by the labeling f. When the combination labeling f^C is injective, we say that it is a *valid* combination labeling. If the graph G has a valid combination labeling, then we say G is a *combination graph.*

The study of graph labelings has been and continues to be an popular topic of graph theory. The dynamic survey by Gillian [2] shows the diversity of graph labelings. Graceful labelings are similar to combination labelings. A *graceful labeling* of a simple graph $G = (V, E)$ is a labeling of its vertices with distinct integers from the set $\{0, 1, ..., |E|\}$, such that each edge is uniquely identified by the absolute difference between its endpoints. Graceful labelings have been extensively studied. A well-known conjecture of graceful labelings, known as the *graceful tree conjecture*, states that all trees have graceful labelings. For a recent survey, see [1]. By comparing the range of allowable (induced) labels on the edges of a graceful labeling versus that of a combination labeling, if a graph has both a valid combination labeling and a graceful labeling, it would seen that it would be more difficult to find a graceful labeling. Finally, it is known that $K_{3,3}$ has a graceful labeling but no valid combination labeling, while the 5-cycle has a valid combination labeling but no graceful labeling.

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In this paper, we concentrate on the labeling of vertices of a graph which induces a combination labeling on the edges of the graph. This problem was introduced by Hedge and Shetty [3] in 2006. In this paper, we will study combination labelings for several classes of graphs and answer some questions that were posed by in [3]. More specifically, we will show that complete k-ary trees, wheel graphs, generalized Petersen graphs $GP(n, 1), GP(n, 2)$, and grid graphs are combination graphs. In addition, we will show that, except for some special cases, complete bi-partite graphs are not combination graphs.

In Section 2 we will show that full k-ary trees, wheel graphs, generalized Petersen graphs $GP(n, 1), GP(n, 2)$, and grid graphs are combination graphs. In Section 3, we show that complete bi-partite graphs are not combination graphs, except for a few special cases.

2 Classes of Combination Graphs

In this section, we will study several classes of graphs and show that they are combination graphs. We begin with rooted trees.

2.1 Trees

A *rooted tree* is a tree where one of the vertices (or nodes) is distinguished from the other. This distinguished vertex is known as the *root* of the tree. The nodes of a tree can be categorized as either non-leaf nodes or leaf nodes. A node is a *leaf node* if it has degree 1. Otherwise, it is a *non-leaf node.* The *depth* of a vertex in a rooted tree is the number of edges on the path from the root to the vertex. The *height* of a tree is the largest depth of any leaf node. A *k-ary* tree is a rooted tree where each node has at most k children. A *complete* k-ary tree is a k-ary tree where each non-leaf node has exactly k children and the leaf nodes have the same depth.

Our approach will be to show that a rooted tree with the property that all leaf nodes have the same depth is a combination graph. This immediately implies that a complete k-ary tree is a combination graph. We begin with a simple, useful fact that can easily be proved by algebraic manipulations.

LEMMA 1. If $n > k > 0$, then $\binom{n+1}{k+1} > \binom{n}{k}$.

LEMMA 2. Let T be a rooted tree with the property that the depth of any two leaf nodes are the same. Then T is a combination graph.

PROOF. Let T be a rooted tree satisfying the assumptions stated in the lemma. We may assume that T has at least three vertices since a tree consisting one or two nodes is a combination graph.

We will find an assignment f of labels for the nodes of T so that the induced edge labels of T are pairwise distinct. To label the nodes, we will visit and label (using the positive integers) the nodes using a breadth-first traversal starting at the root such that:

1. the smallest available value is used to label the current node being visited, and

2. if the depth of two non-leaf nodes u and v are the same and $f(u) < f(v)$, then the labels assigned to the children of u are less than the labels assigned to the children of v.

Note that this labeling process does not necessarily need to a unique labeling since siblings can be labeled in any order. As the labels are assigned in a breadth-first manner, the label of a node at depth k is smaller than the label of a node at depth $k+1$. Figure 1 illustrates one labeling constructed by the labeling process on a given tree.

Consider an edge $e = uv$ in the tree T, where $f(u) < f(v)$ are the labels assigned to the two endpoints of e. The label induced on this edge is $\binom{f(v)}{f(u)}$. Note that u is the parent of v. Suppose the non-leaf node u has children $v_1, v_2, ..., v_{k'}$ with $f(v_1) < f(v_2) < \cdots < f(v_{k'})$ and $k^{'} \geq 1$. Note that by labeling process, it must be that $f(v_1)+1 = f(v_2), f(v_2) = f(v_3)+1, ..., f(v_{k'-1})+1 = f(v_{k'})$. By Lemma 1, $\binom{f(v_{k'})}{f(u)} > \binom{f(v_{k'-1})}{f(u)} > \cdots > \binom{f(v_1)}{f(u)}$. Therefore, edges between a parent and its siblings have distinct labelings.

Now consider two nodes u, w having the same depth and $f(w) = f(u)+1$. Suppose node u is a non-leaf node. As all leaf nodes have the same depth, the node w must also be a non-leaf node. By the labeling process, the children of u must be labeled $a, a+1, ..., a+l$ and the children of w must be labeled $a+l+1, a+l+2, ..., a+l+m$ for some a and $l, m \geq 1$. By Lemma 1, $\binom{a+l}{f(u)} < \binom{a+l+1}{f(u)+1} = \binom{a+l+1}{f(w)}$. Therefore all the edges of the same level of the tree are pairwise distinct.

Finally, consider two non-leaf nodes u, w where the depth of u is d, the depth of w is $d+1$, for some d, such that $f(u)$ is the largest label assigned to nodes of depth d and $f(w)$ is the smallest label assigned to nodes of depth $d+1$. Then we see that $f(w) = f(u)+1$. Suppose the children of u are labeled $a, a+1, ..., a+l$ for some a and $l \geq 1$. Then the children of w are $a+l+1, ..., a+l+m$ for some $m \geq 1$. By Lemma 1, $\binom{a+l}{f(u)} < \binom{a+l+1}{f(w)}$. This shows that the edges at level d have labels less than those at level $d+1$.

Combining these three results, we see that the tree T is a combination graph.

Lemma 2 immediately implies that complete k-ary trees are combination graphs. We state this in the following Corollary.

THEOREM 1. The complete k-ary tree is a combination graph.

2.2 Caterpillars

We now consider another class of trees called *caterpillars*. A tree is a *caterpillar* if, upon removing all leaves and their incident edges, a path is left. This path is called the *central path* of the caterpillar graph. Note that in the caterpillar, the central path can be extended to a longer path since each endpoint of the path must be adjacent to a vertex in the caterpillar. Let us call this path the *extended central path* of the caterpillar. We will call an edge that is not on the extended central path of a caterpillar a *leg*.

We begin by showing that if a caterpillar has enough legs, then it is a combination graph. To do this we start with a simple lemma that can be verified through algebraic

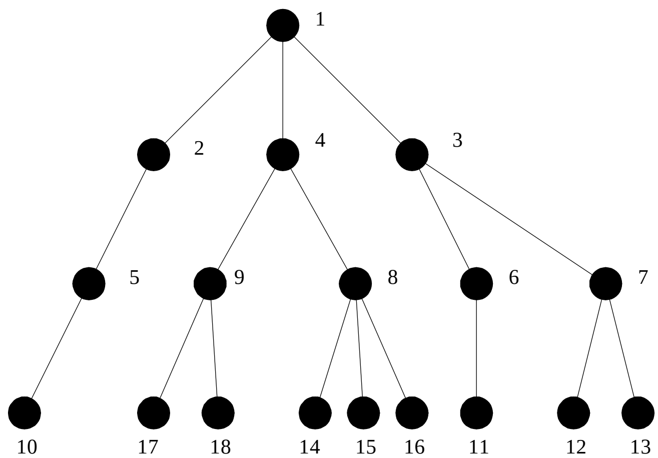

Figure 1: A labeling of a 18 node rooted tree.

manipulation.

LEMMA 3. If $l, m \geq 0$ and $n+1 \geq 2(l+m)$, then $\binom{n}{l} < \binom{n+1}{l+m}$.

THEOREM 2. Let $G = (V, E)$ be a caterpillar with extended central path P consisting of p vertices. If G has at least $3p-6$ vertices, then T is a combination graph.

PROOF. We partition the vertex set of V into two smaller sets A and B by first dividing the path P into two (disjoint) sub-paths Q and R of equal or almost equal length. Then we place a vertex v into A if $v \in Q$ or v is adjacent to a vertex on Q. Otherwise, we place v into B. At least one of $G[A]$ or $G[B]$ contain at least $(2p-6)/2 = p-3$ edges that are not edges of P, where $G[A](G[B])$ denote the subgraph of G induced by the vertex set $A(B)$. Without loss of generality, assume $G[A]$ has this property.

We now construct a labeling f of the vertices of V. Label the path P starting from one end to the other end with labels $1, 2, ..., p$ so that the vertex that is labeled with the value 1 is the end-vertex of P which belongs to A. Now label the remaining vertices of G that are not on the path P so that if u, v are not on P, $up_1, vp_2 \in E$, $p_1, p_2 \in P$ and $f(p_1) < f(p_2)$, then $f(u) < f(v)$. This can be accomplished by starting at the end of P label with value 1, and moving along the path P. As a leg is encountered, we label the vertex of the leg that is not on the path with the next available value. We claim that this labeling is a combination labeling of G.

We see that the edges on the path P have labels $1, 2, ..., p$ and the smallest edge label of an edge not on the path is at least $\binom{p+1}{2} > p$. It is clear that the edge labels of any two edges in $G[A]$ but not on P satisfies Lemma 3 and therefore are pairwise distinct. Finally, the smallest label assigned to a leaf node in B but not on P is at least $p + (p-3) + 1 = 2p-2 = 2(p-1)$. In $G[B]$, the label $p-1$ is the largest label assigned to a vertex on P that can be adjacent to vertices not on P. Therefore, Lemma 3 can be applied to the labels of the legs of the entire graph G to show that the legs of G have pairwise distinct labels.

THEOREM 3. Let $G = (V, E)$ be a caterpillar with extended central path P consisting of p vertices. If each vertex of P, except for its two endpoints, is adjacent to at least one vertex that is not on P, then G is a combination graph.

PROOF. Start at one end of the path P and label the endpoint 1. Follow the path and label each vertex visited with the next available label. To label the vertices that are not on P, use the labeling scheme as in Theorem 2. If $f(u)$ is the label of a vertex u not on P that is adjacent to a vertex v on P with label $f(v)$, then $f(u) \geq 2f(v)$. To see this, note that the smallest value that $f(u)$ can be is $p + f(v) - 1$. Since $p - 1 \geq v$, we have $f(u) \geq p + f(v) - 1 \geq 2f(v)$. Applying Lemma 3, we see that G is a combination graph.

2.3 Generalized Petersen Graph $GP(n, k)$

Suppose k, n are positive integers such that $n > 2k$. The *generalized Petersen graph*, denote by $GP(n, k)$, is the simple graph with vertices $u_1, u_2, .., u_n, v_1, v_2, ..., v_n$ and edges $u_iu_{i+1}, v_iv_{i+k}, u_iv_i, 1 \leq i \leq n$, where the indexes are taken modulo n. We will show that $GP(n, 1)$ and $GP(n, 2)$ are combination graphs.

LEMMA 4. If $n \geq 2$ then $\binom{2n}{2} < \binom{n+3}{3}$.

THEOREM 4. If $n \geq 4$, then $GP(n, 1)$ is a combination graph.

PROOF. Figure 2 gives a valid combination labeling for $GP(4, 1)$. Therefore, assume that $n \geq 5$. Label the vertices of $GP(n, 1)$ as follows: $f(u_1) = 1, f(u_2) = 2, ..., f(u_{n-2}) = n - 2, f(u_{n-1}) = n, f(u_n) = n - 1$, $f(v_1) = n + 1, f(v_2) = n + 2, ..., f(v_{n-2}) = 2n - 2, f(v_{n-1}) = 2n, f(v_n) = 2n - 1$. We claim that this is a valid combination labeling of $GP(n, 1)$. The edges $u_iu_{i+1}, 1 \leq i \leq n$ have labels $2, 3, ..., n, \binom{n}{2}$. The edges $v_iv_{i+1}, 1 \leq i \leq n$ have labels $n+2, ..., 2n-2, 2n, \binom{2n}{2}, \binom{2n-1}{n+1} = \binom{2n-1}{n-2}$. The edges $u_iv_{i+1}, 1 \leq i \leq n$ have labels $\binom{n+1}{1}, \binom{n+2}{2}, ..., \binom{2n}{n}$. By Lemma 4, $\binom{2n-1}{2} < \binom{2n}{2} < \binom{n+3}{3}$. By Lemma 1, $\binom{n+i}{i} < \binom{n+i+1}{i+1}$. In addition, it is easy to see that if $n \geq 5$, then $2n - 2 < \binom{n}{2} < \binom{n+2}{2}$. Therefore, the edge labels $2, 3, ..., 2n-2, \binom{n}{2} < \binom{n+2}{2}, \binom{2n}{2}, \binom{n+3}{3}, \binom{n+4}{4}, ..., \binom{2n-2}{n-2}, \binom{2n-1}{n-2}, \binom{2n-1}{n-1}, \binom{2n}{n}$ are all distinct and are listed in increasing order.

LEMMA 5. If $n \geq 9$, then $\binom{2n-2}{n-2} < \binom{2n-1}{n-3}$. If $n = 8$, then $\binom{2n-2}{n-2} = \binom{2n-1}{n-3}$

LEMMA 6. If $8 \leq n \leq 16$, then $\binom{n+4}{4} < \binom{2n}{3} < \binom{n+5}{5}$. If $n \geq 17$, then $\binom{2n}{3} < \binom{n+4}{4}$.

THEOREM 5. For $n \geq 5$, then $GP(n, 2)$ is a combination graph.

PROOF. Figure 3 gives valid combination labelings for $GP(n, 2)$, where $5 \leq n \leq 8$. Therefore, assume that $n \geq 9$. Label the vertices of $GP(n, 2)$ as follows: $f(u_1) = 1, f(u_2) = 2, ..., f(u_{n-2}) = n-2, f(u_{n-1}) = n, f(u_n) = n-1, f(v_1) = n+1, f(v_2) = n+2, ..., f(v_{n-2}) = 2n-2, f(v_{n-1}) = 2n, f(v_n) = 2n-1$. We claim that this is a valid combination labeling of $GP(n, 2)$. The edges $u_iu_{i+1}, 1 \leq i \leq n$ have labels $2, 3, ..., n, \binom{n}{2}$. The edges $v_iv_{i+2}, 1 \leq i \leq n$ have labels $\binom{n+2}{2}, \binom{n+3}{3}, ..., \binom{2n-2}{2}, \binom{2n-1}{n+2}, \binom{2n-1}{2n-2}, \binom{2n}{n+1}$ and $\binom{2n}{2n-3}$. The edges $u_iv_{i+1}, 1 \leq i \leq n$ have labels $\binom{n+1}{1}, \binom{n+2}{2}, ..., \binom{2n}{n}$. Note that

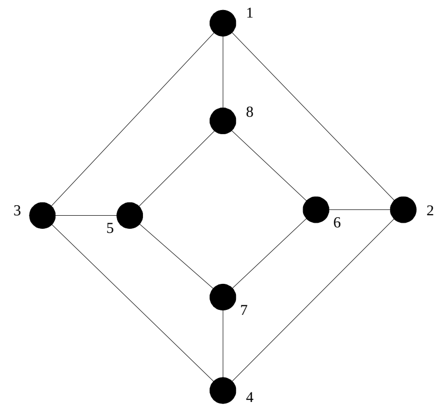

Figure 2: A valid combination labeling of $GP(4,1)$.

$\binom{2n-1}{n+2} = \binom{2n-1}{n-3}$, $\binom{2n-2}{2n-2} = 2n-1$, $\binom{2n}{n+1} = \binom{2n}{n-1}$ and $\binom{2n}{2n-3} = \binom{2n}{3}$. We claim that we can order the edge labels in monotone increasing order. We can order this smallest edge labels as $2 < 3 < \cdots < n < n+1 < 2n-1 < \binom{n}{2}$ where the last inequality holds as $n > 8 \geq 3$. Continuing, we have $\binom{n}{2} < \binom{n+2}{2} < \binom{n+3}{2} < \cdots \binom{2n-2}{2}$. By Lemma 6, we have either $\binom{n+3}{2} < \binom{2n}{3} < \binom{n+4}{4} < \cdots < \binom{2n-2}{n-2} < \binom{2n-1}{nn-3}$ where the last inequality follows from Lemma 5, or $\binom{n+3}{2} < \binom{n+4}{4}\binom{2n}{3} < \binom{n+5}{5} < \cdots < \binom{2n-2}{n-2} < \binom{2n-1}{n-3}$. Finally, we have $\binom{2n-1}{n-1} < \binom{2n}{n-1} < \binom{2n}{n}$. This gives a sequence of strict inequalities involving each edge label. Therefore, $GP(n,2)$ is a combination graph.

2.4 Wheel Graphs

Let n be a positive integer greater than 2. A *wheel graph* on $n+1$ vertices is a graph consisting of a cycle of length n and a vertex not on the cycle that is adjacent to every vertex on the cycle. We denote this graph by W_n. In [3], it was conjectured that for all $n \geq 7$, W_n is a combination graph. We will show that this conjecture is true. We begin with some simple, useful results that can be verified by algebraic manipulations.

LEMMA 7. If $n \geq 6$ is an even number, then $\binom{n}{n/2} < \binom{n+1}{n/2-1}$.

LEMMA 8. If $n \geq 20$ is an even number, then $\binom{n/2+2}{2} < \binom{n+1}{2} < \binom{n/2+2}{3}$. In addition, if $10 \leq n \leq 18$ is an even number, then $\binom{n/2+2}{3} < \binom{n+1}{2} < \binom{n/2+3}{3}$.

LEMMA 9. If $n \geq 7$ is an odd number, then $\binom{n-1}{\lfloor n/2 \rfloor} < \binom{n}{\lfloor n/2 \rfloor - 1}$.

We now proceed to label the wheel graph W_n. We will give a labeling that "almost" works and then modify it slightly to so that it gives a valid combination labeling of W_n.

THEOREM 6. If $n \geq 7$, then W_n is a combination graph.

PROOF. Valid combination labelings for $n = 7, 8$ were given in [3]. Let us assume that $n \geq 9$. Denote the cycle of length n of W_n by C_n. Let x be the vertex that is not

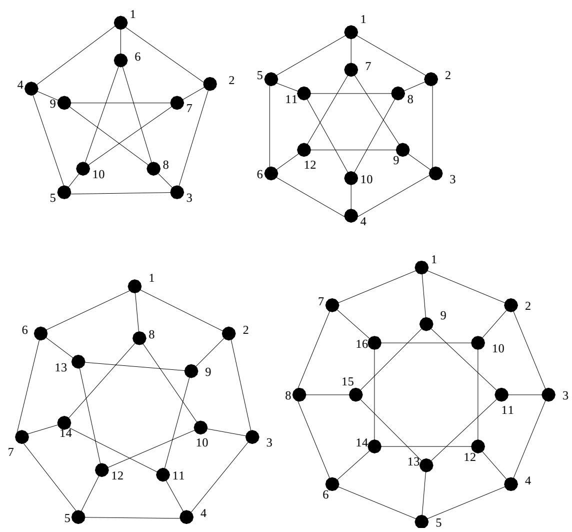

Figure 3: Valid combination labelings for $GP(n,2)$ for $5 \le n \le 8$.

in C_n and is adjacent to each vertex of C_n. Denote the vertices of C_n by $v_0, v_1, ..., v_{n-1}$ where v_i is adjacent to v_{i+1} modulo n. We now give a labeling of the vertices of W_n.

Label vertex x with value 1. On the cycle C_n, label v_0 with 2, v_2 with 3, v_4 with 4, etc. In general, after labeling vertex v_i with value k, we skip over vertex v_{i+1}, and label v_{i+2} with value $k+1$ if it has not already been labeled. If v_{i+2} has already been labeled, then we label v_{i+3} with value $k+1$. The indexes are taken modulo n. Let us denote this labeling by f.

Under this labeling there exists (at least one) i such that $\binom{f(v_i)}{f(v_{i+1})} = \binom{f(v_i)}{f(v_{i-1})}$. When n is odd, we have $f(v_{n-4}) = n$. Therefore $\binom{f(v_{n-4})}{f(v_{n-3})} = \binom{n}{\lceil n/2 \rceil} = \binom{n}{\lfloor n/2 \rfloor} = \binom{f(v_{n-4})}{f(v_{n-5})}$. However, this is the only occurrence because for any vertex whose label l is greater than $\lfloor n/2 \rfloor + 3$, its neighbors on C_n have labels $l - \lfloor n/2 \rfloor$ and $l - \lceil n/2 \rceil$. The only value of l which satisfies $\binom{l}{l-\lfloor n/2 \rfloor} = \binom{l}{l-\lceil n/2 \rceil}$ is $l = n$. Using a similar argument for when n is even, we see that $\binom{f(v_i)}{f(v_{i+1})} = \binom{f(v_i)}{f(v_{i-1})}$ happens only when $i = n-5$ and $f(v_{n-5}) = n-1$.

We make a slight modification to the labeling f by performing the following swaps:

1. If n is odd, we swap the labels n and $n-1$ in the labeling f.

2. If n is even, we swap the labels $n-1$ and $n-2$ in the labeling f.

Let us denote this new labeling by g. We claim that g is a combination labeling of W_n. It is clear that two adjacent edges on C_n do not have labels $\binom{l}{l-k}$ and $\binom{l}{k}$ for $l > k$ because of the modifications made above. Figure 4 gives examples for $n = 10, 11$.

In the case where n is odd, the edge labels of W_n, induced by g are $2, 3, ..., n+1$ and $\binom{\lfloor n/2 \rfloor+3}{2}, \binom{\lfloor n/2 \rfloor+3}{3}, ..., \binom{n-2}{\lfloor n/2 \rfloor-2}, \binom{n-2}{\lfloor n/2 \rfloor-1}, \binom{n-1}{\lfloor n/2 \rfloor}, \binom{n-1}{\lfloor n/2 \rfloor+1}, \binom{n}{\lfloor n/2 \rfloor}, \binom{n}{\lfloor n/2 \rfloor-1}, \binom{n+1}{\lfloor n/2 \rfloor+1},$

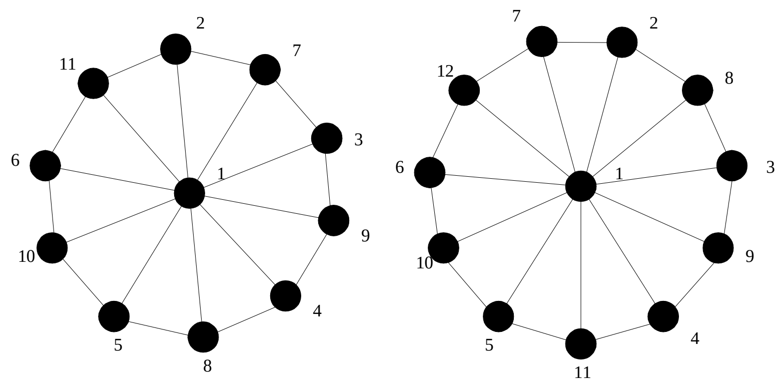

Figure 4: Combination labelings for W_9 and W_{10}.

$\binom{n+1}{\lfloor n/2\rfloor+2}$ and $\binom{\lfloor n/2\rfloor+2}{2}$. Note that $\binom{n+1}{\lfloor n/2\rfloor+2}=\binom{n+1}{\lfloor n/2\rfloor}$. If remove $\binom{n}{\lfloor n/2\rfloor},\binom{n}{\lfloor n/2\rfloor-1}$ from the sequence above, the remaining values are all distinct and in fact, $2<3<\cdots<n+1<\binom{\lfloor n/2\rfloor+2}{2}<\binom{\lfloor n/2\rfloor+3}{2}<\binom{\lfloor n/2\rfloor+3}{3}<\cdots<\binom{n-2}{\lfloor n/2\rfloor-2}<\binom{n-2}{\lfloor n/2\rfloor-1}<\binom{n-1}{\lfloor n/2\rfloor+1}<\binom{n-1}{\lfloor n/2\rfloor}<\binom{n+1}{\lfloor n/2\rfloor+2}(=\binom{n+1}{\lfloor n/2\rfloor})<\binom{n+1}{\lfloor n/2\rfloor+1}$. By Lemma 9, $\binom{n-1}{\lfloor n/2\rfloor+1}<\binom{n-1}{\lfloor n/2\rfloor}<\binom{n}{\lfloor n/2\rfloor-1}<\binom{n}{\lfloor n/2\rfloor}$. Since $\binom{n}{\lfloor n/2\rfloor}<\binom{n+1}{\lfloor n/2\rfloor}<\binom{n+1}{\lfloor n/2\rfloor+1}$, all the edge labels are distinct, when n is odd.

We now consider the case when n is even. The edge labels of W_n,induced by g are $2,3,...,n+1,\binom{n/2+2}{2},\binom{n/2+2}{3},\binom{n/2+3}{3},\binom{n/2+3}{4},...,\binom{n-2}{n/2-1},\binom{n-2}{n/2},\binom{n-1}{n/2-2},\binom{n-1}{n/2-1},\binom{n}{n/2},\binom{n}{n/2+1},\binom{n+1}{n/2+1}$ and $\binom{n+1}{2}$. If we remove $\binom{n-1}{n/2-2},\binom{n-1}{n/2-1}$ and $\binom{n+1}{2}$ from this list, the remaining values are clearly distinct and $2<3<\cdots n+1<\binom{n/2+2}{2}<\binom{n/2+2}{3}<\binom{n/2+3}{3}<\cdots<\binom{n-2}{n/2}=\binom{n-2}{n/2-2}<\binom{n-2}{n/2-1}<\binom{n}{n/2-1}=\binom{n}{n/2+1}<\binom{n}{n/2}<=\binom{n+1}{n/2}=\binom{n+1}{n/2+1}$. Note that as $\binom{n-1}{n/2-2}<\binom{n-1}{n/2-1}<\binom{n}{n/2-1}$ and by Lemma 7, $\binom{n-2}{n/2-1}<\binom{n-1}{n/2-2}$, all edge labels except for possibility $\binom{n+1}{2}$ are distinct. By Lemma 8, $\binom{n/2+2}{2}<\binom{n+1}{2}<\binom{n/2+2}{3}$ for all even $n\geq 20$. For $n=10,12,14,16,18$, we have $\binom{n/2+2}{3}<\binom{n+1}{2}<\binom{n/2+3}{3}$. Therefore all the edge labels are distinct in W_n.

2.5 Grid Graphs

Let k,n be positive integers with $k\leq n$. Let $G=(V,E)$ be the $k\times n$ grid graph. More precisely, $V=\{(i,j):0\leq i\leq k-1,0\leq j\leq n-1\}$ and $E=\{\{(i,j_1),(i,j_2)\}:0\leq j_1\leq n-2,j_2=j_1+1\}\cup\{\{(i_1,j),(i_2,j)\}:0\leq i_1\leq k-2,i_2=i_1+1\}$. Another way of constructing the $k\times n$ grid graph is to take the Cartesian product of the paths P_k and P_n. We claim that for large enough values of k and n, the $k\times n$ grid graph is a combination graph. To prove this, we begin with a useful fact.

LEMMA 10. Let $k\leq n$. If $n\geq\lceil\frac{2k-3+\sqrt{4k^2-12k+1}}{2}\rceil$, then $kn<\binom{n+2}{2}$.

THEOREM 7. Let $k\leq n$. If $n\geq\lceil\frac{2k-3+\sqrt{4k^2-12k+1}}{2}\rceil$, then the $k\times n$ grid graph is a combination graph.

PROOF. Label the vertex (i, j) with the value $in + j + 1$. Then the edges have induced labels in the set $\{2, 3, ..., kn-1, kn, \binom{n+2}{n}, \binom{n+3}{n}, ..., \binom{kn-1}{n}, \binom{kn}{n}\} \setminus \{2n+1, 3n+1, ..., (k-1)n+1\}$. Clearly, the labels $\binom{n+2}{n}, \binom{n+3}{n}, ..., \binom{kn}{n}$ for an increasing sequence and therefore are pairwise distinct. Since $n \geq \lceil \frac{2k-3+\sqrt{4k^2-12k+1}}{2} \rceil$, Lemma 10 implies $\binom{n+2}{n} = \binom{n+2}{2} > kn$. Therefore, the edge labels are pairwise distinct.

3 Other Results

We state several related results.

LEMMA 11. Let G be a graph with $n \geq 3$ vertices. If G is a combination graph, then at most one vertex of G has degree $n - 1$.

PROOF. Suppose there are at least two vertices that have degree $n - 1$ in G and G is a combination graph. Consider a valid combination labeling. Let $x < y$ be vertex labels of two vertices of degree $n-1$. Suppose $y \geq 3$. Then y is adjacent vertices labeled 1 and $y - 1$. But $\binom{y}{y-1} = \binom{y}{1}$, which contradicts assumption that G is a combination graph. Therefore $y \leq 2$ implying $x = 1, y = 2$. Then both x, y are adjacent to the vertex labeled 3. As $\binom{3}{1} = \binom{3}{2}$, this contradicts assumption that G is a combination graph. Therefore, at most one vertex of G can have degree $n - 1$.

This immediately implies that K_n is not a combination graph whenever $n \geq 3$. The proof of Lemma 3 also shows that if a combination graph has a vertex of degree $n - 1$, the label of that vertex must be 1 or 2. We now show that some combination graph on n vertices with the maximum number of edges possible must contain a vertex of degree $n - 1$ whose label is 1.

LEMMA 12. Let m be the maximum number of edges in any combination graph with n vertices. Then there is a combination graph G with n vertices and m edges such the vertex labeled with value 1 is adjacent to all the other vertices.

PROOF. Suppose that G is a combination graph with n vertices, m edges and the vertex v labeled with value 1 does not have degree $n - 1$. Then, let the vertices that are not adjacent to v have labels $a_1, a_2, ..., a_k$, where $k \geq 1$. Remove from G all edges whose induced edge labeling belongs in $\{a_1, a_2, ..., a_k\}$. There are at most k such edges, as G is a combination graph. Finally, add edges to G so that v has degree $n - 1$. The resulting graph is still a combination graph. Since the original graph was a combination graph and has maximum number of edges possible, the number of edges removed must be k.

We can use Lemma 3 to show that any combination graph with 6 vertices can have at most 8 edges. In the contrary, suppose G is a combination graph with 6 vertices and 9 edges. By Lemma 3 there must exists a graph H with 6 vertices and 9 edges such that the vertex labeled with value 1 is adjacent the other 5 vertices. The only remaining edges that are permissible the 5 edges $26, 25, 35, 36, 46$. But since $\binom{6}{2} = \binom{6}{4}$ and $\binom{5}{2} = \binom{5}{3}$, at most 3 of these 5 edges can be in the graph H given a total of 8 edges, which is a contradiction. A similar argument can be applied to obtain the following bound from [3].

$$m \leq \begin{cases} n^2/4 & \text{if } n \text{ is even} \\ (n^2-1)/4 & \text{if } n \text{ is odd.} \end{cases} \quad (1)$$

In [3], it was shown that $K_{r,r}$ is not a combination graph for $r \geq 3$. We now generalize this for complete bipartite graphs $K_{l,k}$.

Note that if a graph $G = (V, E^{'})$ is not a combination graph, then it is clear that if we add additional edges to $E^{'}$ it will not be a combination graph. We record this in the following lemma.

LEMMA 13. Suppose $G = (V, E)$ and $E^{'} \subseteq E$. If $(V, E^{'})$ is not a combination graph, then G is not a combination graph.

THEOREM 8. Let $K_{l,k} = (A, B)$ be the complete bipartite graph with k elements in the partite set A, and l elements in the other partite set B. Then $K_{l,k}$ is a combination graph if and only if $k = 1$ or $l = 1$ or $k = l = 2$.

PROOF. The case where $k = l = 2$ the cycle of length 4 which is clearly a combination graph. Suppose $k = 1$ (or if $l = 1$) and let A denote the partite set with one vertex. Label the lone vertex of the partite set A with value 1 and label the vertices in the other partite set with values $2, 3, ..., l + 1$. Clearly this is a valid labeling.

Suppose $l \geq 2$ and $k > l$. We will show that $K_{l,k}$ is not a combination graph. To do this, suppose to the contrary that $K_{l,k}$ is a combination graph where the vertices of the graph is labeled using a valid combination labeling. Then the vertices with label 1 and $l + k$ must be in the same partite set. For if not, then without loss of generality suppose $1 \in A$, $l + k \in B$. This forces vertex with label $l + k - 1$ to be in B, which in turn forces vertex with label $l + k - 2$ to be in B, and so on. Therefore, the partite set A contains only one vertex, the vertex with label 1. This contradicts our assumption that $|A| \geq 2$. We now have two scenarios: $1, l + k \in A$ or $1, l + k \in B$.

Case 1: Suppose $1, l + k \in A$. If the vertex with label $l + k - 1 \in B$, then using an argument similar to the one above, we have $2, 3, ..., l + k - 1 \in B$. This implies that $l = 2$. As $k > 2$, $\binom{k+2}{2} = \binom{k+2}{k}$ and $2, k \in B$, we have two edges with the same label, which is a contradiction. Therefore, it must be that $l + k - 1 \in A$. By repeatedly applying this argument and the assumption that $|B| = k$, we see that the labels in A are $\{1, k + 2, k + 3, ..., l + k\}$ and the labels in B are $\{2, 3, ..., k + 1\}$. As $k > 2$ and $\binom{k+2}{2} = \binom{k+2}{k}$, there are two edge labels with the same value, which is a contradiction. Therefore, case 1 leads to a contradiction.

Case 2: Suppose $1, l + k \in B$. Then, using an argument similar to that of case 1, we have that A contains vertices with labels $2, 3, ..., l + 1$ and B contains vertices with labels $1, l + 2, ..., l + k$. The labels $l, l + 1 \in A$. Since $k > l$, $2l + 1 \leq l + k$ implying that $2l + 1 \in B$. This along with the fact that $\binom{2l+1}{l} = \binom{2l+1}{l+1}$ implies that two edges have the same labeling, which is a contradiction. Therefore, case 2 also leads to a contradiction.

COROLLARY 1. Suppose G is a complete k-partite with partite sets $A_1, A_2, .., A_k$ where $k \geq 2$ and $|A_i| \geq 2$ for $i = 1$ to k. Then G is not a combination graph.

PROOF. Construct a graph H from G by removing all edges between A_i, A_j where $1 < i \neq j \leq k$. Partition the vertices of H into two sets A_1 and $\cup_{i=2}^{k} A_i$. The graph H is a complete bipartite graph. By Theorem 8, it is not a combination graph. By Lemma 13, G is not a combination graph.

We would like to end by stating the following open problem: *Are all trees combination graphs?*. Based on instances that we have considered, which all turned out to be combination graphs, we believe and conjecture that all trees are combination graphs.

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