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Novel Results On Periodic Solutions Of A Class Of Liénard Type p-Laplacian Equation*

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Abstract

In this study, we investigate the following Liénard type p-Laplacian equation with a deviating argument

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + \beta(t)g(x(t-\tau(t))) = e(t).$$

Some new criteria for guaranteeing the existence and uniqueness of periodic solutions of this equation are given by using the Manásevich–Mawhin continuation theorem and some analysis techniques. Our results hold under weaker conditions than some known results from the literature, and are more effective. In the last section, an illustrative example is provided to demonstrate the applications of our results.

1 Introduction

In the present paper, we consider the following Liénard type p-Laplacian equation with a deviating argument:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + \beta(t)g(x(t-\tau(t))) = e(t), \tag{1}$$

where $p > 1$, $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(s) = |s|^{p-2}s$ is a one-dimensional p-Laplacian; $f, e \in C(\mathbb{R}, \mathbb{R}), \beta, \tau, g \in C^1(\mathbb{R}, \mathbb{R})$, $\beta(t)$, $\tau(t)$ are two T-periodic functions with $\int_0^T e(t)dt = 0$, $T > 0$.

As is well known, the Liénard equation can be derived from many fields, such as physics, mechanics and engineering technique fields, and an important question is whether this equation can support periodic solutions. In the past few years, a lot of researchers have contributed to the theory of this equation with respect to existence of periodic solutions. For example, in 1928, Liénard [8] discussed the existence of periodic solutions of the following equation

$$x''(t) + f(x(t))x'(t) + k(x(t))x(t) = 0, \tag{2}$$

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where $f, k \in C(\mathbb{R}, \mathbb{R})$, some sufficient conditions for securing the existence of periodic solutions were established. Afterward, Levinson and Smith [9] also studied (2) and obtained some new results on the existence of periodic solutions. In 1977, some continuation theorems in [4] were introduced by Gaines and Mawhin. Applying these continuation theorems, many authors discussed the existence of periodic solutions of (2) and generalized the results obtained in [9, 8] (see e.g. [1, 6, 7, 15]); a few authors studied the existence and uniqueness of periodic solutions of (2) (see [10, 17]). In 1998, Manásevich and Mawhin [14] studied periodic solutions for certain nonlinear systems with p-Laplacian-Like operators and provided some new continuation theorems which extended some results in [4]. Subsequently, some authors discussed the existence of periodic solutions of certain Liénard type p-Laplacian equations (see e.g. [2, 3, 11, 12, 13]) using these generalized continuation theorems. However, as far as we know, there exist much fewer results on the existence and uniqueness of periodic solutions of (1). The main difficulty lies in the first term $(\varphi_p(x'(t)))'$ of (1) (i.e., the p-Laplacian operator $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(s) = |s|^{p-2}s$ is nonlinear when $p \neq 2$), the existence of which prevents the usual methods of finding some criteria for guaranteeing the uniqueness of periodic solutions of (2) from working. Recently, Gao and Lu [5] discussed the existence and uniqueness of periodic solution of (1) by translating (1) into a two-dimensional system and got some results as follows:

THEOREM 3.1 ([5]). Assume that the following condition holds:

(H_0) $\beta(t) > 0, g'(x) < 0$ and $\tau(t) \equiv \varepsilon$ (ε is a sufficiently small constant) for all $t, x \in R$.

Then (1) has at most one T-periodic solution.

REMARK 1. However, upon examining their proof of Theorem 3.1, it was found that if $\tau(t) \neq 0$ for $\forall\, t \in \mathbb{R}$, then Theorem 3.1 does not hold; more precisely, for arbitrarily given $\varepsilon > 0$, $v(t^*) = y_1(t^*) - y_2(t^*) > 0$ does not positively imply $v(t^* - \varepsilon) = y_1(t^* - \varepsilon) - y_2(t^* - \varepsilon) > 0$, thus in line 3 on page 377 in [5] the inequality $v''^*) > 0$ is incorrect. On the other hand, if $\tau(t) \equiv 0$, then Theorem 3.1 is correct.

THEOREM 3.2 ([5]). Assume that the following conditions hold:

(H_1) There exist $r_1 > 0, r_2 > 0, m > 0$ and $d \geq 0$ such that

(i) $r_1|u|^m \leq |g(u)| \leq r_2|u|^m$ for all $|u| > d$,

(ii) $ug(u) < 0$ for all $|u| > d$.

(H_2) $$A := \begin{cases} \left[\frac{r_2 T}{r_1 \int_0^T (\beta(t)+1)dt}\right]^{\frac{1}{m}} 2^{\frac{1-m}{m}} < 1, & 0 < m \leq 1, \\ \left[\frac{r_2 T}{r_1 \int_0^T (\beta(t)+1)dt}\right]^{\frac{1}{m}} < 1, & m > 1. \end{cases}$$

(H_3) Suppose one of the following conditions holds:

(i) $m = p - 1$ and $\beta_\infty r_2 T^{m+2-\frac{m+1}{p}}/2(1 - A)^{m+1} < 1$,

(ii) $m < p - 1$,

where $\beta_\infty = \max_{t\in[0,T]} |\beta(t)|$.

Then (1) has at least one T-periodic solution.

REMARK 2. However, upon examining their proof of Theorem 3.2 in [5], we have found that the conditions (H_1)(i), (H_2) and (H_3) can be dropped.

We now reconsider the periodic solutions of (1). The main purpose of this paper is to establish some new criteria for guaranteeing the existence and uniqueness of periodic solution of (1). We obtain some new sufficient conditions for securing the existence and uniqueness of periodic solutions of (1) by using the Manásevich–Mawhin continuation theorem and appropriate analysis techniques. Our results extend and improve the above-mentioned Theorems 3.1 and 3.2 in [5] (see Remarks 3 and 4 and Example 1).

2 Lemmas

For convenience, define

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, \quad |x|_k = \left(\int_0^T |x(t)|^k dt\right)^{1/k}.$$

Let

$$C_T^1 := \left\{x \in C^1(\mathbb{R},\mathbb{R}) : x \text{ is } T\text{-periodic}\right\},$$

which is a Banach space with the norm

$$\|x\| = \max\left\{|x|_\infty, |x'|_\infty\right\}.$$

The following conditions will be used later:

(A_0) $\beta(t) > 0, g'(x) < 0$ and $\tau(t) \equiv 0$ for all $t, x \in R$,

$(\mathrm{A'}_0)$ $\beta(t) < 0, g'(x) > 0$ and $\tau(t) \equiv 0$ for all $t, x \in R$,

(A_1) $\beta(t) > 0$ for all $t \in R$ and there exists $d \geq 0$ such that $ug(u) < 0$ for all $|u| \geq d$,

$(\mathrm{A'}_1)$ $\beta(t) < 0$ for all $t \in R$ and there exists $d \geq 0$ such that $ug(u) > 0$ for all $|u| \geq d$.

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = h(t,x,x'), x(0) = x(T), x'(0) = x'(T), \tag{1}$$

where $h \in C(\mathbb{R}^3,\mathbb{R})$ is T-periodic in the first variable, the following continuation theorem can be induced directly from the theory in [14], and is cited as Lemma 1 in [16].

LEMMA 1 (Manásevich–Mawhin [14]). Let $B = \{x \in C_T^1 : \|x\| < r\}$ for some $r > 0$. Suppose the following two conditions hold:

(i) For each $\lambda \in (0,1)$ the problem $(\varphi_p(x'(t)))' = \lambda h(t,x,x')$ has no solution on ∂B.

(ii) The continuous function F defined on R by $F(a) = \frac{1}{T}\int_0^T h(t,a,0)dt$ is such that $F(-r)F(r) < 0$.
Then the periodic boundary value problem (1) has at least one T-periodic solution on $\bar{B}$.

According to the Theorem 3.1 in [5] and the above-mentioned Remark 1, we have the following results.

LEMMA 2. Suppose (A_0) holds. Then (1) has at most one T-periodic solution.

LEMMA 3. Suppose (A'_0) holds. Then (1) has at most one T-periodic solution.

3 Main Results

Now we are in the position to present our main results.

THEOREM 1. Suppose (A_1) holds. Then (1) has at least one T-periodic solution.

PROOF. Consider the homotopic equation of (1):

$$(\varphi_p(x'(t)))' + \lambda f(x(t))x'(t) + \lambda\beta(t)g(x(t-\tau(t))) = \lambda e(t), \lambda \in (0,1). \tag{1}$$

First, we prove the set of T-periodic solutions of (1) are bounded in C_T^1. Let $S \subset C_T^1$ be the set of T-periodic solutions of (1). If $S = \emptyset$, the proof is ended. Suppose $S \neq \emptyset$, and let $x \in S$. Noticing that $x(0) = x(T)$, $x'(0) = x'(T)$, $\varphi_p(0) = 0$, and $\int_0^T e(t)dt = 0$, it follows from (1) that

$$\int_0^T \beta(t)g(x(t-\tau(t)))dt = 0,$$

which, together with $\beta(t) > 0$, implies that there exists $t_0 \in [0,T]$ such that

$$g(x(t_0 - \tau(t_0))) = 0. \tag{2}$$

Denote $\bar{t}_0 = t_0 - \tau(t_0)$, by ($A_1$), (2) implies

$$|x(\bar{t}_0)| < d. \tag{3}$$

Then, for any $t \in [\bar{t}_0, \bar{t}_0 + T]$,

$$|x(t)| = \left|x(\bar{t}_0) + \int_{\bar{t}_0}^t x'(s)ds\right| < d + \int_{\bar{t}_0}^{\bar{t}_0+T} |x'(s)|ds = d + \int_0^T |x'(s)|ds,$$

which leads to

$$|x|_\infty = \max_{t\in[\bar{t}_0,\bar{t}_0+T]} |x(t)| < d + |x'|_1. \tag{4}$$

Define $E_1 = \{t : t \in [0,T], |x(t-\tau(t))| > d\}, E_2 = \{t : t \in [0,T], |x(t-\tau(t))| \leq d\}$.

Multiplying $x(t)$ and (1) and then integrating from 0 to T, by (A_1) we have

$$\begin{aligned}
\int_0^T x'^p dt &= -\int_0^T (\varphi_p(x'(t)))' x(t)dt \\
&= \lambda\int_0^T \beta(t)g(x(t-\tau(t))x(t)dt - \lambda\int_0^T e(t)x(t)dt \\
&= \lambda\int_{E_1} \beta(t)g(x(t-\tau(t))x(t)dt + \lambda\int_{E_2} \beta(t)g(x(t-\tau(t))x(t)dt \\
&\quad -\lambda\int_0^T e(t)x(t)dt \\
&\le \lambda\int_{E_2} \beta(t)g(x(t-\tau(t))x(t)dt - \lambda\int_0^T e(t)x(t)dt \\
&\le \int_{E_2} |\beta(t)g(x(t-\tau(t))||x(t)|dt + \int_0^T |e(t)||x(t)|dt \\
&\le \left(\max_{t\in[0,T],|x|\le d} |\beta(t)g(x)| + |e|_\infty\right) T|x|_\infty.
\end{aligned}$$

Let $M_0 = \left(\max_{t\in[0,T],|x|\le d} |\beta(t)g(x)| + |e|_\infty\right) T$. Then we obtain

$$|x'|_p \le M_0^{1/p}|x|_\infty^{1/p}. \tag{5}$$

Let $q > 1$ such that $1/p + 1/q = 1$. Then by Hölder inequality we have

$$|x'|_1 \le |x'|_p|1|_q = T^{1/q}|x'|_p. \tag{6}$$

By (4), (5) and (6), we can get

$$|x'|_1 \le T^{1/q}M_0^{1/p}(d + |x'|_1)^{1/p},$$

which yields that there exists $M_1 > 0$ such that $|x'|_1 < M_1$ since $p > 1$, and this together with (4) implies that $|x|_\infty < d + M_1$.

Meanwhile, there exists $\hat{t}_0 \in [0,T]$ such that $x'(\hat{t}_0) = 0$ since $x(0) = x(T)$. Then by (1) we have, for $t \in [\hat{t}_0, \hat{t}_0 + T]$,

$$\begin{aligned}
|\varphi_p(x'(t))| &= \left|\int_{\hat{t}_0}^t (\varphi_p(x'(s)))' ds\right| \\
&= \lambda\left|\int_{\hat{t}_0}^t (f(x(s))x'(s) + \beta(s)g(x(s-\tau(s))) + e(s))ds\right| \\
&\le \int_0^T (|f(x(s))||x'(s)| + |\beta(s)g(x(s-\tau(s)))| + |e(s)|)ds \\
&< FM_1 + (G + |e|_\infty)T,
\end{aligned}$$

where $F = \max\{|f(x)| : |x| \le d + M_1\}$, $G = \max\{|\beta(t)g(x)| : t \in [0,T], |x| \le d + M_1\}$. So we obtain

$$|x'|_\infty = \max_{t\in[0,T]}\{|\varphi_p(x'^{1/(p-1)}\} < (FM_1 + (G + |e|_\infty)T)^{1/(p-1)}.$$

Let $M = \max\{d + M_1, (FM_1 + (G + |e|_\infty)T)^{1/(p-1)}\}$. Then $\|x\| < M$.

Second, we prove the existence of T-periodic solutions of (1). Set

$$h(t, x(t), x'(t)) = -f(x(t))x'(t) - \beta(t)g(x(t - \tau(t))) + e(t). \tag{7}$$

Then (1) is equivalent to the following equation

$$(\varphi_p(x'(t)))' = \lambda h(t, x(t), x'(t)), \lambda \in (0, 1). \tag{8}$$

Set

$$B = \{x : x \in C^1_T, \|x\| < r\} \quad \text{where } r \ge M. \tag{9}$$

By (7), we know that (8) has no solution on ∂B as $\lambda \in (0, 1)$, so condition (i) of Lemma 1 is satisfied. By the definition of F in Lemma 1 we get

$$F(a) = \frac{1}{T}\int_0^T h(t, a, 0)dt = \frac{1}{T}\int_0^T (e(t) - \beta(t)g(a))dt = -\frac{1}{T}\int_0^T \beta(t)g(a)dt.$$

This together with $\beta(t) > 0$ for all $t \in \mathbb{R}$ and (A_1) yields that $F(r)F(-r) < 0$, i.e., condition (ii) of Lemma 1 is satisfied. Therefore, it follows from Lemma 1 that there exists a T-periodic solution $x(t)$ of (1). This completes the proof.

REMARK 3. It is easy to see that Theorem 1 in this study holds under weaker conditions than Theorem 3.2 in [5].

Similar to the proof of Theorem 1, we can also get the following result.

THEOREM 2. Suppose $({A'}_1)$ holds. Then (1) has at least one T-periodic solution.

Together with Lemmas 2 and 3 and Theorems 1 and 2, we can directly obtain two theorems as follows.

THEOREM 3. Suppose (A_0) and (A_1) hold. Then (1) has a unique T-periodic solution.

THEOREM 4. Suppose (A'_0) and (A'_1) hold. Then (1) has a unique T-periodic solution.

4 Example and Remark

In this section, we apply the main results obtained in previous sections to an example.

EXAMPLE 1. Consider the existence and uniqueness of a 2π-periodic solution of the following Liénard type p-Laplacian equation

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + \beta(t)g(x(t)) = e(t), \tag{1}$$

where $p > 1$, $f \in C(\mathbb{R}, \mathbb{R})$, $\beta(t) = 1 + \cos^2 t$, $g(x) = -x^3 - 2x$, $e(t) = \cos t$ and $T = 2\pi$.

PROOF. If $p < 4$, the condition (H_3) in Theorem 3.3 in [5] does not hold any more since $m = 3 > p-1$. Therefore, Theorem 3.3 in [5] fails, while, our criterion in Theorem 3 in this study remains applicable, as we now show. Let d be an arbitrary positive constant, then we can easily check that the conditions (A_0) and (A_1) in Theorem 3 in this study hold. Hence, Theorem 3 shows that there exists a unique 2π-periodic solution of (1).

REMARK 4. This example demonstrates that the conditions in our Theorem 3 are weaker than those conditions in Theorem 3.3 in [5] when $\tau(t) \equiv 0$, and demonstrates the existence of a unique periodic solution to certain Liénard type p-Laplacian equations where the latter cannot be used to decide. Therefore, our results extend and improve the results in [5].

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