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Fixed Points For Weak Contractions In G-Metric Spaces*

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Abstract

In this paper we prove a fixed point theorem for weak contractions in G-metric spaces. Our result is supported by an example.

1 Introduction

The concept of weak contraction is introduced by Alber and Guerre-Delabriere [1]. They proved the existence of fixed points for single-valued maps satisfying weak contractive conditions on Hilbert spaces. Rhoades [14] showed that most results of [1] are still true for any metric spaces. The weak contraction was defined as follows.

DEFINITION 1. A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a weak contraction if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$$

where $x, y \in X$ and $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

In fact Banach contraction is a special case of weak contraction by taking $\phi(t) = (1 - k)t$ for $0 < k < 1$. In this connection Rhoades [14] proved the following very interesting fixed point theorem

THEOREM 1 ([14]). Let (X, d) be a complete metric space, and let T be a weak contraction on X. If $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\phi(t) > 0$ for all $t \in (0, \infty)$ and $\phi(0) = 0$, then T has a unique fixed point.

Gahler [7, 8] coined the term of 2-metric spaces. This is extended to D-metric space by Dhage [4, 5]. In 2003, Mustafa and Sims [11] introduced a new structure called G-metric space as a generalization of the usual metric space. They have studied some fixed point theorems for various types of mappings in this new structure.

DEFINITION 2 ([11]). Let X be a nonempty set, and let $G : X \times X \times X \to R+$, be a function satisfying:

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$(G1)$ $G(x, y, z) = 0$ if $x = y = z$,
$(G2)$ $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$,
$(G3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables), and
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then the function G is called a generalized metric, or, more specially a G-metric on X, and the pair (X, G) is called a G-metric space.

EXAMPLE 1 ([11]). Let (X, d) be a usual metric space. Then (X, G_s) and (X, G_m) are G-metric spaces where

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in X$ and

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$.

DEFINITION 3 ([11]). Let (X, G) be a G-metric space and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. We refer to x as the limit of the sequence (x_n) and write $x_n \xrightarrow{G} x$.

PROPOSITION 1 ([11]). Let (X, G) be a G-metric space. The following statements are equivalent.
(1) (x_n) is G-convergent to x.
(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
(3) $G(x_n, x, x) \to 0$, as $n \to \infty$.

DEFINITION 4 ([11]). Let (X, G) be a G-metric space. A sequence (x_n) is called G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

PROPOSITION 2 ([11]). In a G-metric space (X, G), the following two statements are equivalent.
(1) The sequence (x_n) is G-Cauchy.
(2) For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

DEFINITION 5 ([11]). A G-metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

DEFINITION 6 ([11]). A G-metric space (X, G) is called symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

PROPOSITION 3 ([11]). Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

PROPOSITION 4 ([11]). Every G-metric space (X, G) defines a metric space (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x)$$

for all $x, y \in X$.

Note that if (X, G) is a symmetric G-metric space, then

$$d_G(x, y) = 2G(x, y, y), \forall x, y \in X.$$

2 Main Results

We have the following main theorem.

THEOREM 2. Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z)) \tag{1}$$

for all $x, y, z \in X$. If $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\phi^{-1}(0) = 0$, $\phi(t) > 0$ for all $t \in (0, \infty)$, then T has a unique fixed point in X.

PROOF. Let $x_0 \in X$. We construct the sequence (x_n) by $x_n = Tx_{n-1}, n \in N$. If $x_{n+1} = x_n$ for some n, then trivially T has a fixed point. We assume $x_{n+1} \neq x_n$, for $n \in N$. From (1), we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n) \leq G(x_{n-1}, x_n, x_n) - \phi(G(x_{n-1}, x_n, x_n)). \tag{2}$$

By the property of ϕ, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n).$$

Similarly we can show that

$$G(x_{n-1}, x_n, x_n) \leq G(x_{n-2}, x_{n-1}, x_{n-1}).$$

This shows that $G(x_n, x_{n+1}, x_{n+1})$ is monotone decreasing and consequently there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) \to r \quad \text{as } n \to \infty. \tag{3}$$

By taking $n \to \infty$ in (2), we obtain

$$r \leq r - \phi(r) \tag{4}$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) \to 0 \quad \text{as } n \to \infty. \tag{5}$$

Now we prove that (x_n) is a Cauchy sequence. Suppose (x_n) is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\left(x_{m(k)}\right)$ and $\left(x_{n(k)}\right)$ of (x_n) with $n(k) > m(k) > k$ such that

$$G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \geq \epsilon. \tag{6}$$

Further, corresponding to $m(k)$, we can choose $n(k)$, such that it is the smallest integer with $n(k) > m(k)$ and satisfying (6). Then

$$G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}) < \epsilon. \tag{7}$$

Then we have

$$\begin{aligned}\epsilon &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \epsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).\end{aligned} \tag{8}$$

Setting $k \to \infty$ and using (5),

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{9}$$

Now,

$$\begin{aligned}G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \leq{}& G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \\ &+ G(x_{m(k)-1}, x_{m(k)}, x_{m(k)})\end{aligned}$$

and

$$\begin{aligned}G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \leq{}& G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \\ &+ G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}).\end{aligned}$$

Setting $k \to \infty$ in the above inequality and using (5) and (9), we get

$$\lim_{k\to\infty} G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) = \epsilon.$$

From (1) and (6), we have

$$\begin{aligned}\epsilon &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = G(Tx_{m(k)-1}, Tx_{n(k)-1}, Tx_{n(k)} - 1) \\ &\leq G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) - \phi(G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1})).\end{aligned}$$

Letting $k \to \infty$, we see that

$$\epsilon \leq \epsilon - \phi(\epsilon)$$

clearly it is a contradiction if $\epsilon > 0$. So we must have $\epsilon = 0$. This shows that (x_n) is a Cauchy sequence in X. Since X is a complete G-metric space, so there exist $p \in X$ such that

$$\lim_{n\to\infty} x_n \to p.$$

Now we claim that $Tp = p$. For this we consider

$$\begin{aligned}G(x_n, Tp, Tp) &= G(Tx_{n-1}, Tp, Tp) \\ &\leq G(x_{n-1}, p, p) - \phi(G(x_{n-1}, p, p)).\end{aligned}$$

By taking $n \to \infty$

$$G(p, Tp, Tp) \leq 0.$$

But $G(p, Tp, Tp) \geq 0$. So we have $Tp = p$, i.e. p is a fixed point of T. Suppose T has two fixed points p and q, then

$$\begin{aligned} G(p,q,q) &= G(Tp,Tq,Tq) \\ &\leq G(p,q,q) - \phi(G(p,q,q)), \end{aligned}$$

by the property of ϕ, this is contradiction if $G(p, q, q) > 0$. Hence we must have $G(p, q, q) = 0$ and $p = q$.

EXAMPLE 2. Let $x = [0, 1]$ and $d(x, y) = |x - y|$. Define $G(x, y, z) = |x - y| + |y - z| + |x - z|$. Then (X, G) is a complete G-metric space. Let $T(x) = x - \frac{x^2}{2}$ and $\phi(t) = \frac{t^2}{2}$. Without loss of generality, we assume $x > y > z$. Then

$$\begin{aligned} & G(Tx,Ty,Tz) \\ = \;& |Tx - Ty| + |Ty - Tz| + |Tx - Tz| \\ = \;& \left|\left(x - \frac{x^2}{2}\right) - \left(y - \frac{y^2}{2}\right)\right| + \left|\left(y - \frac{y^2}{2}\right) - \left(z - \frac{z^2}{2}\right)\right| \\ & + \left|\left(x - \frac{x^2}{2}\right) - \left(z - \frac{z^2}{2}\right)\right| \\ = \;& \left(x - \frac{x^2}{2}\right) - \left(y - \frac{y^2}{2}\right) + \left(y - \frac{y^2}{2}\right) - \left(z - \frac{z^2}{2}\right) + \left(x - \frac{x^2}{2}\right) - \left(z - \frac{z^2}{2}\right) \\ = \;& [(x-y) + (y-z) + (x-z)] - \left[\left(\frac{x^2}{2} - \frac{y^2}{2}\right) + \left(\frac{y^2}{2} - \frac{z^2}{2}\right) + \left(\frac{x^2}{2} - \frac{z^2}{2}\right)\right] \\ \leq \;& [(x-y) + (y-z) + (x-z)] - \frac{1}{2}[(x-y)^2 + (y-z)^2 + (x-z)^2] \\ = \;& G(x,y,z) - \phi(G(x,y,z)). \end{aligned}$$

Clearly T satisfies (1). By Theorem 2, T has a unique fixed point i.e. 0.

3 Remarks

In the above theorem, if we define $d_G(x, y) = G(x, y, y) + G(y, x, x)$, then d_G is a metric on X and the above theorem coincide with Theorem 1 of Rhoades.

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