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A Characterization Of A Family Of Semiclassical Orthogonal Polynomials Of Class One*

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Abstract

In this paper, we give another characterization of a non-symmetric semiclassical orthogonal polynomials of class one.

1 Introduction

Our goal is to characterize the set of non-symmetric semiclassical orthogonal polynomials of class one $\{W_n\}_{n\geq 0}$ verifying the three-term recurrence relation with $\beta_n = (-1)^n$, $n \geq 0$ in a concise way as in [5, 6] via the study of the functional equation $(\Phi w)' + \Psi w = 0$ satisfied by its corresponding regular form w. Some information about the shape of polynomials Φ and Ψ intervening in the above functional equation are given due to the quadratic decomposition of $\{W_n\}_{n\geq 0}$ and to a connection between w and a suitable symmetric regular form ϑ. As application, we characterize w by giving the functional equation, the recurrence coefficient γ_{n+1}, $n \geq 0$ and an integral representation.

We denote by $\mathcal{P}$ the vector space of polynomials with coefficients in $\mathbb{C}$ and by $\mathcal{P}'$ its dual space. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For instance, for any form u, any polynomial g and any $(a, b, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^2$, we let $Du = u'$, σu, gu, $h_a u$, $\tau_b u$, $(x - c)^{-1}u$ and δ_c, be the forms defined in [3]:

$$\langle u', f\rangle := -\langle u, f'\rangle,\ \langle \sigma u, f\rangle := \langle u, \sigma f\rangle,\ \langle gu, f\rangle := \langle u, gf\rangle,\ \langle h_a u, f\rangle := \langle u, h_a f\rangle,$$

$$\langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle, \quad \langle (x - c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle, \quad \langle \delta_c, f\rangle := f(c),$$

where $(\sigma f)(x) = f(x^2)$, $(h_a f)(x) = f(ax)$, $(\tau_{-b} f)(x) = f(x+b)$, $(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}$ for all $f \in \mathcal{P}$. It is easy to see that [3, 4]

$$(fu)' = fu' + f'u,\ f \in \mathcal{P},\ u \in \mathcal{P}', \tag{1}$$

$$f(x)\sigma u = \sigma(f(x^2)u),\ f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2}$$

$$\sigma(u') = 2(\sigma(xu))',\ u \in \mathcal{P}', \tag{3}$$

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$$x^{-1}(xu) = u - (u)_0\delta_0,\ x(x^{-1}u) = u,\ u \in \mathcal{P}'. \tag{4}$$

A form w is said to be regular whenever there is a sequence of monic polynomials $\{W_n\}_{n\geq 0}$, $\deg W_n = n$, $n \geq 0$ (MPS) such that $\langle w, W_n W_m\rangle = k_n\delta_{n,m}$, $n, m \geq 0$ with $k_n \neq 0$ for any $n \geq 0$. In this case, $\{W_n\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence (MOPS) and it is characterized by the following three-term recurrence relation [1]

$$\begin{aligned} &W_0(x) = 1, \quad W_1(x) = x - \beta_0, \\ &W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \quad n \geq 0, \end{aligned} \tag{5}$$

where $\beta_n = \frac{\langle w, xW_n^2\rangle}{\langle w, W_n^2\rangle} \in \mathbb{C}$ and $\gamma_{n+1} = \frac{\langle w, W_{n+1}^2\rangle}{\langle w, W_n^2\rangle} \in \mathbb{C} \setminus \{0\},\ n \geq 0$.

When w is regular, $\{W_n\}_{n\geq 0}$ is a symmetric (MOPS) if and only if $\beta_n = 0,\ n \geq 0$ or equivalently $(w)_{2n+1} = 0,\ n \geq 0$. Also, The form w is said to be normalized if $(w)_0 = 1$. In this paper, we suppose that any form will be normalized.

A form w is called semiclassical when it is regular and there exist two polynomials Φ (monic) and Ψ, $\deg\Phi = t \geq 0$, $\deg\Psi = p \geq 1$ such that

$$(\Phi w)' + \Psi w = 0. \tag{6}$$

It's corresponding orthogonal polynomial sequence $\{W_n\}_{n\geq 0}$ is called semiclassical. The semiclassical character is kept by shifting [3, 4, 5]. In fact, let $\{a^{-n}W_n(ax+b)\}_{n\geq 0}$, $a \neq 0$, $b \in \mathbb{C}$; when w satisfies (6), then $(h_{a^{-1}} \circ \tau_{-b})w$ fulfills

$$\big(a^{-t}\Phi(ax+b)(h_{a^{-1}} \circ \tau_{-b})w\big)' + a^{1-t}\Psi(ax+b)(h_{a^{-1}} \circ \tau_{-b})w = 0, \tag{7}$$

and the recurrence coefficients of (5) are

$$\frac{\beta_n - b}{a}\ ,\ \frac{\gamma_{n+1}}{a^2},\ n \geq 0. \tag{8}$$

The semiclassical form w is said to be of class $s = \max(p-1, t-2) \geq 0$ if and only if [3, 4, 5]

$$\prod_{c \in \mathcal{Z}_\Phi} \big\{ (\Psi(c) + \Phi'(c)) + \big(\langle w, (\theta_c\Psi) + (\theta_c^2\Phi)\rangle\big) \big\} > 0, \tag{9}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ. In particular, when $s = 0$ the form w is usually called *classical* Hermite, Laguerre, Bessel and Jacobi, see [3, 4, 5].

LEMMA 1 ([3]). Let w be a symmetric semiclassical form of class s satisfying (6). The following statements hold.
i) When s is odd then the polynomial Φ is odd and Ψ is even.
ii) When s is even then the polynomial Φ is even and Ψ is odd.

Let $\{W_n\}_{n\geq 0}$ be a (MOPS) with respect to the form w fulfilling the three-term recurrence relation (5) with

$$\beta_n = (-1)^n\ ,\ n \geq 0. \tag{10}$$

Such a (MOPS) is characterized by the following quadratic decomposition [4]

$$W_{2n}(x) = P_n(x^2) \quad , \quad W_{2n+1}(x) = (x-1)P_n^*(x^2), \quad n \geq 0, \tag{11}$$

where $\{P_n\}_{n\geq 0}$ is a (MOPS) and $\{P_n^*\}_{n\geq 0}$ is the sequence of monic Kernel polynomials of **K**-parameter 1 associated with $\{P_n\}_{n\geq 0}$ defined by [1, 2]

$$P_n^*(x) = \frac{1}{x-1}\left[P_{n+1}(x) - \frac{P_{n+1}(1)}{P_n(1)}\,P_n(x)\right], \quad n \geq 0. \tag{12}$$

Furthermore the sequences $\{P_n\}_{n\geq 0}$ and $\{P_n^*\}_{n\geq 0}$ satisfy respectively the recurrence relation (5) with

$$\begin{cases} \beta_0^P = \gamma_1 + 1, \\ \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3} + 1, \\ \gamma_{n+1}^P = \gamma_{2n+1}\,\gamma_{2n+2}, \end{cases} \begin{cases} \beta_0^* = \gamma_1 + \gamma_2 + 1, \\ \beta_{n+1}^* = \gamma_{2n+3} + \gamma_{2n+4} + 1, \\ \gamma_{n+1}^* = \gamma_{2n+2}\,\gamma_{2n+3}. \end{cases} \tag{13}$$

for all $n \geq 0$. Denoting by u and v the forms associated with $\{P_n\}_{n\geq 0}$ and $\{P_n^*\}_{n\geq 0}$ respectively, we get [4]

$$u = \sigma w = \sigma(xw), \tag{14}$$

$$v = \gamma_1^{-1}(x-1)\sigma w. \tag{15}$$

The regularity of v means that [1]

$$P_{n+1}(1) \neq 0, \quad n \geq 0. \tag{16}$$

Moreover, the form $(x-1)w$ is antisymmetric, that is,

$$((x-1)w)_{2n} = 0\,, \quad n \geq 0. \tag{17}$$

Let now λ be a non-zero complex number and ϑ be the form such that

$$\lambda x\vartheta = (x-1)w. \tag{18}$$

According to (17)-(18) we get $(x\vartheta)_{2n} = 0\,, n \geq 0$. Hence ϑ is a symmetric form. Multiplying (18) by x, applying the operator σ and using (15) we get $\lambda x\,\sigma\vartheta = \gamma_1 v$. Consequently, according to [3], the form ϑ is regular if and only if

$$\Omega_n(\lambda) = \gamma_1 P_{n-1}^{*(1)}(0) + \lambda P_n^*(0) \neq 0, \quad n \geq 0, \tag{19}$$

with $P_n^{*(1)}(x) = (v\theta_0 P_{n+1}^*)(x)\,, n \geq 0$ and $P_{-1}^{*(1)}(x) := 0$.

LEMMA 2. There exists a non zero constant λ such that the form ϑ given by (18) is regular.

PROOF. According to the following relation [2]

$$P_{n+1}^{*(1)}(x)P_{n+1}^*(x) - P_{n+2}^*(x)P_n^{*(1)}(x) = \prod_{\nu=0}^{n} \gamma_{\nu+1}^* \neq 0, \quad n \geq 0,$$

it is easy to see that

$$|P^{*(1)}_{n-1}(0)| + |P^*_n(0)| \neq 0, \quad \forall n \geq 0. \tag{20}$$

Let n be a fixed nonnegative integer. If $P^{*(1)}_{n-1}(0) = 0$, then $P^*_n(0) \neq 0$ from (20). So, condition (19) is satisfied for $\lambda \neq 0$. If $P^*_n(0) = 0$, then $P^{*(1)}_{n-1}(0) \neq 0$ from (20). So, condition (19) satisfied for $\lambda \neq 0$. If $P^{*(1)}_{n-1}(0) \neq 0$ and $P^*_n(0) \neq 0$, then for all $\lambda \neq \lambda_n$, (20) is satisfied, where we have posed

$$\lambda_n = -\gamma_1 \frac{P^{*(1)}_{n-1}(0)}{P^*_n}, \quad n \geq 0. \tag{21}$$

In any case there exists a constant $\lambda \neq 0$ such that (19) is fulfilled and so ϑ is a regular form.

In what follows we assume that the (MOPS) $\{W_n\}_{n\geq 0}$ associated with (5),(10) is semiclassical of class s_w. Its corresponding regular form w is then semiclassical of class s_w satisfying the functional equation (6). Multiplying the equation (6) by $(x-1)^2$ and on account of (1) and (18), we deduce that the form ϑ, when it is regular, is also semiclassical of class s_ϑ at most $s_w + 2$ satisfying the functional equation

$$(\mathrm{E}\vartheta)' + \mathrm{F}\vartheta = 0, \tag{22}$$

with

$$\mathrm{E}(x) = x(x-1)\Phi(x); \quad \mathrm{F}(x) = x\left((x-1)\Psi(x) - 2\Phi(x)\right). \tag{23}$$

The next technical lemma is needed in the sequel.

LEMMA 3. For all root c of Φ, we have

$$\begin{aligned} &a)\ \left\langle \vartheta, \theta_c^2\mathrm{E} + \theta_c\mathrm{F}\right\rangle = \tfrac{1}{\lambda}(c-1)^2\left\langle w, \theta_c\Psi + \theta_c^2\Phi\right\rangle + (1-\tfrac{1}{\lambda})(c-1)\left(\Phi'(c) + \Psi(c)\right), \\ &b)\ \mathrm{E}'(c) + \mathrm{F}(c) = c(c-1)\left(\Phi'(c) + \Psi(c)\right). \end{aligned} \tag{24}$$

PROOF. Let c be a root of Φ. Write $\Phi(x) = (x-c)\Phi_c(x)$ with $\Phi_c(x) = (\theta_c\Phi)(x)$. From (22)-(23) we have

$$\left(\theta_c^2\mathrm{E} + \theta_c\mathrm{F}\right)(x) = \theta_c\left\{\xi(\xi-1)\left(\Phi_c(\xi) + \Psi(\xi)\right)\right\}(x) - 2x\Phi_c(x). \tag{25}$$

Taking $g(x) = (\Phi_c + \Psi)(x)$ and $f(x) = x(x-1)$ in the following relation

$$\theta_c(fg)(x) = g(x)(\theta_c f)(x) + f(c)(\theta_c g)(x), \quad \text{for all } f, g \in \mathcal{P}, \tag{26}$$

(25) becomes

$$\left(\theta_c^2\mathrm{E} + \theta_c\mathrm{F}\right)(x) = (c-1)\left\{(\Phi_c + \Psi)(x) + c\left(\theta_c(\Phi_c + \Psi)\right)(x)\right\} + x(\Psi - \Phi_c)(x). \tag{27}$$

From the second identity in (4), relation (18) is equivalent to

$$\vartheta = \frac{1}{\lambda}(w - x^{-1}w) + (1 - \frac{1}{\lambda})\delta_0. \tag{28}$$

We may also write

$$\left\langle \frac{1}{\lambda}(w - x^{-1}w), \theta_c^2\mathrm{E} + \theta_c\mathrm{F} \right\rangle = \frac{1}{\lambda}\left\langle w, \theta_c^2\mathrm{E} + \theta_c\mathrm{F} - \theta_0(\theta_c^2\mathrm{E} + \theta_c\mathrm{F})\right\rangle. \tag{29}$$

Taking $f(x) = (\theta_c(\Phi_c + \Psi))(x)$ in the following

$$c\theta_0(\theta_c f) = \theta_c f - \theta_0 f, \quad f \in \mathcal{P}, \quad c \in \mathbb{C}, \tag{30}$$

and applying the operator θ_0 to (27), we obtain

$$\left(\theta_0(\theta_c^2\mathrm{E} + \theta_c\mathrm{F})\right)(x) = (\Psi - \Phi_c)(x) + (c-1)\left(\theta_c(\Phi_c + \Psi)\right)(x). \tag{31}$$

This gives

$$(\theta_c^2\mathrm{E}+\theta_c\mathrm{F})(x) - \left(\theta_0(\theta_c^2\mathrm{E} + \theta_c\mathrm{F})\right)(x) = (c-1)^2\left(\theta_c(\Phi_c + \Psi)\right)(x) + (x+c-2)\Psi - \Phi. \tag{32}$$

Thus (29) becomes

$$\left\langle \frac{1}{\lambda}(w - x^{-1}w), \theta_c^2\mathrm{E} + \theta_c\mathrm{F} \right\rangle = \frac{1}{\lambda}(c-1)^2\left\langle w, \theta_c\Phi_c + \theta_c\Psi\right\rangle, \tag{33}$$

since $\langle w, \Psi\rangle = 0$ and $\langle w, x\Psi(x) - \Phi(x)\rangle = 0$ from (6). Next, by a simple calculation, we have

$$\left\langle (1 - \frac{1}{\lambda})\delta_0, \theta_c^2\mathrm{E} + \theta_c\mathrm{F} \right\rangle = (1 - \frac{1}{\lambda})(c-1)(\Phi_c + \Psi)(c). \tag{34}$$

Adding (33) and (34) we obtain the first relation in (24). From (22)-(23), we have $\mathrm{E}'(c) = c(c-1)\Phi'(c)$ and $\mathrm{F}(c) = c(c-1)\Psi(c)$, hence the second relation in (24) holds.

Let us recall the following result about the class s_ϑ of the form ϑ.

THEOREM 1. The form ϑ is semiclassical and its class depends only on the zero $x = 1$ for any $\lambda \neq \lambda_n$, $n \geq -1$ where λ_n, $n \geq 0$ is given by (21) and

$$\lambda_{-1} = \frac{\langle w, \theta_0\Psi + \theta_0^2\Phi\rangle + \Phi'(0) + \Psi(0)}{\Phi'(0) + \Psi(0)}. \tag{35}$$

Moreover, the semiclassical form ϑ is of class s_ϑ satisfying the functional equation

$$\left(\widetilde{\mathrm{E}}\vartheta\right)' + \widetilde{\mathrm{F}}\vartheta = 0, \tag{36}$$

such that

a) if $\Phi(1) \neq 0$, then $s_\vartheta = s_w + 2$,

$$\widetilde{\mathrm{E}}(x) = x(x-1)\Phi(x) \quad \text{and} \quad \widetilde{\mathrm{F}}(x) = x\left((x-1)\Psi(x) - 2\Phi(x)\right);$$

b) if $\Phi(1) = 0$ and $\Psi(1) \neq 0$, then $s_\vartheta = s_w + 1$,

$$\widetilde{\mathrm{E}}(x) = x\Phi(x) \quad \text{and} \quad \widetilde{\mathrm{F}}(x) = x\left(\Psi(x) - (\theta_1\Phi)(x)\right);$$

c) if $\Phi(1)=0$ and $\Psi(1)=0$, then $s_\vartheta = s_w$,

$$\widetilde{\mathrm{E}}(x) = x(\theta_1\Phi)(x) \quad \text{and} \quad \widetilde{\mathrm{F}}(x) = x(\theta_1\Psi)(x).$$

PROOF. By our assumption, on account of Lemma 2, and by (22)-(23), the form ϑ is regular and so is semiclassical of class $s_\vartheta \le s_w + 2$. Let c be a root of E such that $c \neq 1$. According to (23) we get $c\Phi(c) = 0$. If $c \neq 0$, then c is a root of Φ. We suppose $E'(c) + F(c) = 0$. From (24) we obtain $\Phi'(c) + \Psi(c) = 0$ and $\left\langle \vartheta, \theta_c^2 \mathrm{E} + \theta_c F \right\rangle = \frac{1}{\lambda}(c-1)^2 \left\langle w, \theta_c\Psi + \theta_c^2\Phi \right\rangle \neq 0$, because w is semiclassical and so satisfies (9). If $c = 0$ and $\Phi(0) \neq 0$, then $\mathrm{E}'(0) + \mathrm{F}(0) = -\Phi(0) \neq 0$ from (23). If $c = 0$ and $\Phi(0) = 0$, then $\mathrm{E}'(0) + \mathrm{F}(0) = 0$. We are led to the following: When $\Phi'(0) + \Psi(0) = 0$, we get $\left\langle \vartheta, \theta_0^2 \mathrm{E} + \theta_0 F \right\rangle = \frac{1}{\lambda} \left\langle w, \theta_0\Psi + \theta_0^2\Phi \right\rangle \neq 0$ from (24a). When $\Phi'(0) + \Psi(0) \neq 0$ and because $\lambda \neq \lambda_{-1}$, then according to (24a) with $c = 0$, we obtain $\left\langle \vartheta, \theta_0^2 E + \theta_0 F \right\rangle \neq 0$. Therefore equation (6) is not simplified by $x - c$ for $c \neq 1$. Next, from (23) we have $\mathrm{E}'(1) + \mathrm{F}(1) = -\Phi(1)$.

a) If $\Phi(1) \neq 0$, then $\mathrm{E}'(1) + \mathrm{F}(1) \neq 0$ and the equation (22) cannot be simplified. This means that

$$s_\vartheta = \max(\deg \mathrm{E} - 2, \deg \mathrm{F} - 1) = \max(\deg \Phi - 2, \deg \Psi - 1) = s_w + 2.$$

b) If $\Phi(1) = 0$, then $\mathrm{E}'(1) + \mathrm{F}(1) = 0$ and $\left\langle \vartheta, \theta_1^2 \mathrm{E} + \theta_1 \mathrm{F} \right\rangle = 0$ from (24). Therefore (22) can be simplified by $x - 1$. After simplification, it becomes $\left(\widetilde{\mathrm{E}}\vartheta\right)' + \widetilde{\mathrm{F}}\vartheta = 0$, with $\widetilde{\mathrm{E}}(x) = x\Phi(x)$ and $\widetilde{\mathrm{F}}(x) = x\left(\Psi(x) - (\theta_1\Phi)(x)\right)$. We have $\widetilde{\mathrm{E}}'(1) + \widetilde{\mathrm{F}}(1) = \Psi(1)$. When $\Psi(1) \neq 0$, the above functional equation is not simplified. Consequently, $s_\vartheta = \max(\deg \widetilde{\mathrm{E}} - 2, \deg \widetilde{\mathrm{F}} - 1) = s_w + 1$.

c) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then $\widetilde{\mathrm{E}}'(1) + \widetilde{\mathrm{F}}(1) = \Psi(1) = 0$. By virtue of (18) and (6) we get $\left\langle \vartheta, \theta_1^2 \widetilde{\mathrm{E}} + \theta_1 \widetilde{\mathrm{F}} \right\rangle = \frac{1}{\lambda}\langle w, \Psi \rangle = 0$. Therefore (34) is simplified by $x - 1$, and ϑ fulfils $\left(\widehat{\mathrm{E}}\vartheta\right)' + \widehat{\mathrm{F}}\vartheta = 0$, where $\widehat{\mathrm{E}}(x) = x(\theta_1\Phi)(x)$ and $\widehat{\mathrm{F}}(x) = x(\theta_1\Psi)(x)$. If 1 is a root of $\theta_1\Phi$, then $\Phi'(1) + \Psi(1) = 0$. Assuming that $\widehat{\mathrm{E}}'(1) + \widehat{\mathrm{F}}(1) = 0$, a simple calculation gives $\left\langle \vartheta, \theta_1^2 \widehat{\mathrm{E}} + \theta_1 \widehat{\mathrm{F}} \right\rangle = \frac{1}{\lambda} \left\langle w, \theta_1\Psi + \theta_1^2\Phi \right\rangle \neq 0$ since w is a semiclassical of class 1 satisfying (9). Hence the functional equation $\left(\widehat{\mathrm{E}}\vartheta\right)' + \widehat{\mathrm{F}}\vartheta = 0$ is not simplified and $s_\vartheta = \max(\deg \widehat{\mathrm{E}} - 2, \deg \widehat{\mathrm{F}} - 1) = s_w$.

2 Main Results

In the sequel we deal with the semiclassical sequence $\{W_n\}_{n \ge 0}$ of class one satisfying (10). Its corresponding regular form w is then semiclassical of class $s_w = 1$ fulfilling the functional equation (6) with $0 \le \deg \Phi \le 3$ and $1 \le \deg \Psi \le 2$.

2.1 Characterization of the Polynomials Φ and Ψ

We can usually decompose the polynomials Φ and Ψ through their odd and even parts. Set

$$\begin{aligned}\Phi(x) = \phi(x^2) + x\varphi(x^2), \quad \Psi(x) = \psi(x^2) + x\omega(x^2),\\ (\theta_1\Phi)(x) = \phi_1(x^2) + x\varphi_1(x^2) \quad \text{and} \quad (\theta_1\Psi)(x) = \psi_1(x^2) + x\omega_1(x^2).\end{aligned} \tag{37}$$

PROPOSITION 1. Let w be a semiclassical form of class one satisfying (6) and $\{W_n\}_{n\geq 0}$ be its corresponding MOPS fulfilling (10).

a) If $\Phi(1) \neq 0$, then $\phi(x) = \varphi(x) = \frac{1}{2}\left(x\omega(x) - \psi(x)\right)$.

b) If $\Phi(1) = 0$ and $\Psi(1) \neq 0$, then $\phi(x) = 0$ and $\varphi_1(x) = \omega(x)$.

c) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then $\phi(x) + \varphi(x) = 0$ and $\psi(x) + x\omega(x) = 0$.

PROOF. Set

$$\widetilde{\mathrm{E}}(x) = \widetilde{\mathrm{E}}^e(x^2) + x\widetilde{\mathrm{E}}^o(x^2); \quad \widetilde{\mathrm{F}}(x) = \widetilde{\mathrm{F}}^e(x^2) + x\widetilde{\mathrm{F}}^o(x^2). \tag{38}$$

a) $\Phi(1) \neq 0$. According to (37)-(38) and from Theorem 1., we obtain $\widetilde{\mathrm{E}}^e(x) = x(\phi - \varphi)(x)$, $\widetilde{\mathrm{E}}^o(x) = x\varphi(x) - \phi(x)$, $\widetilde{\mathrm{F}}^e(x) = x(\psi - \omega - 2\varphi)(x)$, $\widetilde{\mathrm{F}}^o(x) = x\omega(x) - \psi(x) - 2\phi(x)$. On account of Lemma 1. and the fact that ϑ is of odd class, we get $\widetilde{\mathrm{E}}^e = \widetilde{\mathrm{F}}^o = 0$. This leads to the result a).

b) $\Phi(1) = 0$ and $\Psi(1) \neq 0$. Similar to a), we have $\widetilde{\mathrm{E}}^e(x) = x\varphi(x)$, $\widetilde{\mathrm{E}}^o(x) = \phi(x)$, $\widetilde{\mathrm{F}}^e(x) = x(\omega - \varphi_1)(x)$ and $\widetilde{\mathrm{F}}^o(x) = (\psi - \phi_1)(x)$. The form ϑ is of odd class, then $\widetilde{\mathrm{E}}^e = \widetilde{\mathrm{F}}^o = 0$. Hence the conclusion.

c) $\Phi(1) = 0$ and $\Psi(1) = 0$. In this case we have $\widetilde{\mathrm{E}}^e(x) = x\varphi_1(x)$, $\widetilde{\mathrm{E}}^o(x) = \phi_1(x)$, $\widetilde{\mathrm{F}}^e(x) = x\omega_1(x)$, $\widetilde{\mathrm{F}}^o(x) = \psi_1(x)$. Since ϑ is of odd class, $\widetilde{\mathrm{E}}^e = \widetilde{\mathrm{F}}^o = 0$. Therefore $\varphi_1 = 0$ and $\psi_1 = 0$. Moreover we can write $\Phi(x) = (x-1)(\theta_1\Phi)(x) = (x-1)\phi_1(x^2)$ and $\Psi(x) = (x-1)x\omega_1(x^2)$. So $\phi = -\phi_1$, $\varphi = -\phi_1$, $\omega = -\omega_1$ and $\psi = x\omega_1$. This gives the desired result.

THEOREM 2. Let w be a semiclassical form of class one satisfying (6) and $\{W_n\}_{n\geq 0}$ be its corresponding (MOPS) fulfilling (10). The functional equation (6) has only one solution given by

$$\Phi(x) = x^3 - x, \quad \Psi(x) = ax^2 + x + c, \quad a \neq 0, \quad (w)_0 = (w)_1 = 1, \tag{39}$$

with

$$a + c + 1 \neq 0; \quad |a+2| + |a+c+3| \neq 0 \quad \text{and} \quad |a+2| + |c-3| \neq 0. \tag{40}$$

PROOF. When $\deg\Phi \leq 2$ and $\deg\Psi = 2$, we consider $a \neq 0$, b and c as three complex numbers such that $\Psi(x) = ax^2 + bx + c$. From Proposition 1, we have the following.

i) If $\Phi(1) \neq 0$, then $\phi(x) = \varphi(x)$, and so $\Phi(x) = (x+1)\phi(x^2)$ from (37). Because Φ is a monic polynomial of degree at most two, then necessarily $\phi(x) = 1$. In addition, we have $x\omega(x) - \psi(x) = 2$. This implies that $a = b$ and $c = -2$. Thus $\Phi(x) = x + 1$ and $\Psi(x) = ax^2 + ax - 2$, $a \neq 0$. According to equation (6), we have $\langle w, \Psi(x)\rangle =$

$\langle w, x\Psi(x) - \Phi(x)\rangle = 0$. Then $\langle w, ax^2 + ax - 2\rangle = \langle w, ax^3 + ax^2 - 3x - 1\rangle = 0$. It is equivalent to

$$a(\gamma_1 + 2) - 2 = 0 \quad \text{and} \quad a(\gamma_1 + 1) - 2 = 0, \tag{41}$$

since $\langle w, x\rangle = 1$ and $\langle w, x^3\rangle = \langle w, x^2\rangle = \gamma_1 + 1$. It is easy to see from (41) that $a = 0$, that is a contradiction with $\deg \Psi = 2$.

ii) If $\Phi(1) = 0$ and $\Psi(1) \neq 0$, then $\phi(x) = 0$. Therefore $\Phi(x) = x$, because Φ is monic and $\deg \Phi \leq 2$. This contradicts $\Phi(1) = 0$.

iii) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then $\Phi(x) = x - 1$ and $\Psi(x) = a(x^2 - x)$ with $a \neq 0$. Writing $\langle w, \Psi(x)\rangle = \langle w, a(x^2 - x)\rangle = 0$, then $a\gamma_1 = 0$ and so $\gamma_1 = 0$. It is a contradiction, by virtue of the regularity of the form w.

When $\deg \Phi = 3$, we obtain $\deg \phi \leq 1$ and $\deg \varphi = 1$ from (37). According to Proposition 1, we have the following.

i) If $\Phi(1) \neq 0$, then $\phi(x) = \varphi(x)$ and $\psi(x) = -2\phi(x) + x\omega(x)$. We obtain $\Phi(x) = (x+1)\varphi(x^2)$ and $\Psi(x) = (x^2+x)\omega(x^2) - 2\varphi(x^2)$. Therefore ω is a constant polynomial and φ is a monic polynomial of degree one since $\deg \Psi \leq 2$ and $\deg \Phi = 3$. Denoting by $\varphi(x) = x + d$ and $\omega(x) = e$. We write $\Phi(x) = (x+1)(x^2+d)$ and $\Psi(x) = (e-2)x^2 + ex - 2d$. As above, we have $\langle w, \Psi\rangle = \langle w, x\Psi(x) - \Phi(x)\rangle = 0$. It follows $(e-2)(\gamma_1 + 1) + e - 2d = 0$ and $(e-2)(\gamma_1 + 1) - 2d = 0$. Hence $e = 0$ and $\gamma_1 + d + 1 = 0$. Again, according to equation (6), we have $\langle (\Phi(x)w)' + \Psi(x)w, x^2\rangle = 0$, then $\langle w, x^2(x^2 + d)\rangle = 0$. Since $x^2 = W_2(x) + \gamma_1 + 1$, we then obtain $\langle w, (W_2(x) + \gamma_1 + 1)W_2(x)\rangle = 0$. This gives $\langle w, W_2^2(x)\rangle = 0$. It is a contradiction with the orthogonality of $\{W_n\}_{n\geq 0}$.

ii) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then $\phi(x) = -\varphi(x)$ and $\psi(x) = -x\omega(x)$. Therefore $\Psi(x) = (x - x^2)\psi(x^2)$, and on account of $1 \leq \deg \Psi \leq 2$, $\deg \psi = 0$. Denoting by $\psi(x) = a_1$, where $a_1 \in \mathbb{C} \setminus \{0\}$, since $\langle w, \Psi\rangle = \langle w, a_1(x - x^2)\rangle = 0$, we have $a_1\gamma_1 = 0$. It is a contradiction.

iii) If $\Phi(1) = 0$ and $\Psi(1) \neq 0$, then $\phi(x) = 0$ and $\omega(x) = \varphi_1(x)$. So $\Phi(x) = x(x^2-1)$ and $\Psi(x) = ax^2 + x + c$. If $a = 0$, then $c + 1 = 0$, since $\langle w, \Psi\rangle = 0$. Thus $\Psi(x) = x - 1$ which contradicts $\Psi(1) \neq 0$. Necessarily $a \neq 0$. Moreover the form w is of class one, we shall have the condition (9) with $\mathcal{Z}_\Phi = \{-1, 0, 1\}$, which leads to relation (40).

2.2 The Computation of γ_{n+1}

We will study the form w given in Theorem 2. Denoting by $\alpha = \frac{1}{2}(c-1)$ and $\beta = -\frac{1}{2}(a + c + 3)$. The form w fulfills the following equation

$$\begin{array}{l} \left(x(x^2-1)w\right)' + \left(-2(\alpha+\beta+2)x^2 + x + 2\alpha + 1\right) w = 0, \\ (w)_0 = (w)_1 = 1, \end{array} \tag{42}$$

where

$$|\alpha + \beta + 1| + |\alpha| \neq 0, \quad \beta + 1 \neq 0, \quad |\alpha + \beta + 1| + |\beta| \neq 0, \quad \alpha + \beta + 2 \neq 0. \tag{43}$$

Applying the operator σ in (42) and on account of (2) and (3), we get

$$\left((x^2 - x)u\right)' + \left(-(\alpha + \beta + 2)x + \alpha + 1\right) u = 0, \quad (u)_0 = 1. \tag{44}$$

Multiplying (44) by $x-1$, we obtain the functional equation satisfied by the form v

$$\big((x^2-x)v\big)' + \left(-(\alpha+\beta+3)x+\alpha+2\right)v, \quad (v)_0 = 1. \tag{45}$$

Therefore the forms u and v are classical. Moreover from a suitable shifting, we obtain

$$u = \left(\tau_{\frac{1}{2}} \circ h_{\frac{1}{2}}\right)\mathcal{J}(\alpha,\beta)\,; \quad v = \left(\tau_{\frac{1}{2}} \circ h_{\frac{1}{2}}\right)\mathcal{J}(\alpha,\beta+1). \tag{46}$$

Where $\mathcal{J}(\alpha,\beta)$ is the Jacobi form of parameters α and β satisfying the following functional equation

$$\big((x^2-1)\mathcal{J}(\alpha,\beta)\big)' + \left(-(\alpha+\beta+2)x+\alpha-\beta\right)\mathcal{J}(\alpha,\beta) = 0, \quad \left(\mathcal{J}(\alpha,\beta)\right)_0 = 1.$$

It is regular if and only if $\alpha \neq -n, \quad \beta \neq -n, \quad \alpha+\beta \neq -n$, $n \geq 1$. Moreover, the coefficients of its corresponding orthogonal polynomials $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ are given by [1]

$$\begin{aligned} \beta_n^{(\alpha,\beta)} &= \tfrac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \quad n \geq 0, \\ \gamma_{n+1}^{(\alpha,\beta)} &= 4\tfrac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)} \quad , n \geq 0. \end{aligned} \tag{47}$$

PROPOSITION 2. Let w be the form of class one satisfying (42). The coefficients of its corresponding (MOPS) $\{W_n\}_{n\geq 0}$ are given by

$$\begin{aligned} \gamma_{2n+1} &= -\tfrac{(n+\alpha+\beta+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 0, \\ \gamma_{2n+2} &= -\tfrac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}, \quad n \geq 0. \end{aligned} \tag{48}$$

PROOF. Let $\{P_n\}_{n\geq 0}$ be a (MOPS) with respect to the regular form u and $\{P_n^*\}_{n\geq 0}$ be the (MOPS) with respect to the regular form v. From (46), we have

$$P_n(x) = 2^{-n}P_n^{(\alpha,\beta)}(2x-1), \quad P_n^*(x) = 2^{-n}P_n^{(\alpha,\beta+1)}(2x-1), \quad n \geq 0. \tag{49}$$

By comparing with (13), (47) and using (8) we get

$$\begin{aligned} \gamma_{2n+1}\gamma_{2n+2} &= \tfrac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}, \quad n \geq 0, \\ \gamma_{2n+2}\gamma_{2n+3} &= \tfrac{(n+1)(n+\alpha+\beta+2)(n+\alpha+1)(n+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)^2(2n+\alpha+\beta+4)}, \quad n \geq 0. \end{aligned} \tag{50}$$

This gives

$$\frac{\gamma_{2n+3}}{\gamma_{2n+1}} = \frac{(n+\alpha+\beta+2)(n+\beta+2)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{(n+\alpha+\beta+1)(n+\beta+1)(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)}, \quad n \geq 0.$$

By virtue of (50) and from a simple calculation we deduce (48).

REMARK 1. In particular, when $\alpha = 2^{-1}$ and $\beta = -2^{-1}$, we obtain the so-called second-order self-associated orthogonal sequence, see [4].

2.3 Integral Representation

Regarding the integral representation of the form w given by (42), we start with the representation of the form u. For $\Re(\alpha) > -1$ and $\Re(\beta) > -1$, we have for all $f \in \mathcal{P}$ [1]

$$\begin{aligned}\langle u, f\rangle &= \left\langle \mathcal{J}(\alpha,\beta), f\left(\frac{x+1}{2}\right)\right\rangle \\ &= \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}(1+x)^{\alpha}(1-x)^{\beta} f\left(\frac{x+1}{2}\right)dx.\end{aligned}$$

Using the substitution $t = \frac{x+1}{2}$, we get

$$\langle u, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{0}^{1} t^{\alpha}(1-t)^{\beta} f(t)dt\,,\; f \in \mathcal{P}. \tag{51}$$

Next, we decompose the polynomial f as follows: $f(x) = f_1(x^2) + (x-1)f_2(x^2)$. From the fact that $(x-1)w$ is antisymmetric, we obtain $\langle w, f\rangle = \langle u, f_1\rangle$. Using again the substitution $t = y^2$ in (51), we obtain

$$\langle w, f\rangle = 2\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{0}^{1} y^{2\alpha+1}(1-y^2)^{\beta} f_1(y^2)dy.$$

Since for $\Re(\alpha) > -\frac{1}{2}$ and $\Re(\beta) > -1$, $\int_{-1}^{1} y \mid y \mid^{2\alpha-1} (1-y^2)^{\beta} f_1(y^2)dy = 0$, the above representation may be written as follows

$$\langle w, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}(y^2+y) \mid y \mid^{2\alpha-1} (1-y^2)^{\beta} f_1(y^2)dy.$$

Moreover, we have

$$\int_{-1}^{1}(y^2+y) \mid y \mid^{2\alpha-1} (1-y^2)^{\beta}(y-1) f_2(y^2)dy = 0.$$

Consequently, we get an integral representation of the form w for all $f \in \mathcal{P}$, $\Re\alpha > -\frac{1}{2}$, $\Re\beta > -1$,

$$\langle w, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}(y^2+y) \mid y \mid^{2\alpha-1} (1-y^2)^{\beta} f(y)dy.$$

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