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Bifurcation From Infinity And Multiple Solutions Of Third Order Periodic Boundary Value Problems*

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Abstract

In this paper, we study the existence and multiplicity of solutions of third order periodic boundary value problems.

1 Introduction

The third-order equations arise in many areas of applied mathematics and physics, such as the deflection of a curved beam having a constant or a varying cross-section, three layer beam, electromagnetic waves or gravity-driven flows [1]. So the third-order boundary value problems were discussed by many authors and existence and multiplicity of solutions have been obtained in recent years, see for instance [2-17] and the references therein. However, most of the boundary conditions in the above mentioned references are two-point or three-point boundary conditions, the periodic boundary conditions are quite rarely seen [2,5,9-12,15].

Recently in [15], Yu and Pei considered the existence of solutions for a third-order periodic boundary value problem

$$u'''(t) + bu''(t) + g(t, u(t)) = e(t), \quad t \in (0, 2\pi), \tag{1}$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2, \tag{2}$$

where b is a nonnegative constant. By using Leray-Schauder continuation theorem, Yu and Pei established the following result:

THEOREM A ([15]). Let $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ be a given L^2-Carathéodory function. Assume that there exist $a_1 \le a_2, r_1 < 0, r < r_2$ such that

$$g(t, u) \ge a_2 \text{ for } u \ge r_2,\ a.e. t \in [0, 2\pi],$$

and

$$g(t, u) \le a_2 \text{ for } u \le r_2,\ a.e. t \in [0, 2\pi].$$

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Suppose also that there exists a function $\gamma(t) \in L^\infty(0, 2\pi)$ with $\|\gamma\|_\infty < 2$ such that

$$\limsup_{|x|\to\infty} \frac{g(t,u)}{u} \le \gamma(t),$$

uniformly in $a.e.t \in [0, 2\pi]$. Then, for every given function $e(t) \in L^2(0, 2\pi)$ with $a_1 \le \frac{1}{2\pi}\int_0^{2\pi} e(t)dt \le a_2$, problem (1)(2) has at least one solution.

However, their result gives no information on the interesting problem that nonlinearity crosses the eigenvalues. It is easy to see that 0 is the first eigenvalue of the following problem

$$u'''(t) + bu''(t) = \lambda u(t), \tag{3}$$

$$u^{(i)}(0) = u^{(i)}(T), i = 0, 1, 2. \tag{4}$$

An interesting problem is what would happen if the nonlinearity crosses the eigenvalue 0.

In this paper, we use Leray-Schauder degree and bifurcation technique to consider the existence and multiplicity of solutions of third order periodic boundary value problems

$$u'''(t) + bu''(t) + \lambda u(t) + g(u(t)) = h(t), \quad t \in (0, T), \tag{5}$$

$$u^{(i)}(0) = u^{(i)}(T), \quad i = 0, 1, 2, \tag{6}$$

where b is a nonnegative constant. $g : \mathbb{R} \to \mathbb{R}$ is continuous, $h \in L^1(0, T)$, and the parameter λ runs near 0 which is the first eigenvalue of (3)(4). In the paper, we make the following assumptions:

(H1) $g : \mathbb{R} \to \mathbb{R}$ is continuous, and there exist $\alpha \in [0, 1)$, $p, q \in (0, \infty)$, such that

$$|g(u)| \le p|u|^\alpha + q, \qquad u \in \mathbb{R}.$$

(H2) There exist constants $A,\ a,\ R, r$ such that $r < 0 < R$ and

$$g(u) \ge A \text{ for all } u \ge R,$$

$$g(u) \le a \text{ for all } u \le r.$$

(H2′) There exist constants $A,\ a,\ R, r$ such that $r < 0 < R$ and

$$g(u) \le A \text{ for all } u \ge R,$$

$$g(u) \ge a \text{ for all } u \le r.$$

(H3)

$$g^{-\infty} < \frac{1}{T}\int_0^T h(s)ds < g_{+\infty},$$

where

$$g^{-\infty} = \limsup_{s\to-\infty} g(s), \quad g_{+\infty} = \liminf_{s\to+\infty} g(s).$$

(H3′)

$$g^{+\infty} < \frac{1}{T}\int_0^T h(s)ds < g_{-\infty},$$

where

$$g^{+\infty} = \limsup_{s\to+\infty} g(s), \quad g_{-\infty} = \liminf_{s\to-\infty} g(s).$$

Our main results are the following

THEOREM 1.1. Assume that (H1)-(H3) hold. Then there exist $\lambda_+,\ \lambda_-$ with $\lambda_+ > 0 > \lambda_-$ such that
(i) (5),(6) has at least one solution if $\lambda \in [0, \lambda_+]$;
(ii) (5),(6) has at least three solutions if $\lambda \in [\lambda_-, 0)$.

THEOREM 1.2. Assume that (H1),(H2$'$) and (H3$'$) hold. Then there exist $\lambda_+,\ \lambda_-$ with $\lambda_+ > 0 > \lambda_-$ such that
(i) (5),(6) has at least one solution if $\lambda \in [\lambda_-, 0]$;
(ii) (5),(6) has at least three solutions if $\lambda \in (0, \lambda_+]$.

Motivated by bifurcation technique and Leray-Schauder degree developed in [16], which is concerned with second order periodic boundary value problem, in this paper we are concerned with the existence and multiplicity solutions of (5),(6). Since (5),(6) has odd order, there exist difficulty in the proof. The rest of the paper is arranged as follows. In Section 2, we discuss the Lyapunov-Schmidt procedure for (5),(6). Section 3 is devoted to the proof of our main results. Finally, we give an example in Section 4.

2 Lyapunov-Schmidt Procedure

Let $X,\ Y$ be respectively the Banach spaces $C^2[0,T]$ and $L^1[0,\ T]$ with respective norms $\|x\| = \max\{\|x\|_0, \|x'\|_0, \|x''\|_0\}$ and $\|u\|_1 = \int_0^T |u(s)|ds$, where $\|x\|_0 = \max\{|x(t)| : t \in [0,T]\}$. Define the linear operator $L : D(L) \subset X \to Y$ by

$$Lu = u''' + bu'',\ u \in D(L), \tag{7}$$

where $D(L) = \{u \in W^{3,1}(0,\ T) : u^{(i)}(0) = u^{(i)}(T), i = 0, 1, 2\}$. Let $N : X \to X$ be the nonlinear operator defined by

$$(Nu)(t) = g(u(t)),\ t \in [0,\ T],\ u \in D(L). \tag{8}$$

It is easy to see that N is continuous. (5),(6) is equivalent to

$$Lu + \lambda u + Nu = h,\ u \in D(L). \tag{9}$$

LEMMA 2.1. Let L be defined as (7). Then

$$\mathrm{Ker}L = \{x \in X : x(t) = c, c \in \mathbb{R}\},$$

$$\mathrm{Im}L = \{y \in Y : \int_0^T y(s)ds = 0\}.$$

PROOF. It is easy to see that $\mathrm{Ker}L = \{x \in X : x(t) = c, c \in \mathbb{R}\}$. The following will prove that $\mathrm{Im}L = \{y \in Y : \int_0^T y(s)ds = 0\}$.

If $y \in \mathrm{Im}L$, then there exists $u \in D(L)$ such that $u'''(t) + bu''(t) = y(t)$. So

$$u''(t) - u''(0) + bu'(t) - bu'(0) = \int_0^T y(s)ds.$$

Combining with $u^{(i)}(0) = u^{(i)}(T), i = 0, 1, 2$, we have $\int_0^T y(s)ds = 0$. The proof is complete.

Define the operator $P : X \to \mathrm{Ker}L$ by

$$(Pu)(t) = \frac{1}{T}\int_0^T u(s)ds,\ u \in X. \tag{10}$$

Let $Q : Y \to Y$ be such that

$$(Qy)(t) = \frac{1}{T}\int_0^T y(s)ds, y \in Y. \tag{11}$$

Then it is easy to check that P and Q are continuous projectors. Let $K(t, s)$ be the Green's function of

$$u'''(t) + bu''(t) = 0, t \in [0, T],$$

$$\int_0^T u(t)dt = 0, u^{(i)}(0) = u^{(i)}(T), i = 0, 1, 2.$$

Then $K : \mathrm{Im}L \to D(L) \cap \mathrm{Ker}P$ can be given by

$$(Ky)(t) = \int_0^T K(t, s)y(s)ds. \tag{12}$$

Obviously, the linear map $K : \mathrm{Im}L \to D(L) \cap \mathrm{Ker}P$ is continuous.

LEMMA 2.2. Let P and Q be defined by (10) and (11) respectively. Then

$$X = \mathrm{Ker}P \oplus \mathrm{Ker}L,\ Y = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Therefore, for every $u \in X$, we have unique decomposition $u(t) = \rho + v(t),\ t \in [0,\ T]$, where $\rho \in \mathbb{R},\ v \in \mathrm{Ker}P$. Similarly, for every $h \in Y$, we have unique decomposition $h(t) = \tau + \bar{h}(t),\ t \in [0,\ T]$, where $\tau \in \mathbb{R},\ \bar{h} \in \mathrm{Im}L$. Let the operators Q and K be defined by (11),(12). Then $K(I - Q)N :\ X \to X$ is completely continuous and (9) is equivalent to the systems

$$v(t) + \lambda Kv(t) + K(I - Q)N(\rho + v(t)) = K\bar{h}(t), \tag{13}$$

$$\lambda\rho + QN(\rho + v(t)) = \tau. \tag{14}$$

LEMMA 2.3 ([17]). Assume that (H1) and (H2) hold. Then for each real number $s > 0$, there exists a decomposition $g(u) = q_s(u) + g_s(u)$ of g by q_s and g_s satisfying the following conditions:

$$uq_s(u) \geq 0, u \in \mathbb{R}, \tag{15}$$

$$|q_s(u)| \leq p|u| + q + s, u \geq 1, \tag{16}$$

there exists σ_s depending on a, A and g such that

$$|g_s(u)| \leq \sigma_s, u \in \mathbb{R}. \tag{17}$$

LEMMA 2.4. Assume that (H1)-(H3) hold. If λ satisfies

$$0 \leq \lambda \leq \eta_1 := \frac{1}{2\|K\|_{\mathrm{Im}\,L \to \mathrm{Ker}\,P}}, \tag{18}$$

then there exists constant $R_0 > 0$ such that any solution u of (5),(6) satisfies $\|u\| < R_0$.

PROOF. We divide the proof into several steps.

Step 1. By the assumption (H1), there exists a constant b such that

$$|g(u)| \leq p\,|u| + b,\ \ u \in \mathbb{R},$$

where $p = \frac{1}{4}\eta_1$. Using Lemma 2.3 with $s = 1$, (5),(6) is equivalent to

$$u'''(t) + bu(t) + \lambda u(t) + g_1(u(t)) + q_1(u(t)) = h(t),\ \ t \in [0, T],\ \ u \in D(L), \tag{19}$$

where q_1 and g_1 satisfy conditions (15) and (17). Moreover, by (16),

$$|q_1(u)| \leq p\,|u| + b + 1. \tag{20}$$

Let $\bar{\delta} > 0$ and choose $B \in \mathbb{R}$ such that

$$(b+1)\left|\frac{1}{u}\right| \leq \frac{1}{4}\bar{\delta} \tag{21}$$

for all $u \in \mathbb{R}$ with $|u| \geq B$. It follows from (20) (21) that

$$0 \leq q_1(u)u^{-1} \leq p + \frac{1}{4}\bar{\delta} \tag{22}$$

for all $u \in \mathbb{R}$ with $|u| \geq B$.

Step 2. Let us define $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(u) = \begin{cases} u^{-1}q_1(u), & |u| \geq B; \\ B^{-1}q_1(B)(\dfrac{u}{B}) + (1 - \dfrac{u}{B})p, & 0 \leq u < B; \\ B^{-1}q_1(-B)(\dfrac{u}{B}) + (1 + \dfrac{u}{B})p, & -B < u \leq 0. \end{cases} \tag{23}$$

It is easy to see that γ is continuous. Moreover, by (22) one has

$$0 \leq \gamma(u) \leq p + \frac{1}{4}\bar{\delta} \tag{24}$$

for all $u \in \mathbb{R}$. Defining $f : \mathbb{R} \to \mathbb{R}$ by

$$f(u) = g_1(u) + q_1(u) - \gamma(u)u, \tag{25}$$

it follows from (22) that for some $\sigma \in \mathbb{R}$,

$$|f(u)| \leq \sigma \tag{26}$$

for all $u \in \mathbb{R}$, where σ depends only on p and h. Finally, (19) is equivalent to

$$u'''(t) + bu(t) + \lambda u(t) + f(u(t)) + \gamma(u(t))u(t) = h(t),\ t \in [0, T],\ u \in D(L).$$

Step 3. It is to see that $(L + \lambda I)|_{\mathrm{Ker}P \cap D(L)} : \mathrm{Ker}P \to \mathrm{Im}L$ is invertible. From (18),

$$\begin{aligned}
\|(L + \lambda I)|^{-1}_{\mathrm{Ker}P \cap D(L)}\|_{\mathrm{Im}L \to \mathrm{Ker}P} &= \|L^{-1}|_{\mathrm{Ker}P \cap D(L)}(I + \lambda K)^{-1}\|_{\mathrm{Im}L \to \mathrm{Ker}P} \\
&= \|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I + \lambda K)^{-1}\|_{\mathrm{Im}L \to \mathrm{Ker}P} \\
&\leq 2\|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}.
\end{aligned}$$

Let $u = \rho + v$ be a solution of (19), where $\rho \in \mathbb{R},\ v \in \mathrm{Ker}P$. Then from (13),

$$\begin{aligned}
\|v\| &= \|(L + \lambda I)|^{-1}_{\mathrm{Ker}P \cap D(L)}(I - Q)(\bar{h} - g(\rho + v(t)))\| \\
&\leq \|(L + \lambda I)|^{-1}_{\mathrm{Ker}P \cap D(L)}\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I - Q)\|_{Y \to \mathrm{Im}L}[\|\bar{h}\|_1 + p(|\rho| + \|v\|)^\alpha + q] \\
&\leq 2\|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I - Q)\|_{Y \to \mathrm{Im}L}[\|\bar{h}\|_1 + p(|\rho| + \|v\|)^\alpha + q] \\
&= 2\|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I - Q)\|_{Y \to \mathrm{Im}L}[\|\bar{h}\|_1 + p(|\rho|)^\alpha(1 + \frac{\|v\|}{|\rho|})^\alpha + q] \\
&\leq 2\|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I - Q)\|_{Y \to \mathrm{Im}L}[\|\bar{h}\|_1 + p(|\rho|)^\alpha(1 + \frac{\alpha\|v\|}{|\rho|}) + q] \\
&= 2\|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I - Q)\|_{Y \to \mathrm{Im}L}[\|\bar{h}\|_1 + p(|\rho|)^\alpha(1 + \frac{\alpha}{(|\rho|)^{1-\alpha}} \cdot \frac{\|v\|}{(|\rho|)^\alpha}) + q].
\end{aligned}$$

Therefore, $\frac{\|v\|}{(|\rho|)^\alpha} \leq \frac{c_0}{(|\rho|)^\alpha} + c_1 + \frac{\alpha c_1}{(|\rho|)^{1-\alpha}} \cdot \frac{\|v\|}{(|\rho|)^\alpha}$, where $c_0 = 2\|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I - Q)\|_{Y \to \mathrm{Im}L}(\|\bar{h}\|_1 + q),\ c_1 = 2p\|K\|_{\mathrm{Im}L \to \mathrm{Ker}P}\|(I - Q)\|_{Y \to \mathrm{Im}L}$.

If

$$|\rho| \geq (2\alpha c_1)^{\frac{1}{1-\alpha}} := \tilde{c},$$

then

$$\frac{\|v\|}{(|\rho|)^\alpha} \leq \frac{2c_0}{(\tilde{c})^\alpha} + 2c_1 := \bar{c}. \tag{27}$$

Step 4. If we now assume that the conclusion of the lemma is false, we obtain a sequence $\{\lambda_n\} : 0 \leq \lambda_n \leq \eta_1,\ \lambda_n \to 0$ and a sequence $\{u_n\} : u_n = \rho_n + v_n,\ \rho_n \in \mathbb{R},\ v_n \in \mathrm{Ker}P$ with $\|u_n\| \to \infty$ such that

$$\lambda_n \rho_n + Qg(\rho_n + v_n(t)) = \tau. \tag{28}$$

It follows immediately from (27) that

$$|\rho_n| \to \infty,\ \|v_n\|(|\rho_n|)^{-1} \to 0,\ n \to \infty. \tag{29}$$

So we infer that there exists sufficiently large $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$|v_n(t)|(|\rho_n|)^{-1} \leq 1, \quad t \in [0, T]. \tag{30}$$

Without loss of generality, let $\rho_n \to +\infty$ if $n \to +\infty$ (the other case be proved by similar method), then there exists sufficiently large $n_0 \in \mathbb{N}$. If $n \geq n_0,\ \lambda_n \rho_n \geq 0$, thus

$$\tau - \frac{1}{T} \int_0^T g(\rho_n + v_n(s))ds \geq 0,$$

$$\tau \geq \frac{1}{T} \liminf_{n\to\infty} \int_0^T g(\rho_n + v_n(s))ds. \tag{31}$$

Now in order to apply the Fatou lemma to (31), we need the existence of a function $\hat{K} \in L^1[0,T]$ such that for $s \in [0,T],\ g\left(u_n(s)\right) \geq \hat{K}(s)$. Indeed, from the relation (30), there exists nonnegative function $k_1 \in L^1[0,T]$ such that for $n \geq n_0$,

$$|v_n(t)|(\rho_n)^{-1} \leq k_1(t),\ t \in [0, T],$$

and for every $s \in [0, T]$,

$$\begin{aligned} \gamma(u_n(s))u_n(s) + f(u_n(s)) &= \gamma(u_n(s))(\rho_n + v_n(s)) + f(u_n(s)) \\ &\geq \gamma(u_n(s))\frac{\rho_n + v_n(s)}{|\rho_n|} + f(u_n(s)) \\ &\geq \gamma(u_n(s))(1 - k_1(s)) - |f(u_n(s))| \\ &\geq -(p + \frac{1}{4}\bar{\delta})(1 - k_1(s)) - \sigma. \end{aligned}$$

Let

$$\hat{K}(s) := -(p + \frac{1}{4}\bar{\delta})(1 - k_1(s)) - \sigma.$$

It follows from $g\left(u_n(s)\right) \geq \hat{K}(s)$ that

$$g(\rho_n + v_n(s)) \geq \hat{K}(s),\ s \in [0, T].$$

Thus, appling Fatou lemma to (31), we have

$$\begin{aligned} \tau &\geq \frac{1}{T} \liminf_{n\to\infty} \int_0^T g(\rho_n + v_n(s))ds \\ &\geq \frac{1}{T} \int_0^T \liminf_{n\to\infty} g(\rho_n + v_n(s))ds \\ &\geq \frac{1}{T} \int_0^T g_{+\infty} ds. \end{aligned}$$

This contradicts with (H3).

LEMMA 2.4′. Assume that (H1),(H2′),(H3′) hold. If λ satisfies

$$0 \leq \lambda \leq \eta_1 := \frac{1}{2\|K\|_{\mathrm{Im}\, L \to \mathrm{Ker}\, P}},$$

then there exists constant $R_0 > 0$ such that any solution u of (5),(6) satisfies

$$\|u\| < R_0.$$

3 The Proof of the Main Result

We have the following

LEMMA 3.1. Assume that (H1)-(H3) hold. Then there exists $R_1 \geq R_0$ such that for $0 \leq \lambda \leq \delta$, and $R \geq R_1$ one has

$$\deg(L + \lambda I + N - h,\ B(R),\ 0) = \deg(L + \delta I,\ B(R),\ 0) = \pm 1,$$

where $B(R) = \{u \in C[0,T] : \|u\| < R\}$, and the "deg" denotes Leray-Schauder degree when $\lambda \neq 0$ and coincidence degree when $\lambda = 0$. Then (5),(6) has a solution in $\bar{B}(R)$ for $0 \leq \lambda \leq \delta$.

PROOF. From Lemma 2.4 and the definition of L, if $\lambda \in [0, \delta]$, then

$$\deg(L + \delta I,\ B(R),\ 0)$$

is defined and is relevant to λ. Let $(\mu, u) \in [0,1] \times X$ be a solution of (9). Then

$$Lu + \delta u + \mu(Nu - h) = 0.$$

So

$$\begin{aligned} \|u\| &= \mu \|(L+\delta)^{-1}(h - Nu)\| \\ &\leq \|(L+\delta)^{-1}\|_{Y \to X}(\|h\|_1 + p\|u\|^{\alpha} + q). \end{aligned}$$

Therefore there exists $R_0' > 0$ such that $\|u\| < R_0'$. Choosing $R_1 = \max\{R_0',\ R_0\}$, then for arbitrary $R > R_1$,

$$\deg(L + \lambda I + N - h,\ B(R),\ 0) = \deg(L + \delta I,\ B(R),\ 0) = \pm 1.$$

LEMMA 3.1′. Assume that (H1),(H2′),(H3′) hold. Then there exists $R_1 \geq R_0$ such that for $0 \leq \lambda \leq \delta$, and $R \geq R_1$ one has

$$\deg(L + \lambda I + N - h,\ B(R),\ 0) = \deg(L + \delta I,\ B(R),\ 0) = \pm 1,$$

where $B(R) = \{u \in C[0,T] : \|u\| < R\}$.

LEMMA 3.2. Assume that (H1)-(H3) hold. Then there exists $\mu \geq 0$ such that for $-\mu \leq \lambda \leq 0$ one has

$$\deg(L + \lambda I + N - h,\ B(R),\ 0) = \deg(L + \delta I,\ B(R),\ 0) = \pm 1,$$

where R is defined in Lemma 3.1. Then (5),(6) has a solution in $B(R)$ for $-\mu \leq \lambda \leq \delta$.

PROOF. Let

$$\tau_0 = \inf_{u \in \partial B(R) \cap X} \|Lu + Nu - h\|.$$

It is easy to verify that $\tau_0 > 0$. Choosing sufficiently small $\mu > 0$ such that $\mu R < \tau_0$, then if $\lambda \in [-\mu,\ \mu]$,

$$\deg(L + \lambda I + N - h,\ B(R),\ 0) = \deg(L + N - h,\ B(R),\ 0).$$

Combined with Lemma 3.1, the result can be proved. That is to say, if $\lambda \in [-\mu,\ \delta]$, then (9) has at least one solution in $\bar{B}(R)$.

LEMMA 3.2$'$. Assume that (H1),(H2$'$)(H3$'$) hold. Then there exists $\mu \geq 0$ such that for $-\mu \leq \lambda \leq 0$ one has

$$\deg(L + \lambda I + N - h,\ B(R),\ 0) = \deg(L + \delta I,\ B(R),\ 0) = \pm 1,$$

where R is defined in Lemma 3.1. Then (5),(6) has a solution in $B(R)$ for $-\mu \leq \lambda \leq \delta$.

REMARK 1 Since g is L-completely continuous and satisfies (H2) and since $\lambda = 0$ is a simple eigenvalue of L, it follows from bifurcation results of [18] that there exist two connected sets $\mathcal{C}_+, \mathcal{C}_- \subset \mathbb{R} \times X$ of solutions of (5),(6) such that for all sufficiently small $\epsilon > 0$,

$$\mathcal{C}_+ \cap U_\epsilon \neq \emptyset,\ \mathcal{C}_- \cap U_\epsilon \neq \emptyset,$$

where $U_\epsilon := \{(\lambda, u) \in \mathbb{R} \times X,\ |\lambda| < \epsilon,\ \|u\| > \frac{1}{\epsilon}\}$.

PROOF OF THEOREM 1.1. Set $\lambda^+ = \delta$. Then it follows from Lemma 3.1 and Lemma 3.2 that (5),(6) has at least one solution in $B(R)$ for $\lambda \in [-\mu,\ \lambda_+]$. On the other hand, Remark 1 shows that there exist two connected sets $\mathcal{C}+$ and $\mathcal{C}-$ of solutions of (5),(6) bifurcating from infinity at $\lambda = 0$. Hence by Lemma 2.4, the connected sets $\mathcal{C}+$ and $\mathcal{C}-$ of Remark 1 must satisfy

$$\mathcal{C}+,\ \mathcal{C}- \subset \{(\lambda,\ u) : \|u\| \geq \frac{1}{\epsilon},\ \ -\mu < \lambda < 0\}.$$

and hence $\frac{1}{\epsilon} \geq R$, i.e. $\epsilon \leq \frac{1}{k}$. Choosing $\lambda_- = \max\{-\mu,\ -\frac{1}{k}\}$, we obtain two solutions $u_1,\ u_2 :\ u_1 \in \mathcal{C}+,\ u_2 \in \mathcal{C}-$, and $\|u_i\| \geq R$, $i = 1, 2$.

Theorem 1.2 can be proved in similar manners.

4 Example

To illustrate Theorem 1.1, we consider the existence and multiplicity of solutions of the following third order periodic boundary value problems

$$u'''(t) + bu''(t) + \lambda u(t) + g(u(t)) = \cos t, \quad t \in (0, T), \tag{32}$$

$$u^{(i)}(0) = u^{(i)}(T), \quad i = 0, 1, 2, \tag{33}$$

where b is a nonnegative constant.

$$g(x) = \begin{cases} 1 + |\sin \dfrac{\pi}{16} x|, & x > 8; \\ \sqrt[3]{x}, & |x| \leq 8; \\ -1 - |\sin \dfrac{\pi}{16} x|, & x < -8. \end{cases} \tag{34}$$

Choose $p = q = 1, \alpha = \frac{1}{2}, A = 1, a = -1, -8 = r < 0 < R = 8$. It is easy to check that (H1) and (H2) hold. It follows from (34) that $g^{-\infty} = 0, g_{+\infty} = 1$. Thus (H3) is also satisfied. By Theorem 1.1, there exist $\lambda_+,\ \lambda_-$ such that $\lambda_+ > 0 > \lambda_-$ and (32),(33) has at least one solution if $\lambda \in [0,\ \lambda_+]$; and (32),(33) has at least three solutions if $\lambda \in [\lambda_-,\ 0)$.

References

[1] M. Greguš, Third Order Linear Differential Equations, Mathematics and its Applications, Reidel Publishing Co., Dordrecht, 1987.

[2] A. R. Aftabizadeh, J. M. Xu and C. P. Gupta, Periodic boundary value problems for third order ordinary differential equations, Nonlinear Anal., 14(1990), 1–10.

[3] R. P. Agarwal, Existence-uniqueness and iterative method for third-order boundary value problems, Comput. Appl. Math., 17(1987), 271–289.

[4] A. Cabada, M. R. Grossinho and F. M. Minhós, Extremal solutions for third-order nonlinear problems with upper and lower solutions in reversed order, Nonlinear Anal., 62(2005), 1109–1121.

[5] J. Chu and Z. Zhou, Positive solutions for singular non-linear third-order periodic boundary value problems, Nonlinear Anal., 86(2006), 1528–1542.

[6] Z. Du, W. Ge and X. Lin, Existence of solutions for a class of third-order nonlinear boundary value problems, J. Math. Anal. Appl., 294(2004), 104–112.

[7] M. R. Grossinho, F. M. Minhós, and A. I. Santos, Existence result for a third-order ODE with nonlinear boundary conditions in presence of a sign-type Nagumo control. J. Math. Anal. Appl. 309 (2005), no. 1, 271–283.

[8] J. Henderson, Best interval lengths for third order Lipschitz equations, SIAM J. Math. Anal., 18(1987), 293–305.

[9] L. Kong and Y. Wang, Positive solution of a singular nonlinear third-order periodic boundary value problems, J. Comput. Appl. Math., 132(2001), 247–253.

[10] F. M. Minhós, Periodic solutions for a third order differential equation under conditions on the potential, Port. Math. (N.S.) 55(1998), 475–484.

[11] S. Mukhigulashvili, On a periodic boundary value problem for third order linear functional differential equations, Nonlinear Anal., 66(2007), 527–535.

[12] P. Omari and M. Trombetta, Remarks on the lower and upper solutions method for second and third-order periodic boundary value problems, Appl. Math. Comput., 50(1992) 1–21.

[13] M. Pei and S. K. Chang, Existence and uniqueness of solutions for third-order nonlinear boundary value problems, J. Math. Anal. Appl., 327(2007), 23–35.

[14] R. Ma. Multiplicity results for a third order boundary value problem at resonance, Nonlinear Anal., 32(1998), 493–499.

[15] H. Yu and M. Pei, Solvability of a nonlinear third-order periodic boundary value problem, Appl. Math. Lett., 23(2010), 892–896.

[16] J. Mawhin and K. Schmitt, Landesman-Lazer type problems at an eigenvalue of odd multiplicity, Result in Math. 14(1988), 138–146.

[17] R. Iannacci, M. N. Nkashama, Unbounded perturbations of forced second order at resonance, J. Diff. Eqns., 69(1987), 289–309.

[18] P. Rabinowitz, On bifurcation from infinity, J. Diff. Eqns., 14(1973), 462–475.