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On The Gamma Function Approximation By Burnside*

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Abstract

The aim of this paper is to improve the Burnside formula for approximation the gamma function.

1 Introduction

The Euler gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

extends the factorial function and it is of great interest in many branches of science. Undoubtedly, one of the most used formula for approximation the big factorials is the following

$$\Gamma(n+1) \approx \sqrt{2\pi e} \left(\frac{n}{e}\right)^{n+\frac{1}{2}} := \sigma_n \tag{1}$$

now known as Stirling formula. Although in probabilities or statistical physics this formula is satisfactory, in pure mathematics more accurate formulas are necessary.

Recently Mortici [4] introduced the approximation

$$\Gamma(n+1) \approx \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}}, \tag{2}$$

being slightly less accurate than Burnside formula [1]

$$\Gamma(n+1) \approx \sqrt{2\pi} \left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} := \beta. \tag{3}$$

Inspired by the Lanczos integral approximations [3] and by a double series representation of Hsu [2], Mortici [4] unified the relations (1)-(2) in the following general approximations family

$$\Gamma(n+1) \approx \sqrt{2\pi e} e^{-p} \left(\frac{n+p}{e}\right)^{n+\frac{1}{2}} \quad (0 \le p \le 1). \tag{4}$$

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As the privileged values $\omega = \left(3-\sqrt{3}\right)/6$, $\zeta = \left(3+\sqrt{3}\right)/6$ provide the best results, there are proven in [4] the following sharp inequalities

$$\sqrt{2\pi e}e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}} < \Gamma\left(x+1\right) \leq \alpha \cdot \sqrt{2\pi e}e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}}$$

and

$$\delta \cdot \sqrt{2\pi e}e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{\zeta+\frac{1}{2}} \leq \Gamma\left(x+1\right) < \sqrt{2\pi e}e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{\zeta+\frac{1}{2}},$$

where $\alpha = 1.072042464...$ and $\delta = 0.988503589...$.

Other recent results about the gamma function and related functions are stated in [5]-[17].

2 The Results

In this paper we continue the direction opened by the family (4) and in particular by the Burnside approximation (3) by replacing the constant 1/2 by a quantity depending on n, which tends to 1/2, as $n \to \infty$.

More precisely, we propose the following under-approximation

$$\Gamma\left(n+1\right) \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}-\frac{1}{24n}}{e}\right)^{n+\frac{1}{2}} := \nu_n,$$

and upper-approximation

$$\Gamma\left(n+1\right) \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}-\frac{1}{24n}+\frac{1}{48n^2}}{e}\right)^{n+\frac{1}{2}} := \mu_n.$$

The superiority of our new formulas over the Stirling and Burnside formulas are proved in the following table.

n	$n!-\sigma_n$	$\beta_n - n!$	$n!-\nu_n$	$\mu_n - n!$
10	30104	14421	730	25
25	5.1615×10^{22}	2.5364×10^{22}	5.1001×10^{20}	7.054×10^{18}
50	5.0647×10^{61}	2.5104×10^{61}	2.5172×10^{59}	1.7305×10^{57}
100	7.7739×10^{154}	3.8700×10^{154}	1.9377×10^{152}	6.6405×10^{149}
500	2.0334×10^{1130}	1.0158×10^{1130}	1.0161×10^{1127}	6.9475×10^{1123}
1000	3.3531×10^{2563}	1.6758×10^{2563}	8.3802×10^{2559}	2.8641×10^{2556}

We prove the following.

THEOREM 1. For every positive integer n, we have

$$\sqrt{2\pi}\left(\frac{n+\frac{1}{2}-\frac{1}{24n}}{e}\right)^{n+\frac{1}{2}} < \Gamma\left(n+1\right) < \sqrt{2\pi}\left(\frac{n+\frac{1}{2}-\frac{1}{24n}+\frac{1}{48n^2}}{e}\right)^{n+\frac{1}{2}}.$$

PROOF. Let us define the sequences

$$a_n = \ln \Gamma(n+1) - \left(n+\frac{1}{2}\right) \ln \left(\frac{n+\frac{1}{2}-\frac{1}{24n}}{e}\right) - \ln \sqrt{2\pi}$$

$$b_n = \ln \Gamma(n+1) - \left(n+\frac{1}{2}\right) \ln \left(\frac{n+\frac{1}{2}-\frac{1}{24n}+\frac{1}{48n^2}}{e}\right) - \ln \sqrt{2\pi}$$

which converge to zero. In order to prove that $a_n > 0$ and $b_n < 0$, we show that a_n is strictly decreasing and b_n is strictly increasing. In this sense, if designate $f(n) = a_{n+1} - a_n$ and $g(n) = b_{n+1} - b_n$, it suffices to show that $f(x) < 0$ and $g(x) > 0$, where

$$f(x) = \ln(x+1) - \left(x+\frac{3}{2}\right) \ln \left(\frac{x+\frac{3}{2}-\frac{1}{24(x+1)}}{e}\right) + \left(x+\frac{1}{2}\right) \ln \left(\frac{x+\frac{1}{2}-\frac{1}{24x}}{e}\right)$$

and

$$g(x) = \ln(x+1) - \left(x+\frac{3}{2}\right) \ln \left(\frac{x+\frac{3}{2}-\frac{1}{24(x+1)}+\frac{1}{48(x+1)^2}}{e}\right)$$

$$+\left(x+\frac{1}{2}\right) \ln \left(\frac{x+\frac{1}{2}-\frac{1}{24x}+\frac{1}{48x^2}}{e}\right).$$

We have $f''(x) < 0$ and $g''(x) > 0$, for every $x \in [1, \infty)$, since

$$f''(x) = -\frac{P(x)}{2x^2(x+1)^2(12x+24x^2-1)^2(60x+24x^2+35)^2}$$

and

$$g''(x) = \frac{Q(x)}{x^2(x+1)^2(24x^2-2x+48x^3+1)^2(190x+168x^2+48x^3+71)^2},$$

where

$$P(x) = 23\,975x + 279\,460x^2 + 1166\,400x^3 + 2468\,928x^4$$

$$+2764\,800x^5 + 1541\,376x^6 + 331\,776x^7 + 1225(x-1)$$

and

$$Q(x) = 6816x + 281\,169x^2 + 3569\,048x^3 + 17\,562\,852x^4 + 46\,653\,696x^5 + 74\,884\,576x^6$$

$$+75\,056\,640x^7 + 45\,988\,608x^8 + 15\,704\,064x^9 + 2267\,136x^{10} + 5041.$$

Finally, f is strictly concave, g is strictly convex, with $f(\infty) = g(\infty) = 0$, so $f < 0$ and $g > 0$ and the theorem is proved.

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