Applied Mathematics E-Notes, 11(2011), 261–273 © ISSN 1607-2510
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Explicit Solutions Of Irreducible Linear Systems Of Delay Differential Equations Of Dimension 2*

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Received 14 January 2011

Abstract

Let A and B be square matrices of dimension 2 with real entries and $r > 0$. We consider the system

$$\acute{X}(t) = AX(t) + BX(t-r),\ t \geq -r,$$

with X specified on the interval $[-r, 0]$. We assume that the system is irreducible in the sense that the matrix A has a single eigenvalue. We give an explicit formula for the general solution of the system by determining a fundamental matrix for the system.

1 Introduction

Ordinary differential equations (in one or several dimensions) are commonly used to model dynamical systems. It is known however that the evolution of real world systems always depends in some way on part or all of their own history. Therefore Delay Differential Equations are preferable as models of such systems. For this reason, there has been active research on the theory of these equations in the recent past. A detailed review of research on the subject and its applicability can be found in Farshid and Ulsoy [2] and Ruan and Wei [8] and the references cited therein.

This research deals mainly with questions of stability and other asymptotic aspects of solutions (see e.g. Pontryagin [7], Hayes [4], Noonburg [6] etc. and the references cited therein). In the multidimensional setting, these studies are done without any knowledge of explicit representations of the solutions in question. On the other hand, knowledge of explicit formulas for solutions of such systems would potentially ease some of the problems involved in studying them. In particular, it is then easy to write computer programmes for the study of properties of solutions.

Our aim in the present paper is to give a closed formula for the solution of a two dimensional linear system of Delay Differential Equations, under an irreducibility assumption.

We consider the following two dimensional Delay Differential System with real coefficients:

$$\acute{x}(t) = a_{11}x(t) + a_{12}y(t) + b_{11}x(t-r) + b_{12}y(t-r),\ t \geq 0 \tag{1}$$

*Mathematics Subject Classifications: 34K05, 34K12, 34K17.

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$$\acute{y}(t) = a_{21}x(t) + a_{22}y(t) + b_{21}x(t-r) + b_{22}y(t-r),\ t \geq 0 \tag{2}$$
$$x(t) = x_1(t),\ t \in [-r, 0] \tag{3}$$
$$y(t) = y_1(t),\ t \in [-r, 0], \tag{4}$$

where x_1 and y_1 are given functions and $r > 0$. We give a closed formula for the solution of this system when the matrix $A = (a_{ij})$ has a single eigenvalue and the functions x_1 and y_1 are integrable.

The results we present here also generalize some results which appear in [3], [5] etc. in the one dimensional setting. In addition, the fundamental matrix which we obtain here reduces to that known in the case of systems of ordinary differential equations, if the Delay Differential System has zero delay.

The rest of the paper is organized as follows: In Section 2 we introduce definitions. We also prove a number of technical Lemmas which we use in Section 3. Our main results Theorem 1 and Theorem 2 are presented in Section 3. The main step in the proof of these theorems is Lemma 6. Other results are Corollaries to these Theorems. We note that the results we obtain here can not be extended trivially to systems for which the matrix $A = (a_{ij})$ has two distinct eigenvalues.

2 Prerequisites

DEFINITION 1. A vector valued function $(x(t)\ y(t))^T_{t \geq -r}$($T$ denoting the transpose) with values in $\mathbb{R}^2$ is called a solution of (1-4) if it is continuous, satisfies (1-2) Lebesgue almost everywhere on $[0, \infty)$ and (3-4).

The system (1-4) can be rewritten as

$$\acute{X}(t) = AX(t) + BX(t-r),\ t \geq 0 \tag{5}$$
$$X(t) = (x_1(t)\ y_1(t))^T, t \in [-r, 0] \tag{6}$$

where $X = (x\ y)^T,\ A = (a_{ij})$ and $B = (b_{ij})$. We assume that the matrices A and B have real coefficients and A is not diagonalizable or is diagonalizable but has a single eigenvalue. In this case, there exists an invertible matrix Q such that

$$Q^{-1}AQ = D,\ \text{where } D = (d_{ij})_{ij \in \{1,2\}} \text{ with } d_{11} = d_{22} = \xi\ ,\ d_{12} = \tau,\ d_{21} = 0, \tag{7}$$

$\xi \in \mathbb{C}$, $\tau = 1$ if A is not diagonalizable and $\tau = 0$ otherwise. We shall use the symbol 0 for the real number 0, the zero vector in $\mathbb{R}^2$ and the zero matrix in $\mathbb{M}(2, 2, \mathbb{R})$, where $\mathbb{M}(2, 2, \mathbb{R})$ denotes the set of real 2×2 matrices. We use the symbol E for the multiplicative identity in $\mathbb{M}(2, 2, \mathbb{R})$. Let $Z := Q^{-1}X,\ H := Q^{-1}BQ,$ then the solution of the system (5-6) is $X := QZ$ where Z solves the system

$$\acute{Z}(t) = DZ(t) + HZ(t-r),\ t \geq 0 \tag{8}$$
$$Z(t) = Q^{-1}(x_1(t)\ y_1(t))^T,\ t \in [-r, 0]. \tag{9}$$

We now introduce the following definition:

DEFINITION 2. We call the function $G : [0,\infty) \to \mathbb{M}(2,2,\mathbb{R})$ the fundamental matrix associated with (5) if for any $\eta \in \mathbb{R}^2$,

$$X(t) := \begin{cases} G(t)\eta & : \quad t \in [0,\infty) \\ \eta 1_{\{0\}}(t) & : \quad t \in [-r,0] \end{cases}$$

is a solution of (5) with initial condition $X(t) = \eta 1_{\{0\}}(t),\ t \in [-r,0]$. Our first observation is the following Lemma which uses the notation in (7):

LEMMA 1. Let D be the matrix in (7) and $H \in \mathbb{M}(2,2,\mathbb{R})$, $g : [-r,0] \to \mathbb{R}^2$ be continuous at least on $[-r,0)$ and bounded on $[-r,0]$. Let $Z(t) := g(t),\ t \in [-r,0]$. For $t \in [kr,(k+1)r),\ k = 0,1,2,\dots$, let

$$Z(t) := e^{\xi(t-kr)}(E+(t-kr)M)Z(kr)+H\int_{kr}^{t}\psi(s,t)ds+(MH)\int_{kr}^{t}\int_{kr}^{s}\psi(u,t)duds, \tag{10}$$

where

$$Z(kr) := \lim_{t\uparrow kr} Z(t),\ k = 1,2,\dots, \tag{11}$$

$$M := \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix} \text{ and } \psi(v,w) := e^{\xi(w-v)}Z(v-r),\ v,w \in \mathbb{R}, \tag{12}$$

then

(i) Z is continuous on $[0,\infty)$

(ii) Z is differentiable on $[kr,(k+1)r)$, $k = 0,1,2,\dots$, where the derivative at the point kr is understood to be the derivative on the right.

(iii) for $k = 0,1,2,\dots$,

$$\acute{Z}(t) = DZ(t) + HZ(t-r) \tag{13}$$

for $t \in [kr,(k+1)r)$ and

$$Z(t) = Z(kr) \tag{14}$$

for $t = kr$.

PROOF. (i) and (ii) follow from the assumptions on g and the definition of Z. Note that by our assumption, $s \mapsto Z(s-r)$ need not be continuous from the left at the point r and hence $t \mapsto Z(t)$ need not be differentiable on the left at r.

(iii) We now show that (13) and (14) hold. For $t = kr$, the right hand side of (10) is $Z(kr)$ and hence (14) is satisfied. On $(kr,(k+1)r)$,

$$\begin{aligned} \acute{Z}(t) &= \xi e^{\xi(t-kr)}(E+(t-kr)M)Z(kr) + e^{\xi(t-kr)}MZ(kr) \\ &\quad + H\xi\int_{kr}^{t}\psi(s,t)ds + HZ(t-r) + \xi(MH)\int_{kr}^{t}\int_{kr}^{s}\psi(u,t)duds \end{aligned}$$

$$+(MH)\int_{kr}^{t}\psi(u,t)du$$

$$= \xi E\left(e^{\xi(t-kr)}(E+(t-kr)M)Z(kr)+H\int_{kr}^{t}\psi(s,t)ds+(MH)\int_{kr}^{t}\int_{kr}^{s}\psi(u,t)duds\right)$$

$$+\left(e^{\xi(t-kr)}MZ(kr)+(MH)\int_{kr}^{t}\psi(s,t)ds\right)+HZ(t-r)$$

$$= \xi EZ(t)+MZ(t)+HZ(t-r)=DZ(t)+HZ(t-r).$$

Further, by computing the limit $\lim\limits_{\substack{h\to 0\\ h>0}}\frac{Z(kr+h)-Z(kr)}{h}$, it is seen that the derivative on the right at kr exists and $\acute{Z}(kr)=DZ(kr)+HZ(kr-r)$. Therefore (13) holds.

In what follows, M shall denote the matrix in (12) and H an arbitrary element of $\mathbb{M}(2,2,\mathbb{R})$. Lemma 1 shows that (10-11) is a solution of the system (8-9) with $Q^{-1}(x_1(t)\ y_1(t))^T=g(t)$. It also shows that when we solve the system on successive intervals $[kr,(k+1)r)$, products of H, M and (MH) will appear in the solution. In the remainder of this section, we give those properties of these products which we will use in the sequel. For this purpose, we introduce some notation.

We define $p(E):=0$, $p(H):=p(M):=1$, $x^0:=E$ and write x^m for $\overbrace{x\cdots x}^{m\times}$, $x\in\mathbb{M}(2,2,\mathbb{R})$. If $n\geq 1$, $x_i\in\{H,M,E\}, i=1,\ldots,n$, then we define $p(x_1\cdots x_n):=\sum\limits_{i=1}^{n}p(x_i)$. $p(x)$ is the number of times that the matrices M and H appear as factors in the given factorization of x over $\{M,H,E\}$. Hence although $M^3=M^2=0$, $p(M^3)=3$ and $p(M^2)=2$. If $n\geq 1,\ x:=x_1\cdots x_n=0$ where $x_i\in\{M,H,E\},\ i=1,\ldots,n$, then we call x a *zero*.

If a set contains one or more *zeroes*, then all the *zeroes* shall be represented by the single symbol 0. Thus for sets A and C, $A=\{x,0\}$ says that the set A contains at least one *zero*, apart from the element x and $A=C\cup\{0\}$ says that A is the union of two sets- C and another set which contains one or more elements all of which are *zeroes*. Note that C may also contain *zeroes*. Therefore for our purposes, the set $\{E,M^2,M^3\}\cup\{M^4,M^5\}$ can be written as $\{E,0\}\cup\{0\}$. For $x\in\mathbb{M}(2,2,\mathbb{R})$, let T_x denote the linear transformation on $\mathbb{M}(2,2,\mathbb{R})$ defined by $T_x(y)=xy,\ y\in\mathbb{M}(2,2,\mathbb{R})$, $T_x(A):=\{T_x(y):y\in A\}$, $A\subseteq\mathbb{M}(2,2,\mathbb{R})$ and for $j\in\{0,1\}$ and $k\geq 0$, define

$$I_k^j:=\begin{cases}\{E\} & : \quad k=0\\ T_{(M^jH)}(I_{k-1}) & : \quad k\geq 1\end{cases},\qquad I_k:=I_k^0\cup I_k^1.$$

LEMMA 2. For $k\geq 1$,

$$T_M(I_k)=I_k^1\cup\{0\}. \tag{15}$$

PROOF. Since $T_M(I_k^1)=\{0\}$, $T_M(I_k)=T_M(I_k^0)\cup\{0\}=T_M(T_H(I_{k-1}))\cup\{0\}=T_{(MH)}(I_{k-1})\cup\{0\}=I_k^1\cup\{0\}$.

LEMMA 3. For $k \geq 1$, $\min\{p(x) : x \in I_k\} = \min\{p(x) : x \in I_k^0\}$ and $\max\{p(x) : x \in I_k\} = \max\{p(x) : x \in I_k^1\}$.

PROOF. $I_k = T_H(I_{k-1}) \cup T_{(MH)}(I_{k-1}) = \{Hx : x \in I_{k-1}\} \cup \{(MH)x : x \in I_{k-1}\}$, hence $\min\{p(x) : x \in I_k\} = \min\{p(Hx) : x \in I_{k-1}\} = \min\{p(x) : x \in I_k^0\}$. The proof that $\max\{p(x) : x \in I_k\} = \max\{p(x) : x \in I_k^1\}$ is similar.

REMARK 1. The assertion of Lemma 3 is also true for $k = 0$.

LEMMA 4. Let $k \geq 1$ then $\min\{p(x) : x \in I_k\} = k$ and $\max\{p(x) : x \in I_k\} = 2k$.

PROOF. If $k = 1$, then $I_1 = \{H, (MH)\}$ and hence since $p(H) = 1$ and $p(MH) = 2$, we have $1 = \min\{p(x) : x \in I_1\} = k$ and $2 = \max\{p(x) : x \in I_1\}$. Assume that the assertion is true for $k = m$. For $k = m + 1$, we have by Lemma 3 and the assumption of the induction that

$$\begin{aligned}
\min\{p(x) : x \in I_{m+1}\} &= \min\{p(x) : x \in I_{m+1}^0\} \\
&= \min\{p(x) : x \in T_H(I_m)\} \\
&= 1 + \min\{p(x) : x \in I_m\} = m + 1.
\end{aligned}$$

The proof that $\max\{p(x) : x \in I_k\} = 2k$ is similar.

REMARK 2. The assertion of Lemma 4 is also true for $k = 0$.

From Lemma 4 we have the following Corollary:

COROLLARY 1. Let $k \geq 1$ then $\max\{p(x) : x \in I_k^0\} = 2k - 1$ and $\min\{p(x) : x \in I_k^1\} = k + 1$.

PROOF. $\min\{p(x) : x \in I_k^1\} = \min\{p(x) : x \in T_{(MH)}(I_{k-1})\} = 2 + \min\{p(x) : x \in I_{k-1}\} = k + 1$, where the last equality follows from Lemma 4. The proof that $\max\{p(x) : x \in I_k^0\} = 2k - 1$ is similar.

REMARK 3. Lemma 4 implies that for $k \geq 1$, $\{x \in I_k : p(x) \leq k - 1\} = \emptyset$ and $\{x \in I_k : p(x) \geq 2k + 1\} = \emptyset$. Also, from Lemma 3 and Lemma 4, for $k \geq 1$, $\min\{p(x) : x \in I_k^0\} = k$ and $\max\{p(x) : x \in I_k^1\} = 2k$. This and Corollary 1 imply that the following sets are empty: $\{x \in I_k^0 : p(x) \leq k - 1\}$, $\{x \in I_k^0 : p(x) \geq 2k\}$, $\{x \in I_k^1 : p(x) \leq k\}, \{x \in I_k^1 : p(x) \geq 2k + 1\}$.

LEMMA 5. For $k \geq 1$, $\{x \in I_k^0 : p(x) = l\}$ is nonempty for $k \leq l \leq 2k - 1$ and $\{x \in I_k^1 : p(x) = l\}$ is nonempty for $k + 1 \leq l \leq 2k$.

PROOF. If $k = 1$, then $I_1^0 = \{H\}$, $I_1^1 = \{(MH)\}$ and the assertion is easily verified. Let the statement be true for $k = m \geq 2$. We now show that it is true for $k = m + 1$

$$\begin{aligned}
\{x &\in I_{m+1}^0 : p(x) = l\} = \{x \in T_H(I_m) : p(x) = l\} \\
&= T_H(\{x \in I_m^0 : p(x) = l - 1\}) \cup T_H(\{x \in I_m^1 : p(x) = l - 1\}).
\end{aligned}$$

By the assumption of the induction, $\{x \in I_m^0 : p(x) = l - 1\}$ is non-empty for $l = m + 1, \ldots, 2m$. Therefore $T_H(\{x \in I_m^0 : p(x) = l - 1\}) \neq \emptyset$, $l = m + 1, \ldots, 2m$. A similar argument shows that $T_H(\{x \in I_m^1 : p(x) = l - 1\}) \neq \emptyset$, $l = m + 2, \ldots, 2m + 1$. Consequently $\{x \in I_{m+1}^0 : p(x) = l\} \neq \emptyset$, $l = m + 1, \ldots, 2m + 1$, proving the first assertion. The second assertion is proven similarly.

REMARK 4. (i) For $l \geq 1,\ k \geq 1$, let $U_k^l := \{x \in I_k : p(x) = l\}$, then from Remark 3 and Lemma 5, $I_k = \cup\{U_k^l : l = k, \ldots, 2k\}$ - a disjoint union of non-empty sets and $U_k^l = \emptyset,\ l \leq k-1$ or $l \geq 2k+1$. By (15) and Lemma 5,

$$T_M(\{x \in I_k : p(x) = l\}) = \begin{cases} \{x \in I_k^1 : p(x) = l+1\} & : \quad l = k, \\ \{x \in I_k^1 : p(x) = l+1\} \cup \{0\} & : \quad k < l \leq 2k-1. \end{cases} \tag{16}$$

Also note that $T_M(\{x \in I_k : p(x) = 2k\}) = T_M((MH)^k) = \{0\}$. Therefore

$$T_M(\{x \in I_k : p(x) = l\}) = \begin{cases} \{x \in I_k^1 : p(x) = l+1\} & : \quad l = k, \\ \{x \in I_k^1 : p(x) = l+1\} \cup \{0\} & : \quad k < l \leq 2k-1, \\ \{0\} & : \quad l = 2k, \\ \emptyset & : \quad l \leq k-1 \text{ or } l \geq 2k+1. \end{cases} \tag{17}$$

(ii) It follows from Lemma 5 and Remark 3 that for $k \geq 2$,

$$(a) \quad T_H(\{x \in I_{k-1} : p(x) = l\}) = \begin{cases} \{x \in I_k^0 : p(x) = l+1\} & : \quad k-1 \leq l \leq 2k-2, \\ \emptyset & : \quad \text{otherwise.} \end{cases}$$

and

$$(b) \quad T_{(MH)}(\{x \in I_{k-1} : p(x) = l\}) = \begin{cases} \{x \in I_k^1 : p(x) = l+2\} & : \quad k-1 \leq l \leq 2k-2, \\ \emptyset & : \quad \text{otherwise.} \end{cases}$$

It is easy to check that (a) and (b) also hold for $k = 1$.

3 The Fundamental Matrix and Solutions of DDSs

The matrices D and H in this section are as in Lemma 1.

LEMMA 6. The system

$$\begin{aligned} \acute{Y}(t) &= DY(t) + HY(t-r),\ t \geq 0 \\ Y(t) &= z1_{\{0\}}(t),\ t \in [-r, 0],\ z \in \mathbb{R}^2 \end{aligned}$$

admits a unique solution given by

$$Y(t) := \begin{cases} z1_{\{0\}}(t) & : \quad t \in [-r, 0] \\ \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} x\left(\frac{(t-kr)^l}{l!} z + \frac{(t-kr)^{(l+1)}}{(l+1)!} w\right) & : \quad t \geq 0 \end{cases} \tag{18}$$

where $w = Mz$.

PROOF. The uniqueness follows from the step method in Lemma 1. Also, by Lemma 1, the solution is continuous. We shall prove the rest of the assertion by induction that $Z(t)$ in (10-11) and $Y(t)$ in (18) coincide on the intervals $[nr, (n+1)r],\ n = 0, 1, 2, \ldots$, where g in Lemma 1 is now given by $g(t) := z1_{\{0\}}(t), t \in [-r, 0]$.

Let $n=0$. By Lemma 1, $Z(0)=z$ and putting $t=0$ in (18), $Y(0)=z$. Let now $t\in(0,r)$, then by Lemma 1,

$$\begin{aligned} Z(t) &= e^{\xi t}(E+tM)z+Hz\int_0^t e^{\xi(t-s)}1_{\{0\}}(s-r)ds \\ &\quad +(MH)z\int_0^t\int_0^s e^{\xi(t-u)}1_{\{0\}}(u-r)duds \\ &= e^{\xi t}(z+tw). \end{aligned}$$

Also, by (18), if $t\in(0,r)$, then $Y(t)=e^{\xi t}(z+tw)$. By Lemma 1, $Z(r)=\lim\limits_{t\uparrow r}e^{\xi t}(z+tw)=e^{\xi r}(z+rw)$. If we set $t=r$ in (18) then

$$\begin{aligned} Y(r) &= \sum_{k=0}^{1}e^{\xi(r-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}}x\left(\frac{(r-kr)^l}{l!}z+\frac{(r-kr)^{l+1}}{(l+1)!}w\right) \\ &= e^{\xi r}\left(z+rw\right). \end{aligned}$$

Therefore $Z(t)=Y(t), t\in[0,r]$. Assume now that the formulas agree on $[(n-1)r,nr]$ for $n\geq 2$, then for $t\in[(n-1)r,nr]$ we have

$$Z(t)=\sum_{k=0}^{n-1}e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}}xF(t,k,l)=Y(t),$$

where $F(t,k,l):=\varphi(t,k,l)z+\varphi(t,k,l+1)w$ and φ is defined by $\varphi(t,k,l):=\frac{(t-kr)^l}{l!}$.

We will now show that for $t\in(nr,(n+1)r]$, we have

$$Z(t)=\sum_{k=0}^{n}e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}}xF(t,k,l).$$

By assumption $Z(nr)=\sum_{k=0}^{n-1}e^{\xi(n-k)r}\sum_{l=k}^{2k}\sum\limits_{\{x\in I_k:p(x)=l\}}xF(nr,k,l)$. If $s\in(nr,(n+1)r]$ then $s-r\in[(n-1)r,nr]$. Also, $\varphi(s-r,k,l)=\varphi(s,(k+1),l)$, hence for $s\in(nr,(n+1)r]$,

$$Z(s-r)=\sum_{k=0}^{n-1}e^{\xi(s-(k+1)r)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}}xF(s,k+1,l).$$

By Lemma 1, we then have that for $t\in(nr,(n+1)r)$,

$$Z(t) \quad = \quad e^{\xi(t-nr)}(E+(t-nr)M)Z(nr)+H\int_{nr}^{t}\psi(s,t)ds+(MH)\int_{nr}^{t}\int_{nr}^{s}\psi(u,t)duds$$

$$
\begin{aligned}
&= e^{\xi(t-nr)}(E+(t-nr)M)\sum_{k=0}^{n-1} e^{\xi(n-k)r}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(nr,k,l) \\
&\quad +H\int_{nr}^{t} e^{\xi(t-s)}\sum_{k=0}^{n-1} e^{\xi(s-(k+1)r)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(s,k+1,l)ds \\
&\quad +(MH)\int_{nr}^{t}\int_{nr}^{s} e^{\xi(t-u)}\sum_{k=0}^{n-1} e^{\xi(u-(k+1)r)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(u,k+1,l)duds.
\end{aligned}
$$

Let $G(t,k,l) := \int_{nr}^{t} F(s,k+1,l)ds$ and $L(t,k,l) := \int_{nr}^{t}\int_{nr}^{s} F(u,k+1,l)duds$, then

$$
\begin{aligned}
Z(t) &= \sum_{k=0}^{n-1} e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(nr,k,l) \\
&\quad +\sum_{k=0}^{n-1} e^{\xi(t-kr)}(t-nr)\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} (Mx)F(nr,k,l) \\
&\quad +\sum_{k=0}^{n-1} e^{\xi(t-(k+1)r)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} (Hx)G(t,k,l) \\
&\quad +\sum_{k=0}^{n-1} e^{\xi(t-(k+1)r)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} ((MH)x)L(t,k,l).
\end{aligned}
$$

Since $F(nr,0,0)=(z+nrw)$, $\{x\in I_0 : p(x)=0\}=E$, $Mz=w$ and $Mw=0$ we have

$$
\begin{aligned}
Z(t) &= e^{\xi t}(z+nrw)+\sum_{k=1}^{n-1} e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(nr,k,l) \\
&\quad +e^{\xi t}(t-nr)w+\sum_{k=1}^{n-1} e^{\xi(t-kr)}(t-nr)\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} (Mx)F(nr,k,l) \\
&\quad +\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k-1}^{2(k-1)}\sum_{\{x\in I_{k-1}:p(x)=l\}} (Hx)G(t,k-1,l) \\
&\quad +\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k-1}^{2(k-1)}\sum_{\{x\in I_{k-1}:p(x)=l\}} ((MH)x)L(t,k-1,l).
\end{aligned}
$$

Using (17), the fact that $Mx=0$ if $x\in I_k$ with $p(x)=2k$, Remark 4 (ii) (a) and Remark 4 (ii) (b), in this order, we have

$$
Z(t) = e^{\xi t}(z+tw)+\sum_{k=1}^{n-1} e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(nr,k,l)
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-1} e^{\xi(t-kr)}(t-nr)\sum_{l=k}^{2k-1}\sum_{\{x\in I_k^1:p(x)=l+1\}} xF(nr,k,l)\\
&+\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k-1}^{2(k-1)}\sum_{\{x\in I_k^0:p(x)=l+1\}} xG(t,k-1,l)\\
&+\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k-1}^{2(k-1)}\sum_{\{x\in I_k^1:p(x)=l+2\}} xL(t,k-1,l)\\
=\;& e^{\xi t}(z+tw)+\sum_{k=1}^{n-1} e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(nr,k,l)\\
&+\sum_{k=1}^{n-1} e^{\xi(t-kr)}(t-nr)\sum_{l=k+1}^{2k}\sum_{\{x\in I_k^1:p(x)=l\}} xF(nr,k,l-1)\\
&+\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k}^{2k-1}\sum_{\{x\in I_k^0:p(x)=l\}} xG(t,k-1,l-1)\\
&+\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k+1}^{2k}\sum_{\{x\in I_k^1:p(x)=l\}} xL(t,k-1,l-2).
\end{aligned}
$$

By Remark 3, $\{x \in I_k^1 : p(x) = k\} = \emptyset$ and $\{x \in I_k^0 : p(x) = 2k\} = \emptyset$. If we define $\sum\limits_{x\in A} g(x) = 0$ whenever $A = \emptyset$, for any function g, then

$$
\begin{aligned}
Z(t) \;=\;& e^{\xi t}(z+tw)+\sum_{k=1}^{n-1} e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k:p(x)=l\}} xF(nr,k,l)\\
&+\sum_{k=1}^{n-1} e^{\xi(t-kr)}(t-nr)\sum_{l=k}^{2k}\sum_{\{x\in I_k^1:p(x)=l\}} xF(nr,k,l-1)\\
&+\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k^0:p(x)=l\}} xG(t,k-1,l-1)\\
&+\sum_{k=1}^{n} e^{\xi(t-kr)}\sum_{l=k}^{2k}\sum_{\{x\in I_k^1:p(x)=l\}} xL(t,k-1,l-2). \qquad (19)
\end{aligned}
$$

Since $\int\limits_{nr}^{t} F(s,k,l)ds = F(t,k,l+1) - F(nr,k,l+1)$, $G(t,k,l) = F(t,k+1,l+1) - F(nr,k+1,l+1)$ and $L(t,k,l) = F(t,k+1,l+2) - F(nr,k+1,l+2) - (t-nr)F(nr,k+1,l+1)$. If we keep in mind that $\{x \in I_k : p(x) = l\} = \{x \in I_k^0 : p(x) = l\} \cup \{x \in I_k^1 :$

$p(x) = l\}$ and that $F(nr, n, l) = 0$ for all $l = 1, 2, \ldots$, then

$$\begin{aligned} Z(t) &= e^{\xi t}(z + tw) + \sum_{k=1}^{n-1} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} xF(nr, k, l) \\ &\quad + \sum_{k=1}^{n-1} e^{\xi(t-kr)}(t - nr) \sum_{l=k}^{2k} \sum_{\{x \in I_k^1 : p(x) = l\}} xF(nr, k, l-1) \\ &\quad + \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k^0 : p(x) = l\}} x(F(t, k, l) - F(nr, k, l)) \\ &\quad + \sum_{k=1}^{n} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k^1 : p(x) = l\}} x(F(t, k, l) - F(nr, k, l) - (t - nr)F(nr, k, l-1)) \\ &= \sum_{k=0}^{n} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} xF(t, k, l) = Y(t). \end{aligned}$$

From this and (11),

$$Z((n+1)r) = \sum_{k=0}^{n} e^{\xi((n+1)r-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} xF((n+1)r, k, l).$$

Also

$$\begin{aligned} Y((n+1)r) &= \sum_{k=0}^{n+1} e^{\xi((n+1)r-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} xF((n+1)r, k, l) \\ &= \sum_{k=0}^{n} e^{\xi((n+1)r-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} xF((n+1)r, k, l) \\ &+ \sum_{l=n+1}^{2(n+1)} \sum_{\{x \in I_{n+1} : p(x) = l\}} xF((n+1)r, n+1, l) \\ &= \sum_{k=0}^{n} e^{\xi((n+1)r-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} xF((n+1)r, k, l) = Z((n+1)r). \end{aligned}$$

Therefore $Z(t) = Y(t),\ t \in (nr, (n+1)r]$. This completes the proof of the Lemma.

The notation we use in the following Theorem is the same as that used in Section 2.

THEOREM 1. The fundamental matrix associated with (5) is given by

$$G(t) := \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x) = l\}} QxQ^{-1} \left(\frac{(t-kr)^l}{l!} E + \frac{(t-kr)^{(l+1)}}{(l+1)!} QMQ^{-1} \right) \tag{20}$$

for $t \geq 0$.

PROOF. By Lemma 6 the solution of (8) with the initial condition $Z(t) = z1_{\{0\}}(t),\ t \in [-r, 0], z \in \mathbb{R}^2$, is given by

$$Z(t) = \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x)=l\}} x \left(\frac{(t-kr)^l}{l!} z + \frac{(t-kr)^{(l+1)}}{(l+1)!} Mz \right).$$

The solution of (8) with the initial condition $Z(t) = Q^{-1}\eta 1_{\{0\}}(t),\ t \in [-r, 0],\ \eta \in \mathbb{R}^2$ is therefore given by

$$Z(t) = \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x)=l\}} x \left(\frac{(t-kr)^l}{l!} Q^{-1} + \frac{(t-kr)^{(l+1)}}{(l+1)!} Q^{-1}QMQ^{-1} \right) \eta.$$

Therefore

$$X(t) = \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x)=l\}} QxQ^{-1} \left(\frac{(t-kr)^l}{l!} E + \frac{(t-kr)^{(l+1)}}{(l+1)!} QMQ^{-1} \right) \eta$$

solves (5) with the initial condition $X(t) = \eta 1_{\{0\}}(t),\ t \in [-r, 0]$. This shows that the fundamental matrix associated with (5) is given by (20).

From Theorem 1, we obtain the following Corollary which generalizes the formula known in one dimension:

COROLLARY 2. If A is a diagonal matrix, i.e. $\tau = 0$, then the fundamental matrix associated with (5) is given by $G(t) = \sum_{k=0}^{[\frac{t}{r}]} \frac{B^k}{k!}(t-kr)^k e^{\xi(t-kr)}$.

PROOF. Note first of all that if A is diagonal, i.e. $\tau = 0$, then $M = 0$, (20) becomes

$$G(t) = \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} \sum_{l=k}^{2k} \sum_{\{x \in I_k : p(x)=l\}} QxQ^{-1} \frac{(t-kr)^l}{l!}.$$

Also,

$$\{x \in I_k : p(x) = l\} = \begin{cases} \{H^k\} & : \quad l = k, \\ \{0\} & : \quad l = k+1, \ldots, 2k. \end{cases}$$

Hence $G(t) = \sum_{k=0}^{[\frac{t}{r}]} e^{\xi(t-kr)} QH^kQ^{-1} \frac{(t-kr)^k}{k!} = \sum_{k=0}^{[\frac{t}{r}]} \frac{B^k}{k!}(t-kr)^k e^{\xi(t-kr)}$. This completes the proof.

We shall now write a general solution for (5). For this purpose, we shall require that the matrices A and B commute. In essence, it is a requirement that QMQ^{-1} and B commute.

LEMMA 7. A and B commute if and only if QMQ^{-1} and B commute.

PROOF.

$$AB = BA \iff Q(\xi E + M)Q^{-1}B = BQ(\xi E + M)Q^{-1}$$

$$\begin{aligned}
&\iff \quad \xi B + QMQ^{-1}B = B\xi + BQMQ^{-1} \\
&\iff \quad QMQ^{-1}B = BQMQ^{-1}.
\end{aligned}$$

THEOREM 2. Let $r > 0$ and $g : [-r, 0] \to \mathbb{R}^2$ be integrable. If the matrices QMQ^{-1} and B commute, then the solution $X(t)$ to the equation (5), with the integrable initial condition $X(t) = g(t),\ t \in [-r, 0]$, is given by

$$X(t) := \begin{cases} g(t) & : \quad t \in [-r, 0] \\ G(t)g(0) + B \int\limits_{-r}^{0} G(t-s-r)g(s)ds & : \quad t \geq 0 \end{cases}.$$

PROOF. From the definition of the fundamental matrix G, it satisfies $\acute{G}(t) = AG(t) + BG(t-r)$ for $t \geq 0$ and $G(t) = E1_{\{0\}}(t),\ t \in [-r, 0]$. For $t \geq 0$, an application of a generalization of [1] Theorem 16.8 gives that Lebesgue a.e. on $[0, \infty)$

$$\begin{aligned}
&\frac{dX(t)}{dt} \\
= \quad & \frac{dG(t)}{dt} g(0) + B \frac{d}{dt} \int\limits_{-r}^{0} G(t-s-r)g(s)ds \\
= \quad & \frac{dG(t)}{dt} g(0) + B \int\limits_{-r}^{0} \frac{\partial}{\partial t} G(t-s-r)g(s)ds \\
= \quad & (AG(t) + BG(t-r))g(0) + B \int\limits_{-r}^{0} (AG(t-s-r) + BG((t-r)-s-r))g(s)ds \\
= \quad & (AG(t)g(0) + BA \int\limits_{-r}^{0} G(t-s-r)g(s)ds) + B(G(t-r)g(0) \\
& + B \int\limits_{-r}^{0} G((t-r)-s-r)g(s)ds \\
= \quad & (AG(t)g(0) + AB \int\limits_{-r}^{0} G(t-s-r)g(s)ds) + B(G(t-r)g(0) \\
& + B \int\limits_{-r}^{0} G((t-r)-s-r)g(s)ds \\
= \quad & A(G(t)g(0) + B \int\limits_{-r}^{0} G(t-s-r)g(s)ds) + B(G(t-r)g(0)
\end{aligned}$$

$$
\begin{aligned}
&+B\int_{-r}^{0} G((t-r)-s-r)g(s)ds \\
= \quad & AX(t)+BX(t-r).
\end{aligned}
$$

From the preceding Theorem, we obtain the following:

COROLLARY 3. If the matrix A is a diagonal matrix, then the solution to the equation (5), with the integrable initial condition $X(t) = g(t),\ t \in [-r, 0]$, is given by

$$
X(t) := \begin{cases} g(t) & : \quad t \in [-r, 0], \\ G(t)g(0) + B\int_{-r}^{0} G(t-s-r)g(s)ds & : \quad t \geq 0. \end{cases}
$$

PROOF. If A is a diagonal matrix, then $M = 0$ and hence QMQ^{-1} and B commute.

Acknowledgment. This work was partly sponsored by the Research Grant of the University of Buea for 2010.

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