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Stability And Stabilizability For Means*

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Abstract

New concepts for means, namely the stability and stabilizability, are introduced in this paper. Applications of these new notions for standard means are provided. At the end, open problems derived from the present work are stated.

1 Introduction

In the recent past, theory of means with scalar, operator and functional arguments has extensive development and interesting applications. Such theory is an useful tool in the theoretical point of view as well as for practical purposes.

The aim of this work is to introduce new concepts for means which we call stability and stabilizability. These notions appeared to us in writing a recent paper about the identric mean involving convex functional variables [10]. The original idea for introducing these notions is explained in the following way. Let a and b be two positive real numbers with $a \leq b$ and define a mean $m(a, b)$ of a and b. Then we have $a \leq m(a, b) \leq b$, and we can repeat the same by taking the mean of a and $m(a, b)$ in a first part and that of $m(a, b)$ and b in a second part. It follows that

$$a \leq m\left(a, m(a, b)\right) \leq m(a, b) \leq m\left(m(a, b), b\right) \leq b.$$

Now, if we consider the positive real numbers $m\left(a, m(a, b)\right)$ and $m\left(m(a, b), b\right)$ and we take their mean, a natural question then arises from this situation:

Question 1. Is it true that

$$m(a, b) = m\Big(m\big(a, m(a, b)\big), m\big(m(a, b), b\big)\Big)? \tag{1}$$

The answer is clearly affirmative for the arithmetic mean A, but no immediate answer for the geometric, harmonic, logarithmic and identric means (which we denoted by G, H, L, I, respectively, and we will recall their definitions in the next section).

A mean m which satisfies (1) will be called a stable mean. We prove that the three familiar means A, G and H are stable while L and I are not stable.

Now, if in the above three steps the taken mean in each iteration is not necessary the same, then the above question remains more difficult. Precisely, let m_1, m_2 and m_3

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be the taken means in the first, second and third step respectively, then we have the following inequalities

$$a \le m_2\Big(a, m_1(a,b)\Big) \le m_3\Big(m_2\big(a, m_1(a,b)\big), m_2\big(m_1(a,b), b\big)\Big) \le m_2\Big(m_1(a,b), b\Big) \le b, \tag{2}$$

$$a \le m_2\Big(a, m_1(a,b)\Big) \le m_2(a,b) \le m_2\Big(m_1(a,b), b\Big) \le b, \tag{3}$$

$$a \le m_2\Big(a, m_1(a,b)\Big) \le m_1(a,b) \le m_2\Big(m_1(a,b), b\Big) \le b. \tag{4}$$

Let us observe the above inequalities in the aim to introduce the following questions.

Question 2. According to (2) and (3), for which means m_1, m_2, m_3 one has

$$m_3\Big(m_2\big(a, m_1(a,b)\big), m_2\big(m_1(a,b), b\big)\Big) = m_2(a,b)? \tag{5}$$

If (5) is satisfied, with m_1 and m_3 stable means, then we say that m_2 is (m_3, m_1)-stabilizable. With this, we will show that the logarithmic mean L is simultaneously (H, A)-stabilizable and (A, G)-stabilizable while the identric mean I is (G, A)-stabilizable.

Question 3. By virtue of (2) and (4), is it possible to have

$$m_3\Big(m_2\big(a, m_1(a,b)\big), m_2\big(m_1(a,b), b\big)\Big) = m_1(a,b)? \tag{6}$$

If (6) holds, the mean m_1 will be called (m_3, m_2)-stabilized. As particular examples, we prove that the geometric mean G is (A, H)-stabilized and (H, A)-stabilized, while the Heron mean H_e, $H_e := (1/2)A + (1/2)G$, is (A, G)-stabilized.

Our above concepts will also be applied for a game of power means. For instance, we will prove that, for all real number p, the power binomial mean B_p is stable, the power logarithmic mean L_p is (B_p, A)-stabilizable, the power difference mean D_p is (A, B_p)-stabilizable, the power exponential mean I_p is (G, B_p)-stabilizable and the second power logarithmic mean l_p is (B_p, G)-stabilizable (for the definition of these power means, see the next section). At the end, we state a list of open problems as purposes for future research.

2 Background Material

Throughout this paper, we understand by mean a binary map m between positive real numbers satisfying the following statements:

(i) $m(a, a) = a$, for all $a > 0$;
(ii) $m(a, b) = m(b, a)$, for all $a, b > 0$;
(iii) $m(ta, tb) = tm(a, b)$, for all $a, b, t > 0$;
(iv) $m(a, b)$ is an increasing function in a (and in b);
(v) $m(a, b)$ is a continuous function of a and b.

The set of all means can be equipped with a partial ordering, called point-wise order, defined by: $m_1 \le m_2$ if and only if $m_1(a, b) \le m_2(a, b)$ for every $a, b > 0$.

Two trivial means are $(a,b) \longmapsto \min(a,b)$ and $(a,b) \longmapsto \max(a,b)$, and every mean m satisfies the following

$$\min(a,b) \leq m(a,b) \leq \max(a,b),$$

for all $a,b > 0$. We denote by min and max the above trivial means which we call lower and upper means, respectively.

The standard examples of means satisfying the above requirements are recited in the following.

- Arithmetic mean, $A(a,b) = \frac{a+b}{2}$;
- Geometric mean, $G(a,b) = \sqrt{ab}$;
- Harmonic mean, $H(a,b) = \frac{2ab}{a+b}$;
- Logarithmic mean, $L(a,b) = \frac{b-a}{ln\ b - ln\ a}$, $\ L(a,a) = a$;
- Identric mean, $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}$, $\ I(a,a) = a$;
- Quadratic mean, $K(a,b) = \sqrt{\frac{a^2+b^2}{2}}$.

As is well known, these means satisfy the following inequalities

$$\min \leq H \leq G \leq L \leq I \leq A \leq K \leq \max.$$

For a given mean m, we set

$$m^*(a,b) = \Big(m\big(a^{-1}, b^{-1}\big)\Big)^{-1},$$

and it is easy to see that m^* is also a mean, called the dual mean of m. The symmetry and homogeneity axioms (ii),(iii) yield

$$m^*(a,b) = \frac{ab}{m(a,b)}.$$

It is easy to see that every mean m satisfies $m^{**} = m$, and if m_1 and m_2 are two means such that $m_1 \leq m_2$ then $m_1^* \geq m_2^*$. A mean m is called self-dual if $m^* = m$. It is clear that the arithmetic and harmonic means are mutually dual and the geometric mean is the unique self-dual mean. The dual of the logarithmic mean is given by

$$L^*(a,b) = ab\frac{\ln b - \ln a}{b-a}, \quad L^*(a,a) = a;$$

while that of the identric mean is

$$I^*(a,b) = e\left(\frac{a^b}{b^a}\right)^{1/(b-a)}, \quad I^*(a,a) = a.$$

From the above, the following inequalities are immediate:

$$\min \le K^* \le H \le I^* \le L^* \le G \le L \le I \le A \le K \le \max .$$

There are many families of means, called power means, which extend the above standard ones. For instance, let p be a given real number, we recall the following:

- Power Binomial mean defined by

$$B_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}.$$

This includes some of the most familiar means in the sense that:

$$\begin{aligned} B_{-\infty} \quad &: \quad = \lim_{p\to-\infty} B_p = \min,\ B_{-1} = H,\ B_1 = A, \\ B_0 \quad &: \quad = \lim_{p\to 0} B_p = G,\ B_\infty := \lim_{p\to+\infty} B_p = \max . \end{aligned}$$

Notice that

$$B_{1/2}(a,b) = \frac{1}{2}\frac{a+b}{2} + \frac{1}{2}\sqrt{ab} := H_e(a,b)$$

is called the Heron mean. However, it is obvious that $B_p^* = B_{-p}$ for all real number p.

- Power Logarithmic mean given by

$$L_p(a,b) = \left(\frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right)^{1/p} = \left(\frac{1}{b-a}\int_a^b t^p dt\right)^{1/p},\ \ L_p(a,a) = a.$$

Some particular special values of p are understood as:

$$L_{-\infty} = \min,\ L_{-2} = G,\ L_{-1} = L,\ L_0 = I,\ L_1 = A,\ L_\infty = \max .$$

Further, the following inequalities are immediate

$$B_p(a,b) \le L_p(a,b) \text{ for } p \le 1, \text{ and } L_p(a,b) \le B_p(a,b) \text{ for } p \ge 1.$$

- Power Difference mean given by

$$D_p(a,b) = \frac{p}{p+1}\frac{a^{p+1} - b^{p+1}}{a^p - b^p},\ \ D_p(a,a) = a.$$

As particular cases of interest, we cite the following:

$$D_{-\infty} = \min,\ D_{-2} = H,\ D_{-1} = L^*,\ D_{-1/2} = G,\ D_0 = L,\ D_1 = A,\ D_\infty = \max .$$

- Power Exponential mean defined by

$$I_p(a,b) = \exp\left(-\frac{1}{p} + \frac{a^p \ln a - b^p \ln b}{a^p - b^p}\right).$$

Particular cases are here included as well:

$$I_{-\infty} = \min,\ I_{-1} = I^*,\ I_0 = G,\ I_1 = I,\ I_{+\infty} = \max .$$

- Second Power Logarithmic mean given by

$$l_p(a,b) = \left(\frac{1}{p}\frac{b^p - a^p}{\ln b - \ln a}\right)^{1/p}.$$

In particular we have:

$$l_{-\infty} = \min,\ l_{-1} = L^*,\ l_0 = G,\ l_1 = L,\ l_{+\infty} = \max.$$

3 Resultant Mean-Map

We start this section by stating some basic notions that will be needed in the sequel. We denote by $\mathcal{M}$ the convex set of all binary means (satisfying the above axioms (i)-(v)). For integer $k \geq 1$, we consider a map $\Phi : \mathcal{M}^k \longrightarrow \mathcal{M}$ i.e. Φ is a map with mean-variables (which we call mean-map). We say that Φ is point-wise convex (in short p-convex) if the following mean-inequality,

$$\Phi\Big((1-t)M_1 + tM_2\Big) \leq (1-t)\Phi(M_1) + t\Phi(M_2),$$

with respect to the point-wise ordering, holds true for every real number $t \in [0,1]$ and all mean-vectors $M_1 = (m_1, m_2, ..., m_k) \in \mathcal{M}^k$ and $M_2 = (m_1^{'}, m_2^{'}, ..., m_k^{'}) \in \mathcal{M}^k$. We say that Φ is p-concave if the above inequality is reversed. A mean-map simultaneously p-convex and p-concave, that is the above inequality is an equality, will be called p-affine. Now, let us observe the following examples in the aim to illustrate the above notions.

EXAMPLE 1. With the above, let $k = 1$ and the mean-map $m \longmapsto m^*$, where m^* is the dual of m. It is not hard to verify that this mean-map is p-convex, that is,

$$\Big((1-t)m_1 + tm_2\Big)^* \leq (1-t)m_1^* + tm_2^*$$

holds for all $t \in [0,1]$ and all means m_1 and m_2.

EXAMPLE 2. Let $k = 3$ and consider the mean-map $\mathcal{C}$ defined by

$$\mathcal{C}(m_1, m_2, m_3) = m_1(m_2, m_3),$$

in the sense that we have

$$\mathcal{C}(m_1, m_2, m_3)(a,b) = m_1\Big(m_2(a,b), m_3(a,b)\Big)$$

for all $a, b > 0$. As particular examples, we can take $\mathcal{C}(A, m_1, m_2) = (1/2)(m_1 + m_2)$ and $\mathcal{C}(G, m_1, m_2) = (m_1 m_2)^{1/2}$. Otherwise, it is easy to see that, for fixed means m_1 and m_2, the mean-map $m \longmapsto \mathcal{C}(m, m_1, m_2)$ is p-affine.

We left to the reader the routine task of formulating some other examples of mean-maps. However, a third example of mean-map having good properties and applications is our aim in what follows.

DEFINITION 1. Let m_1, m_2, m_3 be given means. For all $a, b > 0$, define

$$\mathcal{R}(m_1, m_2, m_3)(a, b) = m_1\Big(m_2\big(a, m_3(a, b)\big), m_2\big(m_3(a, b), b\big)\Big),$$

which we call the resultant mean-map of m_1, m_2 and m_3.

Following the above, the map $\mathcal{R}$ is a mean-map with three mean-variables. The elementary properties of $\mathcal{R}$ are summarized in the following results.

PROPOSITION 1. With the above, the following assertions are met:
(i) The map $(a, b) \longmapsto \mathcal{R}(a, b) := \mathcal{R}(m_1, m_2, m_3)(a, b)$ defines a mean, with the following inequalities

$$min(a, b) \leq m_2\Big(a, m_3(a, b)\Big) \leq \mathcal{R}(a, b) \leq m_2\Big(m_3(a, b), b\Big) \leq max(a, b).$$

(ii) For every means m_1, m_2, m_3 we have

$$\Big(\mathcal{R}(m_1, m_2, m_3)\Big)^* = \mathcal{R}(m_1^*, m_2^*, m_3^*).$$

(iii) The mean-map $\mathcal{R}$ is point-wisely increasing with respect to each of its mean-variables, that is,

$$\Big(m_1 \leq m_1^{'},\ m_2 \leq m_2^{'},\ m_3 \leq m_3^{'},\ \Big) \Longrightarrow \mathcal{R}(m_1, m_2, m_3) \leq \mathcal{R}(m_1^{'}, m_2^{'}, m_3^{'}).$$

PROOF. (i) It is not hard to verify that $\mathcal{R}(m_1, m_2, m_3)$ satisfies the axioms (i)-(v) of a binary mean. Further, if $a \leq b$ then $a \leq m_3(a, b) \leq b$ and so $m_2\big(a, m_3(a, b)\big) \leq m_2\big(m_3(a, b), b\big)$. Then, the desired inequalities follow by the monotonicity axiom (iv) of m_1.

(ii) Apply the definition of $\mathcal{R}$ and that of the dual-mean. The desired result follows after simple manipulations.

(iii) It is immediate from the definition of $\mathcal{R}$ with the monotonicity axiom (iv) for m_1, m_2 and m_3. The details are omitted for the reader.

Now, we will illustrate the above with some examples for computing $\mathcal{R}(m_1, m_2, m_3)$, when m_1, m_2 and m_3 belong to the set of the familiar means. Such computations are straightforward and so the details are omitted.

EXAMPLE 3. It is obvious that $\mathcal{R}(A, A, A) = A$. By simple computations, we verify that $\mathcal{R}(G, G, G) = G$, $\mathcal{R}(H, H, H) = H$ and $\mathcal{R}(A, H, G) = \mathcal{R}(H, A, G) = G$. Routine computations yield $\mathcal{R}(H, L, A) = \mathcal{R}(A, L, G) = L$ and $\mathcal{R}(G, I, A) = I$. As we will see later, these relationships will be interpreted in a general point of view.

EXAMPLE 4. Simple computations lead to

$$\mathcal{R}(H, H, A) = \left(\frac{1}{2}A + \frac{1}{2}H\right)^* = \frac{2AH}{A + H}, \quad \mathcal{R}(H, A, A) = \frac{3}{4}A + \frac{1}{4}H; \tag{7}$$

$$\mathcal{R}(A, G, G) = \left(\frac{1}{2}AG + \frac{1}{2}G^2\right)^{1/2}, \quad \mathcal{R}(A, A, G) = \frac{1}{2}A + \frac{1}{2}G; \tag{8}$$

$$\mathcal{R}(G,G,A)=\sqrt{AG},\quad \mathcal{R}(G,A,A)=\left(\frac{3}{4}A^2+\frac{1}{4}G^2\right)^{1/2}. \tag{9}$$

EXAMPLE 5. Let m_1, m_2, m be given means, it is not hard to verify the next relationships

$$\mathcal{R}(m_1,G,m_2)(a,b)=m_1(\sqrt{a},\sqrt{b})\,(m_2(a,b))^{1/2};$$

$$\mathcal{R}(m_1,m_2,G)(a,b)=m_1(\sqrt{a},\sqrt{b})m_2(\sqrt{a},\sqrt{b});$$

$$\mathcal{R}(B_p,B_p,m)=B_p(B_p,m):=\left(\frac{B_p^p+m^p}{2}\right)^{1/p}.$$

The following result, whose the proof is straightforward, may be stated as well:

PROPOSITION 2. For all mean M, the mean-map $m\longmapsto\mathcal{R}(A,m,M)$ is p-affine, that is, the following equality

$$\mathcal{R}\left(A,(1-t)m+tm^{'},M\right)=(1-t)\mathcal{R}(A,m,M)+t\mathcal{R}(A,m^{'},M)$$

holds for all real number $t\in[0,1]$ and all means $m, m^{'}$.

4 Stable and Stabilizable Means

As already pointed before, the present section is devoted to introduce the stability and stabilizability concepts. We may start with the following.

DEFINITION 2. A mean m is said to be stable if $\mathcal{R}(m,m,m)=m$.

The above definition, when combined with Proposition 1 (ii) immediately gives the following result.

PROPOSITION 3. If m is a stable mean then so is m^*, that is,

$$\mathcal{R}(m,m,m)=m\Longleftrightarrow\mathcal{R}(m^*,m^*,m^*)=m^*.$$

It is easy to see that the lower and upper means are stable. Other nontrivial examples of stable means are given in the following result.

THEOREM 1. For all real number p, the power binomial mean B_p is stable. In particular, the arithmetic, geometric, harmonic and quadratic means are stable.

PROOF. According to the explicit form of B_p, we easily verify that $\mathcal{R}(B_p,B_p,B_p)=B_p$ for all $p\neq0$. Letting $p=0$, in the sense $p\to0$, with an argument of continuity, we deduce the stability for G. Taking particular values of p, $p=1, p=-1, p=1/2$, respectively, we obtain the stability of A, H and K. The proof is complete.

The reader can easily verify by a counter-example that the logarithmic and identric means are not stable. For this, we may state another concept as recited in the following.

DEFINITION 3. A mean m is said to be: (i) Stabilized if there exist two nontrivial stable means m_1 and m_2 such that $\mathcal{R}(m_1,m_2,m)=m$. In this case we say that m is (m_1,m_2)-stabilized. (ii) Stabilizable if there exist two nontrivial stable means m_1

and m_2 satisfying the relation $\mathcal{R}(m_1, m, m_2) = m$. We then say that m is (m_1, m_2)-stabilizable.

In the above definition, the fact that the means m_1 and m_2 are nontrivial is necessary, because if not, the definition will be without any interest since the relations $\mathcal{R}(min, max, m) = m$ and $\mathcal{R}(min, m, max) = m$ are valid for every mean m. Otherwise, it is clear that every nontrivial stable mean m is (m, m)-stabilizable and (m, m)-stabilized.

The following result may be stated as well:

PROPOSITION 4. If m is (m_1, m_2)-stabilizable (resp., stabilized) then m^* is (m_1^*, m_2^*)-stabilizable (resp., stabilized).

PROOF. Follows from the definition with Proposition 1 (ii).

As example of stabilized mean we can derive the following result.

THEOREM 2. For all nontrivial mean m, the geometric mean G is (m, m^*)-stabilized, that is to say, $\mathcal{R}(m, m^*, G) = G$.

PROOF. According to the second relationship of Example 5 we immediately obtain, for all $a, b > 0$

$$\mathcal{R}(m, m^*, G)(a, b) = m\Big(\sqrt{a}, \sqrt{b}\Big)m^*\Big(\sqrt{a}, \sqrt{b}\Big).$$

This, with the fact that $m(a, b)m^*(a, b) = ab$ for all $a, b > 0$, yields

$$\mathcal{R}(m, m^*, G)(a, b) = \sqrt{ab} = G(a, b),$$

so proving the desired result.

The above theorem implies again that G is stable. Also, G is simultaneously (A, H)-stabilized and (H, A)-stabilized. We left to the reader to verify that the Heron mean H_e is (A, G)-stabilized.

In what follows, we will be interested by the stabilizability concept which, as we will see, allows us to obtain good links between the standard means.

THEOREM 3. For all real number p, the power logarithmic mean L_p is (B_p, A)-stabilizable, that is, the following equality

$$\mathcal{R}(B_p, L_p, A) = L_p$$

holds for every real number p.

PROOF. According to the definition of $\mathcal{R}$ and the integral form of L_p we obtain, for all $a, b > 0$ and $p \neq 0$,

$$\begin{aligned}
\mathcal{R}(B_p, L_p, A)(a, b) &= B_p\left(L_p\left(a, \frac{a+b}{2}\right), L_p\left(\frac{a+b}{2}, b\right)\right) \\
&= B_p\left(\left(\frac{2}{b-a}\int_a^{(a+b)/2} t^p dt\right)^{1/p}, \left(\frac{2}{b-a}\int_{(a+b)/2}^{b} t^p dt\right)^{1/p}\right).
\end{aligned}$$

This, with the explicit form of B_p and a simple manipulation, gives the desired result for $p \neq 0$. If $p = 0$, the same equality holds with an argument of continuity, so completes the proof.

COROLLARY 1. For all real number p, the dual power logarithmic mean L_p^* is (B_{-p}, H)-stabilizable.

PROOF. Follows from Theorem 3 with Proposition 1 (ii) and the fact that $B_p^* = B_{-p}$ for all real number p.

As particular cases of interest, we can derive the following result.

COROLLARY 2. The logarithmic mean L is (H, A)-stabilizable while the identric mean I is (G, A)-stabilizable. The dual logarithmic mean L^* is (A, H)-stabilizable and the dual identric mean I^* is (G, H)-stabilizable.

PROOF. Setting $p = -1$, with $L_{-1} = L$, $B_{-1} = H$, Theorem 3 gives the first part of the corollary. Letting $p \to 0$, with $L_0 = I$, $B_0 = G$ and an argument of continuity, we obtain the second part. The rest of the proof follows from the above with Proposition 1 (ii), so completes the proof.

THEOREM 4. For all real number p, the power difference mean D_p is (A, B_p)-stabilizable.

PROOF. Writing $D_p(a, b)$ in the following form

$$D_p(a,b) = \frac{1}{b^p - a^p} \int_{a^p}^{b^p} t^{1/p} dt,$$

and by similar arguments as in the proof of Theorem 3, we deduce the desired result. We omit the routine details.

Similarly to Corollary 1 and Corollary 2, we obtain the next result.

COROLLARY 3. For all real number p, the dual power difference mean D_p^* is (H, B_{-p})-stabilizable.

Taking particular values of p, we may obtain the following.

COROLLARY 4. The logarithmic mean L is (A, G)-stabilizable and so L^* is (H, G)-stabilizable.

PROOF. Taking $p = 0$ ($p \to 0$) in the above theorem we deduce the desired result for L with help of an argument of continuity. This, with Proposition 1 (ii), gives the result for L^*.

REMARK 1. Corollary 2 and Corollary 4 imply that the logarithmic mean L is simultaneously (H, A)-stabilizable and (A, G)-stabilizable, and, L^* is simultaneously (A, H)-stabilizable and (H, G)-stabilizable. We deduce that a given mean m can be (m_1, m_2)-stabilizable and $(m_1^{'}, m_2^{'})$-stabilizable for distinct couples (m_1, m_2) and $(m_1^{'}, m_2^{'})$ of stable means. The reverse situation is an open question, see the next section below.

THEOREM 5. The power exponential mean I_p is (G, B_p)-stabilizable.

PROOF. Here, we write $I_p(a, b)$ in the next form

$$I_p(a,b) = \exp\left(\frac{1}{p(b^p - a^p)} \int_{a^p}^{b^p} \ln t \, dt \right),$$

and the proof is similar to that of the above theorems.

THEOREM 6. The second power logarithmic mean l_p is (B_p, G)-stabilizable.

PROOF. We write $l_p(a,b)$ in the following integral form

$$l_p(a,b) = \left(\frac{1}{\ln b - \ln a} \int_a^b t^{p-1} dt \right)^{1/p}.$$

By definition we have

$$\begin{aligned} & \mathcal{R}(B_p, l_p, G)(a,b) \\ = \; & B_p \left(l_p(a, \sqrt{ab}), l_p(\sqrt{ab}, b) \right) \\ = \; & B_p \left(\left(\frac{2}{\ln b - \ln a} \int_a^{\sqrt{ab}} t^{p-1} dt \right)^{1/p}, \left(\frac{2}{\ln b - \ln a} \int_{\sqrt{ab}}^b t^{p-1} dt \right)^{1/p} \right). \end{aligned}$$

Using the explicit form of B_p we deduce the desired result after a simple reduction.

REMARK 2. Taking $p = -2$ in Theorem 4 we deduce that the harmonic mean H is (A, K^*)-stabilizable. Setting $p = -1/2$ in the same theorem we find that the geometric mean G is (A, H_e^*)-stabilizable, and by Proposition 1 (ii) we deduce that G is also (H, H_e)-stabilizable. In fact, these results are without any greatest interest in a practical context, since the expression of H (resp., G) is more simple than that of K^* (resp., H_e). However, in the theoretical context this shows that the above Remark 1 holds true even for a stable mean m.

Summarizing the above, the stabilizability notion of the standards means L, L^*, I, I^* turns out of the stable means A, G, H. Further, we remark that A, G, H have simple (algebraic) forms while L, L^*, I, I^* are with transcendant expressions. Moreover, the stabilizability of the power means B_p, L_p, D_p, I_p, l_p (and their associate dual means) turns out of the stable power binomial mean B_p. This allows us to arise the following question: Is it possible to characterize a (m_1, m_2)-stabilizable mean m in terms of its related couple (m_1, m_2)? For more details, see the section below.

We end this section by noticing that, to find a mean m which is (m_1, m_2)-stabilizable can be generally reduced to a functional equation. Indeed, assume that we would like to search a mean m that is (A, G)-stabilizable. Following Corollary 4, this mean m is the logarithmic mean L. However, starting from the definition, that is $\mathcal{R}(A, m, G) = m$, we obtain after reduction

$$\forall a, b > 0 \quad m(a, \sqrt{ab}) + m(\sqrt{ab}, b) = 2m(a,b).$$

According to the homogeneity axiom, it is sufficient to research $m(a,1)$ for every $a > 0$, and the above relationship is then reduced to the following one

$$\forall a > 0 \quad \sqrt{a} m(\sqrt{a}, 1) + m(\sqrt{a}, 1) = 2m(a,1),$$

or again

$$\forall a > 0 \quad (\sqrt{a} + 1) m(\sqrt{a}, 1) = 2m(a,1).$$

Our problem can be transformed into the following: Prove or disprove the existence of an increasing continuous function f on $]0,+\infty[$ such that

$$\forall x>0 \quad f(x)>0,\ 2f(x^2)=(x+1)f(x),\ f(1)=1. \tag{10}$$

Further, we can ask if it is possible to explicit $f(x)$.
Following Corollary 4, the function $f(x)=\frac{x-1}{\ln x}$, with $f(1)=1$, is a solution of (10). The question "how to obtain directly the set of all solutions of (10)" is not obvious and appears to be interesting. We conjecture that equation (10) has only the above solution, or in another way, we conjecture that L is the unique mean which is (A,G)-stabilizable. In a general point of view, see section below (Conjecture 2).

5 Some Extensions and Open Problems

In this section, we display some extensions of the above notions and results from the case that the means are with scalar variables to the case that the means are with operator or functional variables. To not lengthen this section, we just mention that the stability and stabilizability notions, already stated for scalar means, can be easily extended in a similar manner for means with operator or functional arguments.

The present section will be divided into subsections as we organize in the following.

5.1 Complements

In what previous, we have seen that, the logarithmic mean L is (H,A)-stabilizable and (A,G)-stabilizable, its dual L^* is (A,H)- stabilizable and (H,G)-stabilizable, with the inequalities $H\le L^*\le G\le L\le A$. The identric mean I is (G,A)-stabilizable, its dual I^* is (G,H)-stabilizable, with $H\le I^*\le G\le I\le A$. Our first open problem can be conjectured as follows.

Conjecture 1. Let m_1 and m_2 be two means such that $m_1\le m_2$. If m is a (m_1,m_2)-stabilizable mean (resp., (m_2,m_1)-stabilizable mean) then we have $m_1\le m\le m_2$.

Furthermore, we have seen that a given mean m can be simultaneously (m_1,m_2)-stabilizable and $(m_1^{'},m_2^{'})$-stabilizable for different couples (m_1,m_2) and $(m_1^{'},m_2^{'})$ of stables means. For the reverse statement, we haven't any affirmative answer and we put our second conjecture.

Conjecture 2. Let m_1 and m_2 be two means such that $m_1\le m_2$. Then, there exists one and only one mean m which is (m_1,m_2)-stabilizable and satisfying that $m_1\le m\le m_2$.

Conversely, we can ask if every given mean is stabilizable? We may put the following.

Conjecture 3. There are many means which are not stabilizable. For instance, we think that the following mean

$$P(a,b) = \frac{b-a}{4\arctan\sqrt{b/a}-\pi}, \quad P(a,a) = a,$$

known in the literature as the Seiffert's mean [11], is not stabilizable.

5.2 Link between Cross Means and Stable Means

In [9] the author introduced the notion of cross means as recalled in the following.

DEFINITION 3. Let m_1 and m_2 be two means. The tensor product of m_1 and m_2 is the map $m_1 \otimes m_2$ defined by

$$\forall a,b,c,d > 0 \qquad m_1 \otimes m_2(a,b,c,d) = m_1\Big(m_2(a,b), m_2(c,d)\Big).$$

A binary mean m will be called cross mean if $m^{\otimes 2} := m \otimes m$ is a mean with four variables, that is,

$$\forall a,b,c,d > 0 \qquad m^{\otimes 2}(a,b,c,d) = m^{\otimes 2}(a,c,b,d).$$

The author proved [9] that the power binomial mean B_p is a cross mean while the power logarithmic mean L_p and difference mean D_p are not always cross means. Further, he proved that L_p and D_p can be approached by recursive algorithms involving the cross mean B_p. The notion of cross mean is stronger than the stability notion as proved in the following.

THEOREM 7. Every cross mean is a stable mean.

PROOF. Let m be a mean, we have by definition

$$\mathcal{R}(m,m,m)(a,b) = m\Big(m\big(a,m(a,b)\big), m\big(b,m(a,b)\big)\Big) = m^{\otimes 2}\Big(a, m(a,b), b, m(a,b)\Big).$$

If m is a cross mean, it becomes that

$$\mathcal{R}(m,m,m)(a,b) = m^{\otimes 2}\Big(a,b,m(a,b),m(a,b)\Big) = m\Big(m(a,b),m(a,b)\Big) = m(a,b),$$

which concludes the proof.

However, we don't know if the reverse of the above theorem is true. Precisely, we put the following.

Problem 1. Prove or disprove that every stable mean is a cross mean.

5.3 Means of Order (p,q)

Let $p,q \in \mathbb{R}$ and $a,b > 0$. We recall the following.

- The Stolarsky mean $E_{p,q}(a,b)$ of order (p,q) of a and b is defined by [12, 13].

$$E_{p,q}(a,b) = \left(\frac{p}{q}\frac{b^q - a^q}{b^p - a^p}\right)^{1/(q-p)}, \quad E_{p,q}(a,a) = a.$$

  This includes some of the most familiar cases in the sense

$$E_{p,p}(a,b) = \exp\left(-\frac{1}{p} + \frac{a^p \ln a - b^p \ln b}{a^p - b^p}\right), \quad E_{p,0}(a,b) = \left(\frac{1}{p}\frac{b^p - a^p}{\ln b - \ln a}\right)^{1/p}$$

  if $p \neq 0$, with $E_{0,0}(a,b) = G(a,b)$. The mean $E_{p,q}$ extends the power binomial, logarithmic and difference means, since the following relations

$$E_{p,2p} = B_p, \; E_{1,p+1} = L_p, \; E_{p,p+1} = D_p$$

  hold for all real number p.

- The Gini mean $G_{p,q}(a,b)$ of order (p,q) of a and b is defined by [3]

$$G_{p,q}(a,b) = \left(\frac{a^q + b^q}{a^p + b^p}\right)^{1/(q-p)}.$$

  Clearly, $G_{0,p} = B_p$ for all real number p. Now, we are in position to put the following.

Problem 2. Determine the set of all couples (p,q) such that the mean $E_{p,q}$ (resp., $G_{p,q}$) is stable (resp., stabilizable).

5.4 Operator Means

The extension of means from the case that the variables are positive real numbers to the case that the variables are positive operators has extensive several developments, see [7] and the related references cited therein. Let T and S be two positive operators acting from a Hilbert space E into its self. We recall that the arithmetic and harmonic operator means are, respectively, defined by

$$\mathcal{A}(T,S) = \frac{T+S}{2}, \quad \mathcal{H}(T,S) = \Big(\mathcal{A}\big(T^{-1}, S^{-1}\big)\Big)^{-1} = 2\Big(T^{-1} + S^{-1}\Big)^{-1},$$

with $T^{-1} = \lim_{\epsilon \downarrow 0} \big(T + \epsilon I\big)^{-1}$ for the sake of convenience.

In the literature, there are two analogues of the scalar geometric mean $G(a,b) := \sqrt{ab}$ for operator variables. The first is the monotone geometric operator mean defined by

$$\mathcal{G}(T,S) = T^{1/2}\Big(T^{-1/2} S T^{-1/2}\Big)^{1/2} T^{1/2},$$

and the second is the chaotic geometric operator mean given by

$$\mathcal{CG}(T,S) = \exp\Big(\frac{1}{2}\log T + \frac{1}{2}\log S\Big).$$

As well known, $\mathcal{CG}(T,S)$ and $\mathcal{G}(T,S)$ are in general different.

Analogously to the standard case, the power binomial operator mean $\mathcal{B}_p$ is defined by

$$\mathcal{B}_p(T,S)=\Big(\frac{T^p+S^p}{2}\Big)^{1/p}.$$

This family includes the following operator means,

$$\mathcal{B}_1(T,S)=\mathcal{A}(T,S),\ \mathcal{B}_{-1}(T,S)=\mathcal{H}(T,S),\ \mathcal{B}_0(T,S)=\lim_{p\to 0}\mathcal{B}_p(T,S)=\mathcal{CG}(T,S).$$

THEOREM 7. For all real number p, the binomial operator mean $\mathcal{B}_p$ is stable. In particular, the arithmetic, harmonic and chaotic geometric operator means are stable.

PROOF. Similar to that of the above scalar case. We omit the routine details.

Throughout the above, we have seen that the scalar geometric mean G is stable, (H,H_e)-stabilizable and (A,H_e^*)-stabilizable. Since $\mathcal{G}$ is an extension of the scalar geometric mean G, it is natural to put the following.

Problem 3. *Prove or disprove that the monotone geometric operator mean $\mathcal{G}$ is stable or stabilizable.*

The power logarithmic operator mean, extending the scalar one, is defined by,

$$\mathcal{L}_p(T,S)=\Big(\int_0^1\big((1-t)T+tS\big)^p dt\Big)^{1/p}.$$

Clearly, $\mathcal{L}_1(T,S)=\mathcal{A}(T,S)$. Further, more familiar operator means are here included, in particular the following:

$$\mathcal{L}_{-1}(T,S)=\Big(\int_0^1\big((1-t)T+tS\big)^{-1} dt\Big)^{-1}:=\mathcal{L}(T,S)$$

called the monotone logarithmic operator mean, and

$$\mathcal{L}_0(T,S)=\exp\int_0^1 log\big((1-t)T+tS\big)dt:=\mathcal{I}(T,S)$$

called the chaotic identric operator mean. For some details about this class of operator means, we refer the reader to [7] and the references cited therein.

After this, and following our above study, it is natural to put the following problem.

Problem 4. Prove or disprove that the power logarithmic operator mean L_p is (B_p,A)-stabilizable.

5.5 Functional Means

Recently, some functional means have been introduced in the literature. Such functional means extend the operator ones in the sense that if $\mathcal{M}(T,S)$ is an operator mean, its extension $\mathbb{M}$ for means with functional variables satisfies the following relationship, see [1, 2, 4, 5, 6, 8]

$$\mathbb{M}(f_T,f_S)=f_{\mathcal{M}(T,S)},$$

where the notation f_T refers to the quadratic function associated to the operator T defined from a Hilbert E into itself, i.e. $f_T(x) = (1/2) < Tx, x >$ for all $x \in E$.

The convex functional means (also called functional means in convex analysis) have been introduced at the first time by Atteia-Raïssouli in [1]. Throughout in what follows, let $f, g : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that $dom\ f \cap dom\ g \neq \emptyset$, where $dom\ f$ stands for the effective domain of f defined by $dom\ f = \{x \in E,\ f(x) < +\infty\}$. The arithmetic and harmonic functional means of f and g are, respectively, defined by

$$\mathbb{A}(f,g) = \frac{f+g}{2}, \quad \mathbb{H}(f,g) = \Big(\frac{1}{2}f^* + \frac{1}{2}g^*\Big)^*,$$

where f^* denotes the Legendre-Fenchel conjugate of f defined as follows

$$\forall x^* \in E \qquad f^*(x^*) := \sup_{x \in E}\{< x^*, x > -f(x)\}.$$

THEOREM 8. The arithmetic and harmonic functional means are stable.

PROOF. It is straightforward and the detail is left to the reader as a simple exercise.

The Geometric functional mean of f and g, denoted here by $\mathbb{G}$, was firstly introduced as the point-wise limit of an iterative process descending from the arithmetic and harmonic functional means. Precisely, for given functionals f and g, we recall the following algorithm, [1].

$$\alpha_{n+1}(f,g) = \frac{1}{2}\alpha_n(f,g) + \frac{1}{2}\Big(\alpha_n(f^*, g^*)\Big)^*, \ \ n \geq 0; \quad \alpha_0(f,g) = \mathbb{A}(f,g).$$

The sequence $\big(\alpha_n(f,g)\big)_n$ converges point-wisely to a convex function $\mathbb{G}(f,g)$ called the convex geometric functional mean of f and g. For the elementary properties of $\mathbb{G}(f,g)$, the reader can consult [1]. For another equivalent definition of $\mathbb{G}(f,g)$, see also [2].

We are in position to state our open problem for functional mean recited as follows.

Problem 5. Prove or disprove that the geometric functional mean G is stable or stabilizable.

The logarithmic mean of two convex functionals has been introduced by the author in [6]. This functional mean extends, at a new angle, that of positive real numbers and positive operators already stated in the literature. Precisely, let f, g be such that $dom\ f \cap dom\ g \neq \emptyset$. We put

$$\mathbb{L}(f,g) = \Big(\int_0^1 (t.f + (1-t).g)^* dt\Big)^*,$$

which is called the logarithmic functional mean of f and g in the sense of convex analysis.

Of course, the functional logarithmic mean is not stable, since it is an extension of the scalar one which is not stable. However, it is natural to state the following.

Problem 6. Prove or disprove that the logarithmic functional mean L is (H, A)-stabilizable and/or (A, G)-stabilizable.

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