Applied Mathematics E-Notes, 11(2011), 278–282 © ISSN 1607-2510
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A New Proof Of The Classical Watson's Summation Theorem*

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Received 11 July 2011

Abstract

The aim of this research note is to provide a new proof of the classical Watson's theorem for the generalized hypergeometric series ${}_3F_2$.

1 Introduction

We start with the classical Watson's summation theorem for the generalized hypergeometric series ${}_3F_2$, [1, P. 16, Eq. 1] viz.

$$
\begin{aligned}
&{}_3F_2\left[\begin{array}{c} a, \quad b, \quad c \\ \frac{1}{2}(a+b+1), \quad 2c \end{array} ; 1\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}
\end{aligned}
\tag{1}
$$

provided $\mathrm{Re}(2c-a-b) > -1$.

The proof of this theorem when one of the parameters a or b is a negative integer was given in Watson [7]. Subsequently, it was established more generally in the non-terminating case by Whipple [8]. The standard proof of the non-terminating case was given in Bailey's tract [1] by employing the fundamental transformation due to Thomae combined with the classical Dixon's theorem of the sum of a ${}_3F_2$.

An alternative and more involved proof was given by MacRobert [4] by employing the well known quadratic transformation for the Gauss's hypergeometric function [5, P. 67, Theorem 25]

$$
{}_2F_1\left[\begin{array}{c} 2a, \quad 2b \\ a+b+\frac{1}{2} \end{array} ; x\right] = {}_2F_1\left[\begin{array}{c} a, \quad b \\ a+b+\frac{1}{2} \end{array} ; 4x(1-x)\right] \tag{2}
$$

valid for $|x| < 1$ and $|4x(1-x)| < 1$.

*Mathematics Subject Classifications: 35C20

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Another proof is due to Bhatt [2], by employing a known relation between F_2 and F_4 Appell functions combined with a comparison of the coefficients in their series expansions.

Very recently, Rathie and Paris [6] have given a very simple and elegant proof of (1) that relies only on the well known Gauss summation theorems for the series ${}_2F_1$.

In this research note, we give a simple proof of (1) by employing the Gauss's second summation theorem. However our method is similar to that given in MacRobert [4] but without using the quadratic transformation (2).

2 Results Required

The following results will be required in our present investigations.

- Finite integral [3]

$$\int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_2F_1\left[\begin{matrix} a, & b \\ & \\ e \end{matrix} ; \; zt\right] dt$$

$$= \frac{\Gamma(d-c)\Gamma(c)}{\Gamma(d)}\,{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & \\ d, & e & \end{matrix} ; \; z\right] \tag{3}$$

provided $\mathrm{Re}(c) > 0, \mathrm{Re}(d-c) > 0$ and $\mathrm{Re}(d+c-a-b-c) > 0$.

- Transformation formula [5, P. 65, Theorem 24]

$${}_2F_1\left[\begin{matrix} a, & b \\ & \\ 2b & \end{matrix} ; \; 2y\right] = (1-y)^{-a}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2} \\ & \\ b+\frac{1}{2} & \end{matrix} ; \; \left(\frac{y}{1-y}\right)^2\right] \tag{4}$$

valid for $|y| < \frac{1}{2}$ and $\left|\frac{y}{1-y}\right| < 1$.

- Integral representation for the hypergeometric function ${}_2F_1$ [5, P. 47, Theorem 16]

$${}_2F_1\left[\begin{matrix} a, & b \\ & \\ c & \end{matrix} ; \; z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \tag{5}$$

valid for $|z| < 1$, and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.

- Gauss's summation theorem [1, P. 2, Eq. 1]

$${}_2F_1\left[\begin{matrix} a, & b \\ & \\ c & \end{matrix} ; \; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{6}$$

provided $\mathrm{Re}(c-a-b) > 0$.

- Gauss's second summation theorem [1, P. 10, Eq. 2]

$$ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) & \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}. \tag{7} $$

- Elementary identity

$$ (a)_{2n} = 2^{2n}\left(\frac{1}{2}a\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n. \tag{8} $$

3 Derivation of (1)

In order to derive (1), we proceed as follows. In (3), taking $e = 2b$, we have

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a, & b, & c \\ d, & 2b & \end{matrix} ; z\right] \\ &= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_2F_1\left[\begin{matrix} a, & b \\ 2b & \end{matrix} ; zt\right] dt \\ &= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 t^{c-1}(1-t)^{d-c-1}\left(1-\frac{1}{2}zt\right)^{-a}{}_2F_1\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} & \end{matrix} ; \left(\tfrac{zt}{2-zt}\right)^2\right] dt, \end{aligned} $$

where the second equality is obtained by using (4) and replacing y by $\frac{1}{2}zt$.

Expressing the ${}_2F_1$ involved in the process as a series and changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series in the interval $(0,1)$, we have, after a little algebra

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a, & b, & c \\ d, & 2b & \end{matrix} ; z\right] \\ &= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}a\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n}{\left(b+\frac{1}{2}\right)_n n!}\left(\frac{z}{2}\right)^{2n}\int_0^1 t^{c+2n-1}(1-t)^{d-c-1}\left(1-\frac{1}{2}zt\right)^{-(a+2n)} dt, \end{aligned} $$

which, by using (5) and simplification, is

$$ \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}a\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n (c)_{2n}}{\left(b+\frac{1}{2}\right)_n (d)_{2n}\, n!}\left(\frac{z}{2}\right)^{2n}{}_2F_1\left[\begin{matrix} a+2n, & c+2n \\ d+2n & \end{matrix} ; \frac{z}{2}\right]. $$

Now, interchanging b and c and taking $d = \frac{1}{2}(a+b+1)$, we have

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c & \end{matrix} ; z\right]
= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n (b)_{2n}}{\left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)_{2n} n!} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n, & b+2n \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2n & \end{matrix} ; \frac{z}{2}\right].
$$

Taking $z = 1$, we have

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c & \end{matrix} ; 1\right]
= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n (b)_{2n}}{\left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)_{2n} n!\, 2^{2n}} {}_2F_1\left[\begin{matrix} a+2n, & b+2n \\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2n & \end{matrix} ; \frac{1}{2}\right],
$$

which, by (7) and (8) and after simplification, is

$$
\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_n \left(\frac{1}{2}b\right)_n}{\left(c+\frac{1}{2}\right)_n n!}.
$$

Summing up the series, we have

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c & \end{matrix} ; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} {}_2F_1\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}b \\ c+\frac{1}{2} & \end{matrix} ; 1\right]
$$

using (6), we finally arrive at (1).

This completes the proof of (1).

Acknowledgement. The author is supported by the Sultan Qaboos University - OMAN (research grant IG/SCI/DOMS/10/03).

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