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Stability Of A 2-Dimensional Irreducible Linear System Of Delay Differential Equations*

Felix Che Shu†

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Abstract

Let $r > 0$. We consider the system $\acute{x}(t) = Ax(t) + Bx(t - r)$ where A and B are real square matrices of dimension 2. We assume that A has a single eigenvalue and give sufficient conditions for the asymptotic stability of the null solution of the system by deriving a pair of one dimensional delay differential equations from the system and comparing the Lyapunov exponents of the corresponding fundamental solutions.

1 Introduction

Let A be an $n \times n$ real matrix. The solution $x(t) \equiv 0$ of the system $\acute{x}(t) = Ax(t), t \geq 0$ of ordinary differential equations is asymptotically stable if and only if all roots of the corresponding characteristic function have negative real parts. Similar statements hold for systems of linear delay differential equations. It is known see e.g., Bellman and Cooke [1] that if all the roots of the characteristic function of the equation

$$\acute{x}(t) = Ax(t) + Bx(t - r) \tag{1}$$

have negative real parts, where A and B are real $n \times n$ matrices, then the zero solution of the equation is asymptotically stable. We are using the following notions of stability for systems of delay differential equations: Let x^g denote the solution of the system (1) with the initial condition $x(t) = g(t), t \in [-r, 0]$ and for $f : [-r, 0] \to \mathbb{R}^n$, let $\|f\| := \sup\{\|f(s)\| : -r \leq s \leq 0\}$. The norm on $\mathbb{R}^n$ is any norm.

DEFINITION 1. The solution x^φ of (1) is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

(i) if $g : [-r, 0] \to \mathbb{R}^n$ and $\|g - \varphi\| < \delta$, then $x^g(t)$ exists for all $t \geq 0$; and

(ii) $\|x^g(t) - x^\varphi(t)\| < \varepsilon$ for all $t \geq 0$.

We say that the stable solution x^φ is asymptotically stable if in addition to (ii), $\lim_{t\to\infty} \|x^g(t) - x^\varphi(t)\| = 0$ for all g for which (i) holds.

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†Department of Mathematics, University of Buea, Box 63 Buea, Cameroon

We say that the asymptotically stable solution x^φ is exponentially stable, if

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x^g(t) - x^\varphi(t)\| < 0$$

for all g for which (i) holds.

Unfortunately, characteristic functions of delay differential equations are not ordinary polynomials as in the case of ordinary differential equations. They are transcendental functions whose roots are not easy to determine. As a consequence, a lot of research has been done on questions related to the distribution of the zeroes of these functions e.g., [9, 6, 4, 5] and the authors cited there. In higher dimensions, these questions are considered in [3].

Given that the task of determining the distribution of characteristic roots in higher dimensions is non-trivial, we are led to attempt to reduce the study of the stability of systems in several dimensions to the study of the stability of suitable one dimensional delay equations.

In what follows, we give sufficient conditions for the stability of a two dimensional irreducible system by deriving a pair of one dimensional delay differential equations from the system and comparing the Lyapunov exponents of the associated fundamental solutions. The method relies on an explicit formula for the solution of such a system presented in [10]. We note that the method we have used here cannot be extended trivially to systems where the matrix A has two distinct eigenvalues. The rest of the paper is organized as follows: In Section 2 we give some notation and in section 3 we present our results and proofs.

2 Prerequisites

We shall use the same symbol 0 for the real number 0, the zero vector in $\mathbb{R}^2$ and the zero matrix in $\mathbb{M}(2,2,\mathbb{R})$, where $\mathbb{M}(2,2,\mathbb{R})$ denotes the set of 2×2 matrices with real entries. The symbol E shall denote the multiplicative identity in $\mathbb{M}(2,2,\mathbb{R})$. Let $r>0$, $A,\ B \in \mathbb{M}(2,2,\mathbb{R})$ and consider the equation (1). We assume that A is irreducible in the sense that it is not diagonalizable or is diagonalizable but has a single eigenvalue $\xi \in \mathbb{C}$. In this case there exists an invertible matrix Q such that

$$Q^{-1}AQ = \xi E + M \tag{2}$$

where $M = (m_{ij})_{ij=1,2}$ with $m_{11} = m_{21} = m_{22} = 0$ and $m_{12} = \tau,\ \tau = 1$ if A is not diagonalizable and $\tau = 0$ otherwise. We define $H := Q^{-1}BQ$. We shall also define $p(E) := 0, p(H) := p(M) := 1$. If $x \in \mathbb{M}(2,2,\mathbb{R})$, then we define $x^0 := E$ and $x^m = \overbrace{x\cdots x}^{m\times}$. If $n \geq 1$, $x_i \in \{H,M,E\}, i = 1,\ldots,n$ and $x = x_1\cdots x_n$, then we define $p(x) = p(x_1\cdots x_n) := \sum_{i=1}^{n} p(x_i)$. $p(x)$ is the number of times that the matrices M and H appear as factors in the given factorization of x over $\{M,H,E\}$. For $x \in \mathbb{M}(2,2,\mathbb{R})$, let T_x denote the linear transformation on $\mathbb{M}(2,2,\mathbb{R})$ defined by $T_x(y) = xy,\ y \in \mathbb{M}(2,2,\mathbb{R})$

and for $A \subseteq \mathbb{M}(2,2,\mathbb{R})$, $T_x(A) := \{T_x(y) : y \in A\}$. For $j \in \{0,1\}$ and $k \geq 0$, we define

$$I_k^j := \begin{cases} \{E\} & : \quad k = 0 \\ T_{(M^j H)}(I_{k-1}) & : \quad k \geq 1 \end{cases}, \qquad I_k := I_k^0 \cup I_k^1.$$

DEFINITION 2. Let $g : [-r,0] \to \mathbb{R}^2$. A function $x^g : [-r,\infty) \to \mathbb{R}^2$ is called a solution of (1) with the initial condition

$$x^g(t) = g(t),\ t \in [-r,0] \tag{3}$$

if it is continuous on $[0\ \infty)$, satisfies (1) Lebesgue almost everywhere on $[0,\infty)$ and (3).

Let $\mathcal{L}^1([-r,0],\mathbb{R}^2) := \left\{ g : [-r,0] \to \mathbb{R}^2 : \int\limits_{-r}^{0} \| g(s) \| \, ds < \infty \right\}$. If $g \in \mathcal{L}^1([-r,0],\mathbb{R}^2)$, then a unique solution x^g of (1) exists. Let $G : [-r,\infty) \to \mathbb{M}(2,2,\mathbb{R})$ be the fundamental matrix associated with (1) i.e., for any $\eta \in \mathbb{R}^2$,

$$x(t) := G(t)\eta,\ T \in [-r,\infty),$$

is the solution of (1) with initial condition $x(t) = \eta 1_{\{0\}}(t),\ t \in [-r,0]$, then it can be shown that the solution x^g of (1) with the initial condition (3) is given by

$$x^g(t) := \begin{cases} g(t) & : \quad t \in [-r,0] \\ G(t)g(0) + \int\limits_{-r}^{0} G(t-s-r)Bg(s)ds & : \quad t \geq 0. \end{cases} \tag{4}$$

In [10] it was shown that if A is irreducible in the sense given above, then the fundamental matrix associated with (1) is the function G defined for $t \geq 0$ by

$$G(t) := \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} QxQ^{-1} \left(\frac{(t-kr)^l}{l!} E + \frac{(t-kr)^{(l+1)}}{(l+1)!} QMQ^{-1} \right). \tag{5}$$

3 Results and Proofs

We begin with the following Lemma.

LEMMA 1. There exists a constant $K_1 < \infty$ such that for all g for which the solution x^g of (1), (3) exists and $t \geq 0$,

$$\|x^g(t)\| \leq K_1 \, \|g\| \, \|G_t\| \ \text{ where } \ \|G_t\| := \sup\{\|G(t+\vartheta)\| : \vartheta \in [-r,0]\}$$

and G is the fundamental matrix associated with (1).

PROOF. From (4), it follows that for $t \geq 0$,

$$\begin{aligned} \|x^g(t)\| &\leq \|G(t)\| \, \|g(0)\| + \int_{-r}^{0} \|G(t-s-r)\| \, \|B\| \, \|g(s)\| \, ds & (6) \\ &\leq \|G(t)\| \, \|g\| + \int_{-r}^{0} \|G(t-s-r)\| \, \|B\| \, \|g\| \, ds. & (7) \end{aligned}$$

If we now set $K_1 := (1 + r\,\|B\|)$, then the assertion follows.

The proof of the next Theorem relies on the following Lemmas:

LEMMA 2. For $k \geq 1$ and $l \in \{k, \ldots, 2k\}$, $\sharp\{x \in I_k : p(x) = l\} = \binom{k}{l-k}$, where we denote the cardinality of a set A by $\sharp A$.

PROOF. Let $k = 1$. Then $I_1 := \{H, (MH)\}$ and $l \in \{1, 2\}$. For $l = 1$ we have $1 = \sharp\{H\} = \sharp\{x \in I_1 : p(x) = 1\}$. Also, $\binom{1}{1-1} = 1$. For $l = 2$, $\sharp\{x \in I_1 : p(x) = 2\} = \sharp\{(MH)\} = 1$. We also have $\binom{1}{2-1} = 1$. Therefore the statement is true for $k = 1$. Assume that the statement is true for some $k \geq 2$. We now show that it is true for $k + 1$.

$$\begin{aligned}
\{x \in I_{k+1} : p(x) = l\} &= \{x \in I^0_{k+1} : p(x) = l\} \cup \{x \in I^1_{k+1} : p(x) = l\} \\
&= \{x \in T_H(I_k) : p(x) = l\} \cup \{x \in T_{(MH)}(I_k) : p(x) = l\},
\end{aligned}$$

where the union is disjoint. Therefore

$$\begin{aligned}
\sharp\{x \in I_{k+1} : p(x) = l\} &= \sharp\{x \in T_H(I_k) : p(x) = l\} + \sharp\{x \in T_{(MH)}(I_k) : p(x) = l\} \\
&= \sharp\{x \in I_k : p(x) = l - 1\} + \sharp\{x \in I_k : p(x) = l - 2\} \\
&= \binom{k}{l-1-k} + \binom{k}{l-2-k} = \binom{k+1}{l-1-k}.
\end{aligned}$$

LEMMA 3. Let $k \geq 1$. If $x \in I_k$ and $p(x) = l$ for any $l \in \{k, \ldots, 2k\}$, then k factors of x are H and $l - k$ factors are M.

PROOF. We will also do this proof by induction on k. Let $k = 1$, then $l \in \{1, 2\}$. If $x \in I_1$ and $p(x) = l = 1$, then $x = H$ which has $k = 1$ factor being H and $l - k = 0$ factors being M. Also, if $p(x) = l = 2$, then $x = (MH)$ which has $k = 1$ factor being H and $l - k = 1$ factor being M.

Assume that the statement is true for some $k \geq 2$ and let now $x \in I_{k+1}$ with $p(x) = l,\ l \in \{k + 1, \ldots, 2(k + 1)\}$. Since $x \in I_{k+1} = I^0_{k+1} \cup I^1_{k+1}$, it follows that $x = Hy$ for some $y \in I_k$ or $x = (MH)y$ for some $y \in I_k$.

If $x = Hy$ with $y \in I_k$, then since $p(x) = l$, it follows that $p(y) = l - 1$. Since $y \in I_k$, it follows from the assumption of the induction that k factors of y are H and $l - 1 - k = l - (k + 1)$ factors are M. Therefore $k + 1$ factors of x are H and $l - (k + 1)$ factors are M.

Alternatively, if $x = (MH)y$ with $y \in I_k$, then since $p(x) = l$, it follows that $p(y) = l - 2$. Since $y \in I_k$, it again follows from the assumption of the induction that k factors of y are H and $l - 2 - k$ factors are M. Therefore $k + 1$ factors of x are H and $l - (k + 1)$ factors are M.

LEMMA 4. Let z^* and y^* denote the fundamental solutions of the scalar equations $\acute{z}(t) = \|M\|\, z(t - r)$ and $\acute{y}(t) = Re(\xi)y(t) + \beta y(t - r)$ respectively, where $Re(\xi)$ denotes the real part of ξ and

$$\beta = \begin{cases} \|B\| & : \quad M = 0 \\ \|H\| & : \quad M \neq 0 \end{cases},$$

then for the fundamental matrix G associated with (1) and $t \geq 0$,

$$\|G(t)\| \leq (\|Q\| \left\|Q^{-1}\right\|)^2 y^*(t) z^*(t)(1 + t\|M\|).$$

PROOF. Let $G_1(t) := \sum_{k=0}^{[\frac{t}{r}]} \frac{B^k}{k!}(t-kr)^k e^{\xi(t-kr)}$ and

$$G_2(t) := \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x)=l\}} QxQ^{-1}\left(\frac{(t-kr)^l}{l!}QQ^{-1} + \frac{(t-kr)^{l+1}}{(l+1)!}QMQ^{-1}\right),$$

then $G(t) = G_1(t)$ if $M = 0$ and $G(t) = G_2(t)$ if $M \neq 0$. Assume first that $M \neq 0$ and let $K(Q) := (\|Q\| \left\|Q^{-1}\right\|)^2$, then

$$\|G(t)\| \leq K(Q) \sum_{k=0}^{[\frac{t}{r}]} |e^{\xi(t-kr)}| \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x)=l\}} \|x\| \frac{(t-kr)^l}{l!}\left(1 + \frac{(t-kr)}{(l+1)}\|M\|\right).$$

By Lemma 3, if $x \in I_k$ and $p(x) = l$, then $\|x\| \leq \|H\|^k \|M\|^{l-k}$. Note that $|e^{\xi(t-kr)}| = e^{Re(\xi)(t-kr)}$ and hence by Lemma 2, we have

$$\begin{aligned}
\|G(t)\| &\leq K(Q) \sum_{k=0}^{[\frac{t}{r}]} e^{Re(\xi)(t-kr)} \|H\|^k \sum_{l=k}^{2k} \binom{k}{l-k} \|M\|^{l-k} \frac{(t-kr)^l}{l!}\left(1 + \frac{(t-kr)}{(l+1)}\|M\|\right) \\
&= K(Q) \sum_{k=0}^{[\frac{t}{r}]} e^{Re(\xi)(t-kr)} \|H\|^k \sum_{l=0}^{k} \binom{k}{l} \|M\|^l \frac{(t-kr)^{l+k}}{(l+k)!}\left(1 + \frac{(t-kr)}{(l+k+1)}\|M\|\right) \\
&= K(Q) \sum_{k=0}^{[\frac{t}{r}]} e^{Re(\xi)(t-kr)} \|H\|^k \frac{(t-kr)^k}{k!} f(t, k, \|M\|),
\end{aligned}$$

where we set $f(t,k,m) := \sum_{l=0}^{k} k! \binom{k}{l} m^l \frac{(t-kr)^l}{(l+k)!}\left(1 + \frac{(t-kr)}{(l+k+1)}m\right)$. Using the convention that for integers $k \geq 0,\ l \geq 0,\ k \geq l$,

$$k(k-1)(k-2)\cdots(k-l+1) = \begin{cases} 1 & : \quad l = 0 \\ k(k-1)(k-2)\cdots(k-l+1) & : \quad 1 \leq l \leq k \end{cases},$$

it is easy to show that $\frac{k!}{(k+l)!}\binom{k}{l} \leq \frac{1}{l!}$. Therefore

$$\begin{aligned}
f(t,k,m) &\leq \sum_{l=0}^{k} \frac{m^l}{l!}(t-kr)^l \left(1 + \frac{(t-kr)m}{l+k+1}\right) \leq \sum_{l=0}^{k} \frac{m^l}{l!}(t-kr)^l(1+tm) \\
&\leq \sum_{l=0}^{k} \frac{m^l}{l!}(t-lr)^l(1+tm) \leq \sum_{l=0}^{[\frac{t}{r}]} \frac{m^l}{l!}(t-lr)^l(1+tm).
\end{aligned}$$

If we now replace m by $\|M\|$, then it follows that $\|G(t)\| \leq K(Q)y^*(t)z^*(t)(1+t\,\|M\|)$.

Assume now that $M = 0$, then it is easy to see that $z^*(t) = 1_{[0\ \infty)}(t),\ t \geq -r$ and hence $z^*(t)(1+t\,\|M\|) = 1,\ t \geq 0$. Therefore

$$\begin{aligned}
\|G(t)\| &\leq K(Q)\sum_{k=0}^{[\frac{t}{r}]} e^{Re(\xi)(t-kr)}\,\|B\|^k\,\frac{(t-kr)^k}{k!} \\
&= K(Q)\sum_{k=0}^{[\frac{t}{r}]} e^{Re(\xi)(t-kr)}\,\|B\|^k\,\frac{(t-kr)^k}{k!}z^*(t)(1+t\,\|M\|) \\
&= K(Q)y^*(t)z^*(t)(1+t\,\|M\|).
\end{aligned}$$

For the scalar delay equation

$$\acute{x}(t) = ax(t) + bx(t-r), \tag{8}$$

we call the function $h(\lambda) := \lambda - a - be^{-\lambda r}$ the characteristic function of (8). Its zeroes are called the characteristic roots of (8). Let Λ denote the set of characteristic roots of (8), $\nu_0 = \nu_0(a,b,r) := \max\{Re(\lambda) : \lambda \in \Lambda\}$ and x_0 be its fundamental solution (matrix). The following is known:

LEMMA 5.

(i) For every real c, $\Lambda \cap \{\lambda \in \mathbb{C} : Re(\lambda) > c\}$ is finite. In particular, $\nu_0 < \infty$.

(ii) For every $\nu > \nu_0$ there exists a constant $K(\nu)$ such that $|x_0(t)| \leq K(\nu)e^{\nu t}$.

PROOF. See Hale [7], Lemma 4.1 and Theorem 5.2 of Chapter 1.

From Lemma 4 and Lemma 5 we have the following theorem:

THEOREM 1. Consider the scalar delay differential equations

$$\begin{aligned}
\acute{z}(t) &= \|M\|\,z(t-r),\ t \geq 0 \\
\acute{y}(t) &= Re(\xi)y(t) + \beta y(t-r),\ t \geq 0.
\end{aligned}$$

If $\nu_0(Re(\xi),\beta,r) < -\nu_0(0,\|M\|,r)$, then the null solution $x^\varphi,\ \varphi \equiv 0$ of (1) is exponentially stable.

PROOF. Let $\lambda^* := \nu_0(0,\|M\|,r)$, then $\lambda^* \geq 0$. Further let $\mu,\ \nu \in \mathbb{R}$ be chosen such that $\nu_0(Re(\xi),\beta,r) < \nu$ and $\nu < \mu < -\lambda^*$ and set $d := -\lambda^* - \mu$ and $\lambda := \lambda^* + d$, then $\lambda^* < \lambda$ and $\lambda + \nu = \nu - \mu < 0$. By Lemma 5, $|y^*(t)| \leq K(\nu)e^{\nu t}$ and $|z^*(t)| \leq K(\lambda)e^{\lambda t}$. Therefore by Lemma 4,

$$\|G(t)\| \leq K(Q)y^*(t)z^*(t)(1+t\,\|M\|) \leq K(Q)K(\nu)K(\lambda)e^{(\lambda+\nu)t}(1+t\,\|M\|).$$

Let $\tilde{K} = K(Q)K(\nu)K(\lambda)$. Then

$$\begin{aligned}
\|G_t\| &= \sup\{\|G(t+\vartheta)\| : \vartheta \in [-r,0]\} \\
&\leq \tilde{K}\sup\{e^{(\lambda+\nu)(t+\vartheta)}(1+(t+\vartheta)\,\|M\|) : \vartheta \in [-r,0]\} \\
&\leq \tilde{K}e^{(\lambda+\nu)t}\sup\{e^{(\lambda+\nu)\vartheta}((1+t\,\|M\|)+\vartheta\,\|M\|) : \vartheta \in [-r,0]\} \\
&\leq \tilde{K}e^{(\lambda+\nu)t}e^{-(\lambda+\nu)r}(1+t\,\|M\|).
\end{aligned}$$

By Lemma 1, there exists a constant K_1 such that for arbitrary g for which the solution x^g exists, $\|x^g(t)\| \leq K_1 \|g\| \|G_t\|$. Let $K_2 := \tilde{K} K_1 e^{-(\lambda+\nu)r}$. Then

$$\|x^g(t)\| \leq K_2 \|g\| e^{(\lambda+\nu)t}(1 + t \|M\|). \tag{9}$$

If now t^* is the point at which the function $t \mapsto \exp\{(\lambda+\nu)t\}(1+t\|M\|),\ t \geq 0$, takes its global maximum and we set $C := K_2 \exp\{(\lambda+\nu)t^*\}(1+t^*\|M\|)$, then $C < \infty$ and for $t \geq 0$,

$$\|x^g(t)\| \leq C \|g\|. \tag{10}$$

For $\varepsilon > 0$ chosen arbitrarily, let $\delta := \frac{\varepsilon}{C}$ and $g : [-r, 0] \to \mathbb{R}^2$ be such that $\|g\| < \delta$, then $g \in \mathcal{L}^1([-r, 0], \mathbb{R}^2)$ and hence $x^g(t)$ exists for all $t \geq 0$. From (10), it follows that $\|x^g(t)\| < \varepsilon$ for all $t \geq 0$ and hence $x^\varphi,\ \varphi \equiv 0$ is stable. Since $\lambda + \nu < 0$, it follows from (9) that $\lim_{t\to\infty} \|x^g(t)\| = 0$, hence $x^\varphi,\ \varphi \equiv 0$ is asymptotically stable. Finally, $\frac{1}{t} \ln \|x^g(t)\| \leq (\lambda+\nu) + \frac{1}{t} \ln[K_2\delta(1+t\|M\|)]$ and hence $\limsup_{t\to\infty} \frac{1}{t} \ln \|x^g(t)\| \leq \lambda+\nu < 0$, i.e., $x^\varphi,\ \varphi \equiv 0$ is exponentially stable.

From the preceding Theorem we have the following Corollary which in essence generalizes the one dimensional case:

COROLLARY 1. If A is a diagonal matrix, then a sufficient condition for the exponential stability of $x^\varphi, \varphi \equiv 0$ is that $\nu_0(Re(\xi), \|B\|, r) < 0$.

PROOF. Note that from (2), A is a diagonal matrix if and only if $M = 0$ and that if $M = 0$, then $\lambda^* = 0$ and $\beta = \|B\|$. The assertion then follows immediately from Theorem 1.

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