Applied Mathematics E-Notes, 12(2012), 1-4 © ISSN 1607-2510
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On A Sum Related By Non-trivial Zeros Of The Riemann Zeta Function*

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Received 27 September 2010

Abstract

In this note we obtain explicit upper and lower bounds for the sum $\sum_{0<\gamma\leq T}\gamma^{-1}$, where γ is the imaginary part of the non-trivial zero $\rho=\beta+i\gamma$ of $\zeta(s)$.

1 Introduction

The Riemann zeta function is defined for $s\in\mathbb{C}$ with $\Re(s)>1$ by $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$, and extended by analytic continuation to the whole complex plan with a simple pole with residues 1 at $s=1$. A symmetric functional equation for $\zeta(s)$ is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}dt$ is the Euler's gamma function. $\Gamma(s)$ is meromorphic with simple poles at $s=0,-1,-2,\dots$ (see [3]). By using these facts, we may see that trivial zeros (zeros on real line $\mathbb{R}$) of $\zeta(s)$ are $s=-2,-4,-6,\dots$. Furthermore, we get symmetry of non-trivial zeros (other zeros $\rho=\beta+i\gamma$ with properties $0\leq\beta\leq 1$ and $\gamma\neq 0$) with respect to the vertical line $\Re(s)=1/2$. Our intention in writing this paper is to approximate the function

$$A(T)=\sum_{\substack{0<\gamma\leq T\\ \zeta(\beta+i\gamma)=0}}\frac{1}{\gamma}.$$

More precisely, we show the following.

THEOREM 1. Let $\gamma_1=\min\{\gamma>0:\zeta(\beta+i\gamma)=0\}\approx 14.13472514$. Then, for any $T>\gamma_1$ we have

$$\frac{3}{50}<A(T)-\left(\frac{1}{4\pi}\log^2 T-\frac{\log(2\pi)}{2\pi}\log T\right)<\frac{109}{250}. \tag{1}$$

Our strategy to prove this result is to consider the zero counting function $N(T)$ defined by

$$N(T)=\sum_{\substack{0<\gamma\leq T\\ \zeta(\beta+i\gamma)=0}}1,$$

*Mathematics Subject Classifications: 11M06, 11M26.

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and then translate known approximations about it to get desired approximations for $A(T)$. The key for doing such translation is using Stieljes integration and integrating by parts. Indeed, if we assume that $1 < U \leq V$ and $\Phi(t) \in C^1(U,V)$ is a non-negative function, then we have

$$\sum_{U<\gamma\leq V}\Phi(\gamma) = \int_U^V \Phi(t)dN(t) = -\int_U^V N(t)\Phi'(t)dt + N(V)\Phi(V) - N(U)\Phi(U).$$

Among his various conjectures about the function $\zeta(s)$ and its non-trivial zeros, B. Riemann [5] claimed that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{2}$$

This conjecture of Riemann is proved by H. von Mangoldt more than 30 years later [1, 2]. An immediate consequence of (2), which follows by partial summation, is the asymptotic approximation

$$A(T) = O(\log^2 T).$$

To obtain a more precise approximation, we use the relation (2) by replacing $\Phi(\gamma) = \frac{1}{\gamma}$, and putting $0 < U < \gamma_1$ and $V = T$. We have

$$A(T) = \int_U^T \frac{dN(t)}{t} = \int_U^T \frac{N(t)}{t^2}dt + \frac{N(T)}{T}. \tag{3}$$

We substitute $N(T)$ from (2) to obtain

$$A(T) = \frac{1}{2\pi}\int_U^T \frac{\log\left(\frac{t}{2\pi}\right)}{t}dt - \frac{1}{2\pi}\int_U^T \frac{dt}{t} + \frac{1}{2\pi}\log\frac{T}{2\pi} - \frac{1}{2\pi} + O\left(\int_U^T \frac{\log(t)}{t^2}dt\right) + O\Big(\frac{\log T}{T}\Big).$$

Then, we simplify the right hand side of this relation, and we let $U \to \gamma_1^-$. Therefore, we get

$$A(T) = \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + O(1).$$

Now, we are very close to the truth of Theorem 1. Our remaining duty is to estimate the constant in error term $O(1)$ in the last relation.

2 Proof of Theorem

The working engine of our paper is a result due to J. B. Rosser (Theorem 19 of [6]), which asserts that

$$|N(T) - F(T)| \leq R(T) \qquad (\text{for } T \geq 2), \tag{4}$$

where

$$F(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}, \qquad \text{and} \qquad R(T) = \frac{137}{1000}\log T + \frac{433}{1000}\log\log T + \frac{397}{250}.$$

This is indeed an explicit version of (2), and it allows us to obtain some explicit approximations of $A(T)$. In fact, by considering (4) and by using (3) with $2 \leq U < \gamma_1$, for every $T > \gamma_1$ we obtain

$$-\int_U^T \frac{R(t)}{t^2}dt + \frac{F(T)-R(T)}{T} \leq A(T) - \int_U^T \frac{F(t)}{t^2}dt \leq \int_U^T \frac{R(t)}{t^2}dt + \frac{F(T)+R(T)}{T}.$$

A simple calculation shows

$$\frac{F(t)}{t^2} = \frac{d}{dt}\left(\frac{1}{4\pi}\log^2 t - \frac{1+\log(2\pi)}{2\pi}\log t + \frac{\log^2(2\pi)-2\log(2\pi)}{4\pi} - \frac{7}{8t}\right).$$

Also, we have

$$\frac{R(t)}{t^2} = \frac{d}{dt}\left(-\frac{433}{1000}\frac{\log\log t}{t} - \frac{137}{1000}\frac{\log t}{t} - \frac{69}{40t} - \frac{433}{1000}E(t)\right),$$

where $E(t) = \int_1^\infty \frac{ds}{st^s}$. This integral converges for $t > 1$; in fact $E(t) \sim \frac{1}{t\log t}$ as $t \to \infty$. Moreover, by using the relation $\frac{d}{dt}E(t) = -\frac{1}{t^2\log t}$, we obtain

$$\frac{1}{t\log t} - \frac{1}{t\log^2 t} < E(t) < \frac{1}{t\log t} - \frac{31}{95t\log^2 t} \qquad (\text{for } t \geq 2).$$

Therefore, by letting $U \to \gamma_1^-$ we get the following explicit upper bound

$$A(T) < \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + \mathfrak{c}_{\mathfrak{au}} - \frac{137\log^2 T + 433\log T - 433}{1000T\log^2 T} \qquad (\text{for } T > \gamma_1),$$

where $\mathfrak{c}_{\mathfrak{au}} = 0.43596427\cdots < \frac{109}{250}$. Since $137\log^2 T + 433\log T > 433$ is valid for $T \geq 2.222$, we obtain

$$A(T) < \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + \frac{109}{250} \qquad (\text{for } T > \gamma_1).$$

This completes the proof of the right hand side of (1). To prove the left hand side of (1), we follow same steps to get

$$\begin{aligned} A(T) > &\frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T \\ &+ \mathfrak{c}_{\mathfrak{al}} + \frac{274\log^3 T + 866(\log\log T)\log^2 T + 3313\log^2 T + 433\log T - 433}{1000T\log^2 T}, \end{aligned}$$

for $T > \gamma_1$, where $\mathfrak{c}_{\mathfrak{al}} = 0.06058187\cdots > \frac{3}{50}$. We note that for $T \geq 2$ the last fraction in the above inequality is strictly positive. Thus, we obtain

$$A(T) > \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + \frac{3}{50} \qquad (\text{for } T > \gamma_1).$$

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