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Fractional Order Riemann-Liouville Integral Equations with Multiple Time Delays*

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Abstract

In the present article we investigate the existence and uniqueness of solutions for a system of integral equations of fractional order by using some fixed point theorems. Also we illustrate our results with some examples.

1 Introduction

The idea of fractional calculus and fractional order integral equations has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [9, 11, 16, 17, 19]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas *et al.* [14], Miller and Ross [18], Samko *et al.* [21], the papers of Abbas and Benchohra [1, 2], Abbas *et al.* [3], Belarbi *et al.* [4], Benchohra *et al.* [5, 6, 7], Diethelm [8], Kilbas and Marzan [15], Mainardi [16], Podlubny *et al* [20], Vityuk [22], Vityuk and Golushkov [23], and Zhang [24] and the references therein.

In [13], R. W. Ibrahim and H. A. Jalab studied the existence of solutions of the following fractional integral inclusion

$$u(t)-\sum_{i=1}^{m} b_i(t)u(t-\tau_i)\in I^{\alpha}F(t,u(t)) \text{ if } t\in[0,T],$$

where $\tau_i < t \in [0,T], b_i : [0,T] \to \mathbb{R},\ i = 1,\dots,n$ are continuous functions, and $F : [0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a given multivalued map.

This paper concerned with the existence and uniqueness of solutions for the following fractional order integral equations for the system

$$u(x,y)=\sum_{i=1}^{m} g_i(x,y)u(x-\xi_i,y-\mu_i)+I_{\theta}^{r}f(x,y,u(x,y)) \text{ if } (x,y)\in J:=[0,a]\times[0,b], \quad (1)$$

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$$u(x,y) = \Phi(x,y);\ \text{if } (x,y) \in \tilde{J} := [-\xi, a] \times [-\mu, b] \backslash (0,a] \times (0,b], \tag{2}$$

where $a, b > 0$, $\theta = (0,0)$, $\xi_i, \mu_i \geq 0$; $i = 1, \ldots, m$, $\xi = \max_{i=1,\ldots,m}\{\xi_i\}$, $\mu = \max_{i=1,\ldots,m}\{\mu_i\}$, I_θ^r is the left-sided mixed Riemann-Liouville integral of order $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $g_i : J \to \mathbb{R}$; $i = 1, \ldots, m$ are given continuous functions, and $\Phi : \tilde{J} \to \mathbb{R}^n$ is a given continuous function such that

$$\Phi(x,0) = \sum_{i=1}^{m} g_i(x,0)\Phi(x - \xi_i, -\mu_i);\ x \in [0,a],$$

and

$$\Phi(0,y) = \sum_{i=1}^{m} g_i(0,y)\Phi(-\xi_i, y - \mu_i);\ y \in [0,b].$$

We present three results for the problem (1)-(2), the first one is based on Schauder's fixed point theorem (Theorem 1), the second one is a uniqueness of the solution by using the Banach fixed point theorem (Theorem 2) and the last one on the nonlinear alternative of Leray-Schauder type (Theorem 4).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$. Also, $C := C([-\xi, a] \times [-\mu, b])$ is a Banach space endowed with the norm

$$\|w\|_C = \sup_{(x,y)\in[-\xi,a]\times[-\mu,b]} \|w(x,y)\|.$$

As usual, by $L^1(J)$ we denote the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^a \int_0^b \|w(x,y)\| dy dx.$$

DEFINITION 1 ([23]). Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, $\theta = (0,0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t) dt ds.$$

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y),\ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t) dt ds \text{ for almost all } (x,y) \in J,$$

where $\sigma = (1, 1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover

$$(I_\theta^r u)(x, 0) = (I_\theta^r u)(0, y) = 0;\ x \in [0, a],\ y \in [0, b].$$

EXAMPLE 1. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2} \text{ for almost all } (x, y) \in J.$$

3 Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1)-(2).

DEFINITION 2. A function $u \in C$ is said to be a solution of (1)-(2) if u satisfies equation (1) on J and condition (2) on $\tilde{J}$.

Set

$$B = \max_{i=1,\dots,m} \left\{ \sup_{(x,y)\in J} |g_i(x, y)| \right\}.$$

THEOREM 1. Assume

(H_1) There exists a positive function $h \in C(J)$ such that

$$\|f(x, y, u)\| \le h(x, y),\ \text{for all } (x, y) \in J \text{ and } u \in \mathbb{R}^n.$$

If $mB < 1$, then problem (1)-(2) has at least one solution u on $[-\xi, a] \times [-\mu, b]$.

PROOF. Transform problem (1)-(2) into a fixed point problem. Consider the operator $N : C \to C$ defined by,

$$N(u)(x, y) = \begin{cases} \Phi(x, y); & (x, y) \in \tilde{J}, \\ \sum_{i=1}^m g_i(x, y) u(x - \xi_i, y - \mu_i) + I_\theta^r f(x, y, u(x, y)); & (x, y) \in J. \end{cases} \tag{3}$$

The problem of finding the solutions of problem (1)-(2) is reduced to finding the solutions of the operator equation $N(u) = u$. Let $R \ge \frac{R^*}{1-mB}$ where

$$R^* = \frac{a^{r_1} b^{r_2} h^*}{\Gamma(1+r_1)\Gamma(1+r_2)},$$

and $h^* = \|h\|_\infty$, and consider the set

$$B_R = \{u \in C : \|u\|_C \le R\}.$$

It is clear that B_R is a closed bounded and convex subset of C. For every $u \in B_R$ and $(x, y) \in J$ we obtain by (H_1) that

$$\begin{aligned}
\|N(u)(x,y)\| &\leq \sum_{i=1}^{m} |g_i(x,y)|\, \|u(x-\xi_i, y-\mu_i)\| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u(s,t))\| dtds \\
&\leq mB\|u\|_C + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t) dtds \\
&\leq mB\|u\|_C + h^* \frac{a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&\leq mBR + (1-mB)R = R.
\end{aligned}$$

On the other hand, for every $u \in B_R$ and $(x, y) \in \tilde{J}$, we obtain

$$\|N(u)(x,y)\| = \|\Phi(x,y)\| \leq R.$$

So we obtain that

$$\|N(u)\|_C \leq R.$$

That is, $N(B_R) \subseteq B_R$. Since f is bounded on B_R, thus $N(B_R)$ is equicontinuous and the Schauder fixed point theorem shows that N has at least one fixed point $u^* \in B_R$ which is solution of (1)-(2).

For the uniqueness we prove the following Theorem

THEOREM 2. Assume that following hypothesis holds:

(H_2) There exists a positive function $l \in C(J)$ such that

$$\|f(x,y,u) - f(x,y,v)\| \leq l(x,y)\|u - v\|,$$

for each $(x, y) \in J$ and $u, v \in \mathbb{R}^n$.

If

$$\frac{mB\Gamma(1+r_1)\Gamma(1+r_2) + a^{r_1} b^{r_2} l^*}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{4}$$

where $l^* = \|l\|_\infty$, then problem (1)-(2) has a unique solution on $[-\xi, a] \times [-\mu, b]$.

PROOF. Consider the operator N defined in (3). Then by (H_2), for every $u, v \in C$

and $(x,y) \in J$ we have

$$\begin{aligned}
\|N(u)(x,y) - N(v)(x,y)\| &\leq \sum_{i=1}^{m} |g_i(x,y)|\,\|u(x-\xi_i, y-\mu_i) - v(x-\xi_i, y)\| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
&\times \|f(s,t,u(s,t)) - f(s,t,v(s,t))\| dtds \\
&\leq mB\|u-v\|_\infty \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
&\times l(s,t)\|u-v\|_C dtds \\
&\leq mB\|u-v\|_\infty + l^* \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|u-v\|_C \\
&= \Big(mB + \frac{l^* a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big)\|u-v\|_C.
\end{aligned}$$

Thus

$$\|N(u) - N(v)\|_C \leq \frac{mB\Gamma(1+r_1)\Gamma(1+r_2) + a^{r_1}b^{r_2}l^*}{\Gamma(1+r_1)\Gamma(1+r_2)}\|u-v\|_C$$

Hence by (4), we have that N is a contraction mapping. Then in view of Banach fixed point Theorem, N has a unique fixed point which is solution of problem (1)-(2).

THEOREM 3 ([10]). (*Nonlinear alternative of Leray-Schauder type*) By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U respectively. Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to C$ continuous and compact operator. Then either

(a) T has fixed points, or

(b) there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T(u)$.

In the sequel we use the following version of Gronwall's Lemma for two independent variables and singular kernel.

LEMMA 1 ([12]). Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that

$$\upsilon(x,y) \leq \omega(x,y) + c\int_0^x \int_0^y \frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dtds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(x,y) \leq \omega(x,y) + \delta c\int_0^x \int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dtds,$$

for every $(x,y) \in J$.

Now, we present an existence result for the problem (1)-(2) based on the Nonlinear alternative of Leray-Schauder type.

THEOREM 4. Assume

(H_3) There exist positive functions $p, q \in C(J)$ such that

$$\|f(x,y,u)\| \leq p(x,y) + q(x,y)\|u\|, \text{ for all } (x,y) \in J \text{ and } u \in \mathbb{R}^n.$$

If $mB < 1$, then problem (1)-(2) has at least one solution on $[-\xi, a] \times [-\mu, b]$.

PROOF. Consider the operator N defined in (3). We shall show that the operator N is completely continuous. By the continuity of f and the Arzela-Ascoli Theorem, we can easily obtain that N is completely continuous.

A priori bounds. We shall show there exists an open set $U \subseteq C$ with $u \neq \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$, we have

$$u(x,y) = \lambda \sum_{i=1}^{m} g_i(x,y) u(x-\xi_i, y-\mu_i) + \lambda I_\theta^r f(x,y,u(x,y)).$$

This implies by (H_3) that, for each $(x,y) \in J$, we have

$$\begin{aligned}\|u(x,y)\| &\leq mB\|u(x,y)\| + \frac{p^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &+ \frac{q^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dt\,ds,\end{aligned}$$

where $p^* = \|p\|_\infty$ and $q^* = \|q\|_\infty$. Thus, for each $(x,y) \in J$, we get

$$\begin{aligned}\|u(x,y)\| &\leq \frac{p^* a^{r_1} b^{r_2}}{(1-mB)\Gamma(1+r_1)\Gamma(1+r_2)} \\ &+ \frac{q^*}{(1-mB)\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dt\,ds \\ &\leq w + c\int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dt\,ds,\end{aligned}$$

where

$$w := \frac{p^* a^{r_1} b^{r_2}}{(1-mB)\Gamma(1+r_1)\Gamma(1+r_2)}$$

and

$$c := \frac{q^*}{(1-mB)\Gamma(r_1)\Gamma(r_2)}.$$

From Lemma 1, there exists $\delta := \delta(r_1, r_2) > 0$ such that, for each $(x,y) \in J$, we get

$$\begin{aligned}\|u\|_\infty &\leq w\left(1 + c\delta \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\,dt\,ds\right) \\ &\leq w\left(1 + \frac{c\delta a^{r_1} b^{r_2}}{r_1 r_2}\right) := \widetilde{M}.\end{aligned}$$

Set $M^* := \max\{\|\Phi\|, \widetilde{M}\}$ and

$$U = \{u \in C : \|u\|_C < M^* + 1\}.$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 3, we deduce that N has a fixed point u in $\overline{U}$ which is a solution to problem (1)-(2).

4 Examples

We provide two examples.

EXAMPLE 1. As an application of our results we consider the following system of fractional integral equations of the form

$$u(x,y) = \frac{x^3y}{8}u(x-\frac{3}{4},y-3) + \frac{x^4y^2}{12}u(x-2,y-\frac{1}{2}) + \frac{1}{4}u(x-1,y-\frac{3}{2})$$
$$+I_\theta^r f(x,y,u);\ \text{if } (x,y) \in J := [0,1]\times[0,1], \tag{5}$$
$$u(x,y) = 0;\ \text{if } (x,y) \in \tilde{J} := [-2,1]\times[-3,1]\backslash(0,1]\times(0,1], \tag{6}$$

where $m = 3,\ r = (\frac{1}{2},\frac{1}{5})$ and

$$f(x,y,u) = e^{x+y}\frac{1}{1+|u|}.$$

Set

$$g_1(x,y) = \frac{x^3y}{8},\ g_2(x,y) = \frac{x^4y^2}{12},\ g_3(x,y) = \frac{1}{4}.$$

We have $B = \frac{1}{4}$ and

$$|f(x,y,u)| \le e^{x+y};\ \text{for all } (x,y) \in J \text{ and } u \in \mathbb{R}.$$

Then condition (H_1) is satisfied and $mB = \frac{3}{4} < 1$. In view of Theorem 1, problem (5)-(6) has a solution defined on $[-2,1]\times[-3,1]$.

EXAMPLE 2. Consider the fractional integral equation

$$u(x,y) = \frac{x^3y}{8}u(x-1,y-\frac{1}{2}) + \frac{x^4y^2}{12}u(x-\frac{2}{5},y-\frac{3}{4}) + \frac{1}{8}u(x-3,y-2)$$
$$+I_\theta^r f(x,y,u);\ \text{if } (x,y) \in J := [0,1]\times[0,1], \tag{7}$$
$$u(x,y) = \Phi(x,y);\ \text{if } (x,y) \in \tilde{J} := [-3,1]\times[-2,1]\backslash(0,1]\times(0,1], \tag{8}$$

where $m = 3,\ r = (\frac{1}{2},\frac{1}{5}),\ f(x,y,u) = \frac{x+y}{20}\frac{|u|}{1+|u|}$ and $\Phi : \tilde{J} \to \mathbb{R}$ is continuous with

$$\Phi(x,0) = \frac{1}{8}\Phi(x-3,-2),\ \Phi(0,y) = \frac{1}{8}\Phi(-3,y-2);\ x,y \in [0,1]. \tag{9}$$

Notice that condition (9) is satisfied by $\Phi \equiv 0$.

Set

$$g_1(x,y) = \frac{x^3y}{8},\ g_2(x,y) = \frac{x^4y^2}{12},\ g_3(x,y) = \frac{1}{8}.$$

We have $B = \frac{1}{8}$. It is clear that f satisfies (H_2) with $l^* = \frac{1}{10}$. A simple computation shows that condition (4) is satisfied. Hence by Theorem 2, problem (7)-(8) has a unique solution defined on $[-3,1]\times[-2,1]$.

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