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On A Metaharmonic Boundary Value Problem*

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Abstract

In this paper we develop maximum principles for solutions of metaharmonic equations defined on arbitrary n dimensional domains. As a consequence we obtain an uniqueness result for the corresponding metaharmonic boundary value problem.

1 Introduction

In the paper [4] we showed that if a_1, $a_3 \geq 0$ (a_1, a_3 constants), $a_2(x) \geq 0$, $a_4(x) > 0$ in $\Omega \subset \mathbb{R}^2$ and the curvature of $\partial\Omega \in C^{2+\varepsilon}$ is strictly positive, then the boundary value problem

$$\begin{cases} \Delta^4 u - a_1\Delta^3 u + a_2(x)\Delta^2 u - a_3\Delta u + a_4(x)u = f & \text{in } \Omega, \\ u = g,\ \Delta u = h,\ \Delta^2 u = i,\ \Delta^3 u = j & \text{on } \Omega \end{cases} \tag{1}$$

has at most a classical solution in $C^8(\Omega) \cap C^6(\overline{\Omega})$.

Using a generalized maximum principle we are able here to extend the above mentioned result for a the m metaharmonic problem

$$\begin{cases} \Delta^m u - a_{m-1}(x)\Delta^{m-1} u + a_{m-2}(x)\Delta^{m-2} u + \cdots + (-1)^m a_0(x)u = f & \text{in } \Omega, \\ u = g_1,\ \Delta u = g_2, \ldots,\ \Delta^{m-1} u = g_m & \text{on } \Omega \end{cases} \tag{2}$$

where a_i, $i = 0, \ldots, m-1$, are bounded in the bounded domain $\Omega \subset \mathbb{R}^2$, $n \geq 2$. Here we deal with classical solutions u of (2), i.e., $u \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$, $m \geq 3$.

This result generalizes the result of Dunninger [5] (the case $m = 2$, $n \geq 2$, $a_1 = 0$, $a_0 \equiv$ constant ≥ 0 and Ω arbitrary), Schaefer [7] (the case curvature of $\partial\Omega > 0$, $m = n = 2$), Schaefer [8] (the case a_2, $a_1 \geq 0$, $a_0 > 0$ with $m = 3, n = 2$, curvature of $\partial\Omega > 0$), S. Goyal and V. Goyal [6] and Danet [3] (the variable coefficient case with $m = 3$ and $\Omega \subset \mathbb{R}^n$ arbitrary).

Throughout this paper we shall assume that $\Omega \subset \mathbb{R}^n, n \geq 2$ is a bounded domain, $m \geq 3$ and the coefficients a_i, $i = 0, \ldots, m-1$ are bounded in Ω. Also we shall suppose that $a_0 \not\equiv 0$. diamΩ will denote the diameter of Ω.

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2 Main Results

The uniqueness result will be a consequence of the following generalized maximum principle and the next lemmas.

THEOREM 1 ([4]). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality $\mathrm{L}u \equiv \Delta u + \gamma(x)u \geq 0$ in Ω, where $\gamma \geq 0$ in Ω. Suppose that

$$\sup_{\Omega} \gamma < \frac{4n+4}{(\operatorname{diam}\Omega)^2} \tag{3}$$

holds. Then, the function u/w_1 satisfies a generalized maximum principle in Ω, i.e., either the function u/w_1 assumes its maximum value on $\partial\Omega$ or is constant in $\overline{\Omega}$. Here $w_1(x) = 1 - \alpha(x_1^2 + \cdots + x_n^2) \in C^\infty(\mathbb{R}^n)$ and $\alpha = \sup_\Omega \gamma/2n$.

If Ω lies in strip of width d and if we impose the restriction

$$\sup_{\Omega} \gamma < \frac{\pi^2}{d^2}, \tag{4}$$

we obtain that u/w_2 satisfies a generalized maximum principle in Ω. Here

$$w_2 = \cos \frac{\pi(2x_i - d)}{2(d+\varepsilon)} \prod_{j=1}^{n} \cosh(\varepsilon x_j) \in C^\infty(\overline{\Omega}),$$

for some $i \in \{1, \ldots, n\}$, where $\varepsilon > 0$ is small.

For simplicity, we shall consider only the case when m is even, i.e., we shall deal with the equation

$$\Delta^m u - a_{m-1}(x)\Delta^{m-1}u + a_{m-2}(x)\Delta^{m-2}u - \cdots + a_0(x)u = 0 \quad \text{in } \Omega. \tag{5}$$

Similar results will hold if m is odd.

LEMMA 1. Let u be a classical solution of (5). Let

$$P_1 = \frac{1}{2}(\Delta^{m-1}u)^2 + \frac{a_{m-1}}{2}(\Delta^{m-2}u)^2 + (\Delta^{m-3}u)^2 + \cdots + u^2.$$

Suppose that $a_{m-3}, \ldots, a_1 \geq 0$, $a_2, a_0 > 0$ and $\Delta(1/a_{m-2}) \leq 0$ in Ω. If one of the following conditions is satisfied

(a)

$$4a_{m-1} - a_{m-3} - a_{m-4} - \cdots - a_0 \geq 0 \quad \text{in } \Omega \tag{6}$$

and

$$A = \max\left\{1 + \sup_\Omega a_0, 2 + \sup_\Omega a_1, \ldots, 2 + \sup_\Omega a_{m-3}, \max\left\{1, \sup_\Omega \frac{a_{m-2}}{2}\right\}\right\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2}; \tag{7}$$

(b)

$$a_{m-1} \geq 0 \quad \text{in } \Omega \tag{8}$$

and

$$\max\left\{A, \sup_{\Omega} \frac{a_{m-3}+\cdots+a_0}{2}\right\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2}, \tag{9}$$

then either the function P_1/w_1 assumes its maximum value on $\partial\Omega$ or is constant in $\overline{\Omega}$.

PROOF. A computation (using equation (5)) shows that in Ω,

$$\begin{aligned} \frac{1}{2}\Delta\left((\Delta^{m-1}u)^2\right) &\geq \Delta^{m-1}u\Delta^m u \\ &= a_{m-1}(\Delta^{m-1}u)^2 - a_{m-2}\Delta^{m-2}u\Delta^{m-1}u \\ &\quad -a_{m-3}\Delta^{m-3}u\Delta^{m-1}u - \cdots - a_0 u\Delta^{m-1}u. \end{aligned}$$

From the inequalities

$$(-1)^i a_{i-3}\Delta^{i-3}u\Delta^{m-1}u \geq -\frac{a_{i-3}}{4}(\Delta^{m-1}u)^2 - a_{i-3}(\Delta^{i-3}u)^2,\ i = 3,\ldots,m, \tag{10}$$

and

$$\frac{1}{2}\Delta\left(a_{m-2}(\Delta^{m-2}u)^2\right) \geq a_{m-2}\Delta^{m-1}u\Delta^{m-2}u,$$

we get

$$\begin{aligned} &\frac{1}{2}\Delta\left(((\Delta^{m-1}u)^2 + a_{m-2}(\Delta^{m-2}u)^2)\right) \\ &\geq (a_{m-1} - a_{m-3}/4 - a_{m-4}/4 - \cdots - a_0/4)(\Delta^{m-1}u)^2 \\ &\quad -a_{m-3}(\Delta^{m-3}u)^2 - a_{m-4}(\Delta^{m-4}u)^2 - \cdots - a_0 u^2. \end{aligned}$$

Since

$$\Delta\left((\Delta^{m-3}u)^2\right) \geq 2\Delta^{m-2}u\Delta^{m-3}u \geq -(\Delta^{m-2}u)^2 - (\Delta^{m-3}u)^2,$$

$$\Delta\left((\Delta^{m-4}u)^2\right) \geq 2\Delta^{m-3}u\Delta^{m-4}u \geq -(\Delta^{m-3}u)^2 - (\Delta^{m-4}u)^2,$$

$$\ldots,$$

$$\Delta u^2 \geq 2u\Delta u \geq -\Delta u^2 - u^2,$$

we deduce that P_1 satisfies the differential inequality

$$\begin{aligned} \Delta P_1 &\geq (a_{m-1} - a_{m-3}/4 - a_{m-4}/4 - \cdots - a_0/4)(\Delta^{m-1}u)^2 - (\Delta^{m-2}u)^2 \\ &- (2 + a_{m-3})(\Delta^{m-3}u)^2 - \cdots - (2 + a_1)(\Delta u)^2 - (1 + a_0)u^2. \end{aligned}$$

Hence

$$\Delta P_1 + \gamma P_1 \geq 0 \quad \text{in } \Omega,$$

where

$$\gamma = \max\{1 + \sup_{\Omega} a_0, 2 + \sup_{\Omega} a_1, \ldots, 2 + \sup_{\Omega} a_{m-3}, \max\{1, \sup_{\Omega} a_{m-2}/2\}\}.$$

By (7) we have

$$\gamma < \frac{4n+4}{(\operatorname{diam}\Omega)^2}.$$

Now the proof of (a) follows from Theorem 1. The proof for (b) is similar.

LEMMA 2. Let u be a classical solution of (5). Let

$$P_2 = \frac{1}{2}(\Delta^{m-1}u)^2 + (\Delta^{m-2}u)^2 + (\Delta^{m-3}u)^2 + \cdots + u^2.$$

Suppose that $a_{m-1}, \ldots, a_1 \geq 0$ and $a_0 > 0$ in Ω. If

$$\max\left\{\sup_\Omega \frac{a_0}{2} + \sup_\Omega \frac{a_1}{2}, \ldots, 2 + \sup_\Omega \frac{a_{m-2}}{2}, A_1\right\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2}, \tag{11}$$

where $A_1 = \max\{1 + \sup_\Omega a_0, 2, \sup_\Omega a_1, \ldots, 2 + \sup_\Omega a_{m-2}\}$, then either the function P_2/w_1 assumes its maximum on $\partial\Omega$ or is a constant in $\overline{\Omega}$.

PROOF. As in the proof of Lemma 1, we get

$$\begin{aligned}\frac{1}{2}\Delta\left((\Delta^{m-1}u)^2\right) &\geq \Delta^{m-1}u\Delta^m u \\ &= a_{m-1}(\Delta^{m-1}u)^2 - a_{m-2}\Delta^{m-2}u\Delta^{m-1}u - \cdots - a_0 u\Delta^{m-1}u.\end{aligned}$$

Since

$$-a_0 u\Delta^{m-1}u \geq -\frac{a_0}{4}(\Delta^{m-1}u)^2 - a_0u^2,$$

$$\ldots$$

$$-a_{m-2}u\Delta^{m-1}u\Delta^{m-2}u \geq -\frac{a_{m-2}}{4}(\Delta^{m-1}u)^2 - a_{m-2}(\Delta^{m-2}u)^2,$$

and

$$\Delta\left((\Delta^{m-2}u)^2\right) \geq -(\Delta^{m-2}u)^2 - (\Delta^{m-1}u)^2,$$

$$\ldots,$$

$$\Delta u^2 \geq -\Delta u^2 - u^2,$$

we get that

$$\begin{aligned}\Delta P_2 \geq\ & -(1 + a_{m-2}/4 + a_{m-3}/4 + \cdots + a_1/4 + a_0/4)(\Delta^{m-1}u)^2 - (2 + a_{m-2})(\Delta^{m-2}u)^2 \\ & -(2 + a_{m-3})(\Delta^{m-3}u)^2 - \cdots - (2 + a_1)(\Delta u)^2 - (1 + a_0)u^2.\end{aligned}$$

Hence

$$\Delta P_2 + \gamma P_2 \geq 0 \quad \text{in } \Omega,$$

where

$$\gamma = \max\{A_1, \{\sup_\Omega a_0/2 + \sup_\Omega a_1/2 + \cdots + \sup_\Omega a_{m-2}/2 + 2\}\}.$$

LEMMA 3. Let u be a classical solution of (5). Suppose that $a_{m-2}, \ldots, a_0 \geq 0$ in Ω. If one of the following conditions is fulfilled

(a)

$$\max\{1 + \sup_\Omega a_0^2, 2 + \sup_\Omega a_1^2, \ldots, 2 + \sup_\Omega a_{m-2}^2\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2} \tag{12}$$

and $4a_{m-1} \geq m+3$ in Ω; or

(b)

$$\max\left\{1+\sup_{\Omega} a_0^2, 2+\sup_{\Omega} a_1^2, \ldots, 2+\sup_{\Omega} a_{m-2}^2, 2+\frac{m-1}{2}\right\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2} \tag{13}$$

and $a_{m-1} \geq 0$ in Ω,
then either the function P_2/w_1 assumes its maximum on $\partial\Omega$ or is a constant in $\overline{\Omega}$.

This may be proved exactly as Lemma 2, except the inequalities (10) are replaced by

$$(-1)^i a_{i-3}\Delta^{i-3}u\Delta^{m-1}u \geq -\frac{1}{4}(\Delta^{m-1}u)^2 - a_{i-3}^2(\Delta^{i-3}u)^2, \; i=3,\ldots,m.$$

It is clear that Lemma 3 remains valid if the coefficients $a_{m-2},\ldots,a_0$ have arbitrary sign in Ω.

The following particular result becomes sharper than Lemma 2 if we choose a_0 and a_1 appropriately.

LEMMA 4. Let u be a classical solution of (5). Let

$$P_3 = \frac{1}{2}(\Delta^{m-1}u - a_1 u)^2 + P_2.$$

Suppose that $a_{m-1} = \cdots = a_2 = 0$ and $a_0 > 0$ in Ω. If $a_1 \equiv$ constant > 0 and if

$$\max\left\{2+2\sup_{\Omega}\frac{a_0}{a_1}+2a_1, 2+\frac{a_1}{4}\right\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2}, \tag{14}$$

then, the function P_3/w_1 assumes its maximum on $\partial\Omega$ or is a constant in $\overline{\Omega}$.

PROOF. A calculation gives

$$\begin{aligned}
&\Delta\left((\frac{1}{2}(\Delta^{m-1}u - a_1u)^2 + \frac{1}{2}(\Delta^{m-1}u)^2\right)\\
\geq\ & -2a_0u\Delta^{m-1}u + a_1\Delta u\Delta^{m-1}u + a_0a_1u^2\\
=\ & a_0a_1\left(u^2 - \frac{2}{a_1}u\Delta^{m-1}u + \frac{1}{a_1^2}(\Delta^{m-1}u)^2\right) - \frac{a_0}{a_1}(\Delta^{m-1}u)^2 + a_1\Delta u\Delta^{m-1}u\\
\geq\ & -\frac{a_0}{a_1}(\Delta^{m-1}u)^2 - \frac{a_1}{4}(\Delta u)^2 - a_1(\Delta^{m-1}u)^2
\end{aligned}$$

in Ω. It follows that

$$\begin{aligned}
\Delta P_3 \geq\ & -\left(\frac{a_0}{a_1}+a_1+1\right)(\Delta^{m-1}u)^2 - 2(\Delta^{m-2}u)^2 - \cdots - 2(\Delta^3u)^2 -\\
& -\left(\frac{a_1}{4}+2\right)(\Delta u)^2 - u^2
\end{aligned}$$

in Ω. Hence

$$\Delta P_3 + \gamma P_3 \geq 0 \quad \text{in}\Omega,$$

where $\gamma = \max\{2 + 2\sup_\Omega(a_0/a_1) + 2a_1, 2 + a_1/4\}$.

We now state our main result.

THEOREM 2. There is at most one classical solution of the boundary value problem (2) provided the coefficients $a_{m-1}, \dots, a_0$ satisfy the conditions imposed in Lemma 1, Lemma 2, Lemma 3 or Lemma 4.

PROOF. Suppose that the hypothesis of Lemma 1 is satisfied. Define $u = u_1 - u_2$, where u_1 and u_2 are solutions of (2). Then u_1 and u_2 satisfy the equation (5) and

$$u = \Delta u = \cdots = \Delta^{m-1} u = 0 \quad \text{on } \partial\Omega. \tag{15}$$

Hence, by Theorem 1 either

i). there exists a constant $k \in \mathbb{R}$ such that

$$\frac{P_1}{w_1} \equiv k \quad \text{in } \Omega, \tag{16}$$

or

ii). P_1/w_1 does not attain a maximum in Ω.

Case i). By continuity (16) holds in $\overline{\Omega}$. By the boundary conditions (15) we obtain $P_1 = 0$ on $\partial\Omega$, i.e., $k = 0$. It follows that $P_1 \equiv 0$ in Ω, which means $u \equiv 0$ in Ω. Hence $u_1 = u_2$ in Ω.

Case ii). From

$$\max_{\overline{\Omega}} \frac{P_1}{w_1} = \max_{\partial\Omega} \frac{P_1}{w_1}$$

and (15) we get

$$0 \leq \max_{\overline{\Omega}} \frac{P_1}{w_1} = 0,$$

i.e., $u_1 = u_2$ in Ω.

We can argue similarly if we are under the hypotheses of Lemma 2, Lemma 3 or Lemma 4. The proof is complete.

Of course, our method can also be applied to the problem (1) to get results in arbitrary domains Ω.

Next, we consider classical solutions of the equation

$$\Delta^4 u + a_2(x)\Delta^2 u - a_3(x)\Delta u + a_4(x) u = 0 \quad \text{in } \Omega. \tag{17}$$

LEMMA 5. Let u be a classical solution of (17). Assume that

$$a_2 > 0, \quad \Delta(1/a_2) \leq 0 \quad \text{in } \Omega, \tag{18}$$

$$a_4 > 0, \quad \Delta(1/a_4) \leq 0 \quad \text{in } \Omega, \tag{19}$$

$$a_2 - 2a_4 - 1 > 0, \quad \Delta(1/(a_2 - 2a_4 - 1)) \leq 0 \quad \text{in } \Omega. \tag{20}$$

If

$$\max\left\{\sup_\Omega a_3, \sup_\Omega \frac{1}{a_4}, \sup_\Omega \frac{a_4^2}{a_2 - 2a_4 - 1}\right\} < \frac{2n+2}{(\operatorname{diam}\Omega)^2}, \tag{21}$$

then, the function P_4/w_1 assumes its maximum on $\partial\Omega$ or is a constant in $\overline{\Omega}$. Here

$$P_4 = \frac{1}{2}(\Delta^3 u + \Delta u)^2 + a_4(\Delta^2 u + u)^2 + \frac{a_2 - 2a_4 - 1}{2}(\Delta^2 u)^2 + \frac{a_2 - 2a_4 - 1}{2}(\Delta u)^2 + \frac{1}{2}(\Delta^3 u)^2 + \frac{a_2}{2}(\Delta^2 u)^2 + \frac{1}{2}a_4 u^2.$$

Under the hypotheses of Lemma 5, an uniqueness result follows for problem (1). We note that this uniqueness result is not a particular result of Theorem 2. Moreover we do not impose any convexity assumption on $\partial\Omega$.

Finally, we give an application of the uniqueness result that follows from Lemma 5. We see that the boundary value problem

$$\begin{cases} \Delta^4 u + 4(x^2 + y^2 + 3)\Delta^2 u - ((x^2 + y^2 + 3)^2/4)\Delta u + (x^2 + y^2 + 3)u = 0 & \text{in } \Omega \\ u = 13/4,\ \Delta u = 4,\ \Delta^2 u = 0,\ \Delta^3 u = 0 & \text{on } \partial\Omega, \end{cases}$$

has the solution $u(x, y) = x^2 + y^2 + 3$ in $\Omega = \{(x, y)|x^2 + y^2 \leq 1/4\}$.

Since (18), (19), (20) and (21) are satisfied, we get by the uniqueness result that follows from Lemma 5 that $u(x, y) = x^2 + y^2 + 3$ is the unique solution.

As our final remarks, for some domains we may improve the maximum principle, i.e. the constant $C(n, \operatorname{diam}\Omega) = (4n + 4)/(\operatorname{diam}\Omega)$ can be taken larger (see for details [2] and [3]).

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